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A NOTE ON POSITIVELY EXPANSIVE ENDOMORPHISMS

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It is known that a compact connected Lie group G is toral if there exists an expansive transformation group of automorphisms of G (see [2]). In particular, such a group is toral if it has an expansive automorphism. Wu [4] has generalized the latter result by showing that a compact connected finite-dimensional group is abelian if it has an expansive automorphism. (All automorphisms and endomorphisms we consider are assumed to be continuous.)

An endomorphism φ of a topological group G is called *positively expansive* if there exists some neighborhood U of the identity element e of G such that, given $x \in U$ with $x \neq e$, then $\varphi^i(x) \notin U$ for some positive integer i. (If φ is an automorphism, and if the word "positive" is replaced by "nonzero" in the preceding sentence, then one obtains the condition for φ to be *expansive*.) We shall prove the following consequence, announced in [1], of Wu's result.

THEOREM. *Let G be a compact connected finite-dimensional group, and let there exist a positively expansive surjective endomorphism φ of G. Then G is abelian.*

Here we may use either covering dimension or inductive dimension, since these two notions coincide in general for locally compact groups and in particular in our case according to:

LEMMA 1. [2, Theorem 1.] *A compact group is metrizable if it has either a positively expansive endomorphism or an expansive automorphism.*

We shall use the following result, reference to which was kindly provided by Professor Jack Segal.

LEMMA 2. [3, Theorem 2a.] *If X is the projective limit of an inverse sequence of compact metric spaces each of dimension at most n, where n is a positive integer, then X is of dimension at most n.*

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PROOF OF THEOREM. Consider the inverse sequence

$$\ldots \leftarrow G \leftarrow G \leftarrow G \ldots$$

in which each of the connecting maps is φ, and let H be the projective limit of this sequence. Then H is a compact connected group of which G is a homomorphic image, and it is enough to show that H is abelian. By the two lemmas, H is finite-dimensional. In view of Wu's result mentioned above, it remains only to exhibit an expansive automorphism of H. Such an automorphism σ is constructed as in [2, section 4]. Denote by Z the set of all integers. If $x=(x_n \mid n \in Z) \in H$, let

$$\sigma(x) = (\varphi(x_n) \mid n \in Z) = (x_{n-1} \mid n \in Z)\,.$$

It is not known whether the hypothesis that G be finite-dimensional is really needed. However, the theorem fails if either the compactness or the connectedness of G is dropped. In [2] we construct a positively expansive automorphism of the noncompact connected simply-connected nonabelian nilpotent three-dimensional Lie group. As a counterexample for disconnected G, let C be the circle group, let $\sigma(x)=x^2$, $x \in C$, and let D be the discrete nonabelian group of order 6; then the endomorphism $(x,y) \to (\sigma(x),y)$ of $G=C \times D$ is positively expansive since σ is positively expansive on C.

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