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SOME RESULTS ON NARROW SPECTRAL ANALYSIS

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Introduction.

Let B be a commutative Banach algebra with an identity e. The dual Banach space of bounded linear functionals on B is denoted B^*. Without changing the topology on B we can assume (see for instance Loomis [5, p. 48]) that

$$(1) \qquad \|e\| = 1$$

and that

$$(2) \qquad \|fg\| \leqq \|f\| \, \|g\| ,$$

if f and g are arbitrary elements in B.

M denotes the space of all non-trivial complex-valued homomorphisms of B. The image of an element $f \in B$ by a homomorphism $x \in M$ is denoted $f(x)$. As is well known from the elementary theory of Banach algebras the homomorphisms in M are bounded linear functionals with norm 1. Thus M is a subset of the unit sphere $S = \{F \mid \|F\|^* = 1\}$ in B^*. $\| \ \|^*$ denotes the norm in B^*.

For any $F \in B^*$ and any $f \in B$ we define the functional $F \circ f$ by the relation

$$(F \circ f)(g) = F(fg) ,$$

for every $g \in B$. Using (2), it is easy to see that $F \circ f \in B^*$ and that

$$(3) \qquad \|F \circ f\|^* \leqq \|F\|^* \, \|f\| .$$

With this operation B^* can be interpreted as a module over the algebra B.

To every $F \in B^*$ we associate the linear subspace L_F of B^* which consists of all functionals of the form $F \circ f$, where $f \in B$. Since $F = F \circ e$, we have $F \in L_F$. We form the two subsets of M

$$\Lambda_F = \bar{L}_F \cap M$$

and

$$\Lambda_F' = \overline{L_F \cap S} \cap M ,$$

where the closure operations refer to weak* closure in B^*. Obviously $\Lambda_F' \subset \Lambda_F$. Using a terminology which goes back to the work of Beurling,

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originating in [1], we call Λ_F the *spectrum of F* and Λ_F' the *narrow spectrum of F*.

Let us, for the moment, assume that $F \neq 0$. It is then easy to see that the annihilator of L_F is a proper ideal in B (in fact closed), hence it is included in a maximal ideal. According to the general theory of Banach algebras this maximal ideal is the kernel of a certain homomorphism $x_0 \in M$. Since x_0 belongs to the annihilator of the annihilator of L_F, $x_0 \in \bar{L}_F$. Thus $F \neq 0$ implies that Λ_F is non-empty.

This well-known result is often referred to using the formulation that *spectral analysis holds for commutative Banach algebras with identity*. Our main objective is to show that in a large class of Banach algebras the same is true for *narrow spectral analysis*, i.e. with Λ_F' instead of Λ_F (Theorems 2 and 3). As a by-product of the investigation, we show that in certain algebras $\Lambda_F = \Lambda_F'$ (Theorem 1). The question whether these results are true for every commutative Banach algebra with identity remains open.

We now state our theorems:

THEOREM 1. *Let B be semi-simple and assume that there exists, to every $x_0 \in M$ and every neighborhood V of x_0, an $f \in B$ such that $|f(x)| \leqq 1$ on M, $f(x)=0$ outside V, and $f(x)=1$ on a neighborhood of x_0. Then $\Lambda_F' = \Lambda_F$.*

THEOREM 2. *We assume that to every x_1 and $x_2 \in M$ there exists an $f \in B$ such that $f(x)$ is real for every $x \in M$ and such that $f(x_1)$, $f(x_2)$ and 0 are all different. Then Λ_F' is non-empty for every $F \neq 0$.*

THEOREM 3. *We assume that B has one generator f_0 such that $\{f_0(x) \mid x \in M\}$ is the disc $\{z \mid |z| \leqq 1\}$. Then Λ_F' is non-empty for every $F \neq 0$.*

REMARK. Using standard terminology the assumption in Theorem 2 means that the subclass of all *real-valued* Gelfand transforms of elements in B separates the points on M strongly. This condition is obviously fulfilled if the class of all Gelfand transforms is closed under complex conjugation.

The origin of our work can be found in chapter 4 of [2], which contains generalizations of Beurling's theorem in [1] on the narrow closure of linear combinations of translates of uniformly continuous bounded functions on R. For other methods to extend Beurling's theorem see [3] and Koosis [4]. In the case when B is the group algebra of a discrete abelian group G, Theorem 2 is equivalent to Beurling's theorem. The essential new feature in our investigation is that we do not need regularity and semi-simplicity of B in order to show narrow spectral anaysis.

Theorem 3 is applicable to the case when B is the Banach algebra of complex sequences $\{a_n\}_0^\infty$ with the norm $\sum_0^\infty |a_n|$ and the convolution operation. Nyman [7, p. 50], has obtained a result in the same direction for this particular algebra. A close look at his investigation shows that his method proves that spectral analysis holds for this algebra if the spectral set is defined as $\Lambda_F'' = \overline{H_F \cap S} \cap M$, where H_F is the linear closure of $\overline{L_F \cap S}$. Obviously $\Lambda_F' \subset \Lambda_F'' \subset \Lambda_F$.

Preliminaries

As topology on M we introduce, as usual, the relativization of the weak* topology of B^* to M. M is then a compact Hausdorff space.

LEMMA 1. *Let C be a compact subset of B. Then*

$$\overline{\lim_{n\to\infty}} \sup_{f_\nu \in C} \left\| \prod_{\nu=1}^{n} f_\nu \right\|^{1/n} = \sup_{f\in C} \sup_{x\in M} |f(x)| \, .$$

PROOF. Put $\sup_{f\in C} \sup_{x\in M} |f(x)| = d$, and let ε be any positive number. C can be covered by a finite number of open balls $\|f - g_k\| < \varepsilon$, where $g_k \in C$. Using the relation

$$\lim_{n\to\infty} \|h^n\|^{1/n} = \sup_{x\in M} |h(x)| \, ,$$

which is true for any $h \in B$ by elementary Banach algebra theory, we see that there exists a constant D such that

$$\|g_k{}^n\| \leqq D(d+\varepsilon)^n$$

for every g_k and every n. The inequality (2) gives then that

$$\left\| \prod_1^n h_\nu \right\| \leqq D^N (d+\varepsilon)^n, \qquad n = 1, 2, \ldots \, ,$$

if the elements h_ν are chosen among the elements g_k, and N denotes the number of elements g_k.

We now let f_ν be arbitrary elements in C. To every f_ν there is an element h_ν of the kind introduced above such that

$$\|f_\nu - h_\nu\| < \varepsilon \, .$$

Then

$$\left\| \prod_1^n f_\nu \right\| = \left\| \prod_1^n (h_\nu + (f_\nu - h_\nu)) \right\|$$
$$\leqq \sum_{m=0}^{n} \binom{n}{m} D^N (d+\varepsilon)^m \varepsilon^{n-m} = D^N (d+2\varepsilon)^n \, .$$

Since ε was arbitrary, this proves Lemma 1.

Before stating Lemmas 2 and 3 we need some definitions.

DEFINITION 1. Let $F \in B^*$. A compact subset E of M is called an *F-determining subset* if for every compact subset C of B

$$\overline{\lim_{n\to\infty}} \sup_{f_\nu \in C} \left(\left\| F \circ \prod_1^n f_\nu \right\|^* \right)^{1/n} \leqq \sup_{f \in C} \sup_{x \in M} |f(x)| .$$

By Lemma 1 and (3) the set M is an F-determining subset. It is of interest to observe that the same is true for the Shilov boundary of M (see Naimark [6]).

DEFINITION 2. Let $F \in B^*$ and let E be an F-determining subset of M. We say that a subset V of E has the property $A(F)$ with respect to E if there exists, for every $\varepsilon > 0$, a compact set $C \subset B$ and elements $f_{n_\nu} \in C$, where $\{n_\nu\}$ are strictly increasing integers, such that

$$|f_{n_\nu}(x)| \leqq 1 \quad \text{on } E ,$$

$$|f_{n_\nu}(x)| \leqq \varepsilon \quad \text{on } E - V ,$$

and

$$(\|F \circ (f_{n_\nu})^{n_\nu}\|^*)^{1/n_\nu} \to 1, \quad \text{as } n_\nu \to \infty .$$

DEFINITION 3. Let $F \in B^*$ and let E be an F-determining subset of M. We say that a point $x_0 \in E$ has the property $A(F)$ with respect to E if every neighborhood of x_0 with respect to E has the property $A(F)$ with respect to E.

LEMMA 2. *We assume that B satisfies the assumption in Theorem* 1. *Then every $x_0 \in \Lambda_F$ has the property $A(F)$ with respect to M.*

PROOF. Let $x_0 \in \Lambda_F$ and let $g \in B$ be any element such that $g(x_0) \neq 0$. If $F \circ g = 0$ we have

$$F \circ f(g) = F \circ g(f) = 0$$

for every f, that is, g is in the annihilator of L_F. But then it also annihilates x_0 which gives a contradiction. Hence $F \circ g \neq 0$.

Now let V be an arbitrary neighborhood of x_0. By the assumption there exists an $f \in B$ such that

$$f(x) = 0, \qquad x \notin V$$

$$|f(x)| \leqq 1, \qquad x \in M$$

and such that $f(x) = 1$ in a neighborhood of x_0. It is enough to show that

$$(\|F\circ f^n\|^*)^{1/n} \to 1, \quad \text{as } n \to \infty ,$$

for the condition in Definition 2 is then fulfilled with $C = \{f\}$.

Let $g_1 \in B$ have the property that $g_1(x)$ vanishes outside the set where $f(x) = 1$ and that $g_1(x_0) \neq 0$. Since $F\circ g_1 \neq 0$ there exists a $g_2 \in B$ such that

$$F(g_1 g_2) = F\circ g_1(g_2) \neq 0 .$$

By the semisimplicity

$$f^n g_1 g_2 = g_1 g_2$$

for every n. Hence

$$\|F\circ f^n\|^* \|g_1 g_2\| \geqq |F\circ f^n(g_1 g_2)| = |F(f^n g_1 g_2)| = |F(g_1 g_2)| ,$$

and thus

$$\varliminf_{n\to\infty} (\|F\circ f^n\|^*)^{1/n} \geqq 1 .$$

That

$$\varlimsup_{n\to\infty} (\|F\circ f^n\|^*)^{1/n} \leqq 1$$

follows directly from (3).

Lemma 3. *Let $F \in B^*$ and suppose that $x_0 \in M$ has the property $A(F)$ with respect to some F-determining subset E of M. Then $x_0 \in \Lambda_F'$.*

Proof. We have to show the follwing: Given $\varepsilon > 0$ and $g_0, g_1, \ldots, g_q \in B$, there exists an element $G \in L_F \cap S$ such that

$$|G(g_p) - g_p(x_0)| < \varepsilon, \qquad p = 0, 1, \ldots, q .$$

Obviously it is no restriction to assume that $g_0 = e$ and that

$$g_p(x_0) = 0, \qquad p = 1, \ldots, q .$$

Then we have to show that $G \in L_F \cap S$ can be chosen in such a way that

$$(4) \qquad |G(e) - 1| < \varepsilon ,$$

$$(5) \qquad |G(g_p)| < \varepsilon, \qquad p = 1, \ldots, q .$$

We claim that this follows if we can find an element $H \in L_F$ such that

$$(6) \qquad \|H\circ g_p\|^* < \varepsilon \|H\|^* \neq 0, \qquad p = 1, \ldots, q .$$

To prove this, let us start from an arbitrary $h \in B$ and the immediate relations

$$\begin{aligned} |H(h)| = |H(he)| &= |H\circ h(e)| \\ &\leqq \|H\circ h\|^* \|e\| = \|H\circ h\|^* \leqq \|H\|^* \|h\| . \end{aligned}$$

By the definition of the norm in B^*, we can for any $\delta > 0$ find h, such that

$$\|H\|^* \, \|h\| \leqq 1+\delta \tag{7}$$

and

$$H(h) = H\circ h(e) \geqq 1-\delta \, . \tag{8}$$

By the relations above, we can normalize h in such a way that

$$\|H\circ h\|^* = 1 \, , \tag{9}$$

hence, also by these relations,

$$H\circ h(e) \leqq 1 \, . \tag{10}$$

Furthermore, by (7)

$$\begin{aligned} \|(H\circ h)\circ g_p\|^* &= \|(H\circ g_p)\circ h\|^* \\ &\leqq \|H\circ g_p\|^* \, \|h\| \\ &= \frac{\|H\circ g_p\|^*}{\|H\|^*} \|h\| \, \|H\|^* \\ &\leqq (1+\delta)\frac{\|H\circ g_p\|^*}{\|H\|^*}, \qquad p=1,\ldots,q \, . \end{aligned}$$

If δ is sufficiently small, the last relation and (6) show that

$$\|(H\circ h)\circ g_p\|^* < \varepsilon, \qquad p=1,\ldots,q \, . \tag{11}$$

Now consider $G=H\circ h$. By (9) it belongs to $L_F \cap S$; by (8) and (10), assuming $\delta < \varepsilon$, it safisfies (4), by (11) it satisfies (5). Hence our claim is justified.

In order to find $H \in L_F$ which satisfies (6), put

$$D = \sup_p \sup_{x\in E} |g_p(x)| \, ,$$

and let V denote the open neighborhood of x_0, with respect to E, where

$$|g_p(x)| < \tfrac{1}{4}\varepsilon \quad \text{for every } p \, .$$

By the assumption there exists a compact set $C \subset B$, and a sequence f_{n_ν} of elements in C such that

$$\begin{aligned} |f_{n_\nu}(x)| &\leqq 1 && \text{on } V \, , \\ |f_{n_\nu}(x)| &\leqq \tfrac{1}{4}\varepsilon/D && \text{on } E-V \, , \end{aligned}$$

and such that

$$(\|F\circ(f_{n_\nu})^{n_\nu}\|^*)^{1/n_\nu} \to 1 \quad \text{as } n_\nu \to \infty \, .$$

The elements of the form $g_p f_{n_\nu}$ are obviously contained in a compact subset of B. They satisfy

$$|g_p(x) f_{n_\nu}(x)| < \tfrac{1}{4}\varepsilon$$

on E. Hence, since E is F-determining (Def. 1), there exists an N_ε, such that

$$\left\| F\circ\left(\prod_{m=1}^{n_\nu} g_{p_m}\right) f_{n_\nu}^{n_\nu}\right\|^* < (\tfrac{1}{2}\varepsilon)^{n_\nu} \tag{13}$$

if $n_\nu \geqq N_\varepsilon$, and if p_m are arbitrary integers, $1 \leqq p_m \leqq p$. We put

$$\sup\left\| F\circ\left(\prod_{m=1}^{k} g_{p_m}\right) f_{n_\nu}^{n_\nu}\right\|^* = \delta_{k,n_\nu},$$

where the supremum is taken over all choices of p_m, $1 \leqq p_m \leqq p$. From (12) and (13) we see that there exists an n_ν such that

$$\delta_{0,n_\nu} \geqq (\tfrac{1}{2})^{n_\nu}$$

and

$$\delta_{n_\nu,n_\nu} < (\tfrac{1}{2}\varepsilon)^{n_\nu}.$$

Hence, it is possible to find a k_0 such that

$$\delta_{k_0+1,n_\nu} < \varepsilon\delta_{k_0,n_\nu} \neq 0 .$$

We define

$$g = \left(\prod_{1}^{k_0} g_{p_m}\right) f_{n_\nu}^{n_\nu},$$

as the element which gives the supremum in the definition of δ_{k_0,n_ν}. Then $H = F\circ g$ belongs to L_F and fulfills (6), and Lemma 3 is proved.

Proof of Theorem 1. The theorem is a direct consequence of Lemma 2 and Lemma 3.

Proof of Theorem 2

Theorem 2 is proved, by Lemma 3, if we can always find a point $x_0 \in M$ which has the property $A(F)$ with respect to M. If no such point exists, every point in M has an open neighbourhood which does not have the property $A(F)$. Thus M can be covered with a finite number of open sets which do not have the property $A(F)$. On the other hand, the set M itself has the property $A(F)$, which is seen by choosing $f_{n_\nu} = e$. Hence Theorem 2 is proved if we can prove the following lemma.

Lemma 4. *Under the assumptions in Theorem 2 the following is true: If O_1 and O_2 are open subsets of M and if $O_1 \cup O_2$ has the property $A(F)$*

with respect to M, then either O_1 or O_2 has the property $A(F)$ with respect to M.

Proof. We put $O_1 \cup O_2 = O$. Let $\varepsilon > 0$ be arbitrary and choose the compact set $C \subset B$ and $\{f_{n_\nu}\}$ in C in such a way that

$$|f_{n_\nu}(x)| \leqq 1 \quad \text{on } M\,,$$
$$|f_{n_\nu}(x)| \leqq \tfrac{1}{2}\varepsilon \quad \text{outside } O\,,$$

and such that

$$a_{n_\nu}^{1/n_\nu} = (\|F \circ (f_{n_\nu})^{n_\nu}\|^*)^{1/n_\nu} \to 1\,, \tag{14}$$

as $\nu \to \infty$. This can obviously be done by Definition 2. By C_0 we denote the class of all $f \in C$ such that $|f(x)| \leqq \frac{1}{2}\varepsilon$ outside O. Obviously, C_0 is compact in B, hence the corresponding class of functions $f(x) \in C(M)$ is compact. Using this compactness, which has the consequence that the functions $f(x)$ in the class considered are uniformly equi-continuous, it follows that the set K of all x such that $f(x) \geqq \varepsilon$ for at least some $f \in C_0$, is a compact subset of O.

The sets $K \cap CO_1$ and $K \cap CO_2$ are compact and disjoint. Since M is a compact Hausdorff space, we can apply Urysohn's lemma which shows that there exists a continuous real-valued function on M which is 0 on $K \cap CO_1$ and $\frac{1}{2}\pi$ on $K \cap CO_2$.

We consider the real sub-algebra of B which consists of those $f \in B$ for which $f(x)$ is real on M. By the assumption, the corresponding functions $f(x)$ form a point-separating real algebra of real-valued continuous functions on M. By the Stone–Weierstrass theorem, this algebra is dense in the algebra of real continuous functions on M with the uniform norm. Hence there exists an element $g \in B$ such that $g(x)$ is real and such that

$$-\frac{\pi}{12} \leqq g(x) \leqq \frac{\pi}{12} \quad \text{on } K \cap CO_1$$

and

$$\frac{5\pi}{12} \leqq g(x) \leqq \frac{7\pi}{12} \quad \text{on } K \cap CO_2\,.$$

We introduce the elements $\cos g$ and $\sin g$, defined by means of power series in g. Obviously,

$$(\cos g)(x) = \cos g(x)$$

and

$$(\sin g)(x) = \sin g(x)$$

for every $x \in M$. It is easy to see that for every real α the function

$$\cos\alpha \cos g(x) + \sin\alpha \sin g(x) = \cos(g(x)-\alpha)$$

has the modulus $\leqq \frac{1}{2}\sqrt{3}$ on at least one of the sets $K \cap CO_1$ and $K \cap CO_2$. We choose an integer $N > 0$ so large that $(\frac{3}{4})^N \leqq \varepsilon$. Then, for every real α, the element.

$$h_\alpha = (\cos\alpha \cos g + \sin\alpha \sin g)^{2N}$$

satisfies $0 \leqq h_\alpha(x) \leqq 1$ on M and satisfies $h_\alpha(x) \leqq \varepsilon$ on at least one of the sets $K \cap CO_1$ and $K \cap CO_2$.

Let ν be arbitrary and temporarily fixed. We form, for every real α, the element $h_\alpha^{n_\nu}$. It can be expanded in a strongly convergent series

$$h_\alpha^{n_\nu} = \sum_{k=0}^{\infty} D_k(\alpha) g^k ,$$

where $g^0 = e$. For every real t we have

$$(15) \quad \frac{1}{2\pi}\int_0^{2\pi} (\cos\alpha \cos t + \sin\alpha \sin t)^{2Nn_\nu}\, d\alpha = \frac{1}{2\pi}\int_0^{2\pi} (\cos\alpha)^{2Nn_\nu}\, d\alpha = E_{n_\nu} ,$$

which thus is independent of t. Hence

$$(16) \qquad \frac{1}{2\pi}\int_0^{2\pi} D_0(\alpha)\, d\alpha = E_{n_\nu}$$

whereas

$$(17) \qquad \frac{1}{2\pi}\int_0^{2\pi} D_k(\alpha)\, d\alpha = 0, \quad \text{if } k \geqq 1 .$$

By the definition of the norm in B^* there exists an element $f_0 \in B$ with $\|f_0\| \leqq 1$, such that

$$(F \circ (f_{n_\nu})^{n_\nu})(f_0) = \tfrac{1}{2} a_{n_\nu} ,$$

where a_{n_ν} is defined in (14). We form, for every real α, the element

$$G_\alpha = F \circ (f_{n_\nu})^{n_\nu} (h_\alpha)^{n_\nu}$$

in B^*. Using (16) and (17) we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} G_\alpha(f_0)\, d\alpha = \sum_0^{\infty} \int_0^{2\pi} D_k(\alpha)\, d\alpha F(g^k f_{n_\nu}^{n_\nu} f_0) = E_{n_\nu} F(f_{n_\nu}^{n_\nu} f_0) = \tfrac{1}{2} a_{n_\nu} E_{n_\nu} .$$

Hence there exists a value α_ν such that

$$|G_{\alpha_\nu}(f_0)| \geqq \tfrac{1}{2} a_{n_\nu} E_{n_\nu} ,$$

i.e., such that

$$||G_{\alpha_\nu}||^* \geqq \tfrac{1}{2} a_{n_\nu} E_{n_\nu} .$$

By (15)

$$(E_{n_\nu})^{1/n_\nu} \to 1 \quad \text{as } \nu \to \infty ,$$

for every N. Together with (14) we thus obtain

$$(18) \qquad (||G_{\alpha_\nu}||^*)^{1/n_\nu} = (||F \circ (f_{n_\nu} h_{\alpha_\nu})^{n_\nu}||^*)^{1/n_\nu} \to 1 \quad \text{as } \nu \to \infty .$$

Since $|h_{\alpha_\nu}(x)| \leqq \varepsilon$ on at least one of the sets $K \cap CO_1$ and $K \cap CO_2$, there exists an index i, $i = 1$ or 2, and a subsequence $\{\nu'\}$ of $\{\nu\}$, such that

$$(19) \qquad |f_{n_{\nu'}}(x)\, h_{\alpha_{\nu'}}(x)| \leqq \varepsilon$$

on $K \cap CO_i$, for every ν'. By the definition of K, and since

$$(20) \qquad |h_{\alpha_{\nu'}}(x)| \leqq 1$$

on M, (19) is true outside O_i. (20) implies that

$$|f_{n_{\nu'}}(x)\, h_{\alpha_{\nu'}}(x)| \leqq 1 ,$$

on M. Furthermore, it is easy to see that the subset of B consisting of all functions of the form fh_α, where $f \in C$ and α is real, is compact. All these properties, together with (18) show that O_i satisfies the conditions in Definition 2 for the number $\varepsilon > 0$ which was chosen. At least one of O_1 and O_2 must then satisfy the conditions for a sequence of arbitrarily small ε, hence fulfil the definition for every $\varepsilon > 0$.

Proof of Theorem 3

Since f_0 is a generator, the mapping f_0 is one-to-one between M and the closed unit disc. It is continuous in one direction, hence a homeomorphism. Hence we can assume that M is the unit disc $\{z \mid |z| \leqq 1\}$ and that $f_0(z) = z$ for every z.

Lemma 5. *Let $F \in B^*$ and put*

$$\overline{\lim_{n\to\infty}}\, (||F \circ f_0{}^n||^*)^{1/n} = a .$$

Then $a \leqq 1$ and the set $\{z \mid |z| = a\}$ is an F-determining subset of M.

Proof. That $a \leqq 1$ is a direct consequence of the relations

$$a = \overline{\lim_{n\to\infty}}\, (||F \circ f_0{}^n||^*)^{1/n} \leqq \lim_{n\to\infty} (||F||^* \, ||f_0{}^n||)^{1/n} \leqq \sup_{z \in M} |f_0(z)| = 1 .$$

As remarked before, M is itself an F-determining subset. For every polynomial $P(f_0)$ in f_0 the function $|P(f_0(z))| = |P(z)|$ on M attains its maximum on $\{z \mid |z|=1\}$, and the same must then be true for all $f(z)$, $f \in B$, since the polynomials $P(f_0)$ are dense in B and since the mapping $B \to C(M)$ is norm-decreasing. Hence $\{z \mid |z|=1\}$ is F-determining, and there remains only to consider the case when $0 \leqq a < 1$.

Let b be any number such that $a < b \leqq 1$. We can find a constant D such that

$$\|F \circ f_0{}^n\|^* \leqq Db^n \quad \text{for every } n . \tag{21}$$

We form the Banach algebra B' of all power series

$$f(z) = \sum_0^\infty a_\nu z^\nu$$

with the norm

$$\|f(z)\|' = \sum_0^\infty |a_\nu| b^\nu < \infty ,$$

and with multiplication (convolution of the sequences of coefficients) as operation. The maximal ideal space of this algebra can be identified with $\{z \mid |z| \leqq b\}$, as is well known ([5, p. 72]).

Every polynomial $P(f_0)$ can be associated with the element $P(z)$ in B', and by (21) we have, if $P(f_0)=\sum_0^N a_n f_0^n$,

$$\|F \circ P(f_0)\|^* = \left\| \sum_0^N a_n \, F \circ f_0{}^n \right\|^* \leqq D \|P(z)\|' . \tag{22}$$

Let $\varepsilon > 0$ be arbitrary and C an arbitrary compact subset of B. Since the polynomials $P(f_0)$ are dense in B, we can find a finite number of open balls

$$\{f \mid \|f - P_k(f_0)\| < \varepsilon\} ,$$

where P_k are polynomials, and such that the union of these balls covers C. It is no restriction to assume that, for every k,

$$\sup_{|z| \leqq b} |P_k(z)| \leqq d + \varepsilon ,$$

where

$$d = \sup_{f \in C} \sup_{|z| \leqq b} |f(z)| .$$

By Lemma 1, there exists a constant E such that

$$\left\| \prod_1^n R_\nu(z) \right\|' \leqq E(d+2\varepsilon)^n, \qquad n = 1, 2, \dots ,$$

if the polynomials R_ν are chosen among the polynomials P_k. Hence, by (22)

$$\left\| F \circ \prod_1^n R_\nu(f_0) \right\|^* \leqq DE(d+2\varepsilon)^n, \qquad n=1,2,\ldots,$$

with the same possibilities in the choice of R_ν. We now use the same way of arguing as in the last part of the proof of Lemma 1. Let f_ν be arbitrary elements in C. To every f_ν there is an element $R_\nu(f_0)$ of the kind introduced above, such that

$$\|f_\nu - R_\nu(f_0)\| < \varepsilon .$$

Then

$$\left\| F \circ \prod_1^n f_\nu \right\|^* = \left\| F \circ \prod_1^n \left(R_\nu(f_0) + (f_\nu - R_\nu(f_0))\right) \right\|^*$$

$$\leqq \sum_{m=0}^n \binom{n}{m} DE(d+2\varepsilon)^m \varepsilon^{n-m} = DE(d+3\varepsilon)^n .$$

Since ε was arbitrary, this implies that $\{z \mid |z| \leqq b\}$ is an F-determining set. Since this is true for every $b > a$, and since obviously

$$\sup_{f\in C} \sup_{|z|\leqq b} |f(z)|$$

is a continuous function of b in the interval $0 \leqq b \leqq 1$, the set $\{z \mid |z| \leqq a\}$ is also F-determining. As in the case $a=1$ we can conclude that $\{z \mid |z| = a\}$ is F-determining, too.

Proof of Theorem 3. If $a=0$, then by Lemma 5, the conditions in Definition 2 are fulfilled for any $\varepsilon > 0$ and with $C = \{e\}$. Hence, by Lemma 3, the theorem is true in this case.

Let $a > 0$, and let n be temporarily fixed. For every real α we form the element

$$h_\alpha = \frac{e}{4}\exp(i\alpha) + \frac{1}{2a} f_0 + \frac{f_0^2}{4a^2}\exp(-i\alpha) .$$

We obtain

$$h_\alpha(z) = \frac{z}{a}\left(\frac{a}{4z}\exp(i\alpha) + \frac{1}{2} + \frac{z}{4a}\exp(-i\alpha)\right).$$

With $z = ae^{i\varphi}$, we have

$$h_\alpha(ae^{i\varphi}) = e^{i\varphi}\left(\tfrac{1}{2} + \tfrac{1}{2}\cos(\alpha - \varphi)\right),$$

hence, if $|z| = a$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} h_\alpha^n(z)\, d\alpha = \frac{z^n}{a^n} A_n , \tag{23}$$

where A_n is a constant such that $\lim_{n\to\infty} A_n n^{\frac{1}{2}}$ exists and is different from 0. The left hand member of (23) is a polynomial in z. Hence (23) is true for every z with $|z| \leqq 1$.

From this we can see, in exactly the same way as in the corresponding discussion in the proof of Theorem 2, that

$$h_\alpha^n = \sum_{k=0}^{2n} D_k(\alpha) \frac{f_0{}^k}{a^k},$$

where

$$\frac{1}{2\pi} \int_0^{2\pi} D_n(\alpha)\, d\alpha = A_n ,$$

whereas

$$\frac{1}{2\pi} \int_0^{2\pi} D_k(\alpha)\, d\alpha = 0, \quad \text{if } k \neq n .$$

Proceeding in the same way as in the proof of Theorem 2 we find that there exists a real number α_n, such that

$$\|F \circ (h_{\alpha_n})^n\|^* \geqq \tfrac{1}{2} a_n A_n a^{-n} ,$$

where

$$a_n = \|F \circ f_0{}^n\|^* .$$

Hence there exists a subsequence $\{n_\nu\}$, such that α_{n_ν} converges to a number α, and such that

$$\lim_{\nu \to \infty} \left(\|F \circ (h_{\alpha_{n_\nu}})^{n_\nu}\|^*\right)^{1/n_\nu} = 1 , \tag{24}$$

as $\nu \to \infty$. Obviously we may assume that $\{n_\nu\}$ is a sub-sequence of $\{n!\}$.

We now claim that $z = ae^{i\alpha}$ has the property $A(F)$ with respect to the F-determining set $E = \{z \mid |z| = a\}$.

To prove this, let V be any neighbourhood of $ae^{i\alpha}$ with respect to E, and let $\varepsilon > 0$ be arbitrary. We can choose N such that, for sufficiently large ν

$$|h_{\alpha_\nu}(x)|^N < \varepsilon$$

on the set $E - V$. On V we have

$$|h_{\alpha_\nu}(x)|^N \leqq 1 .$$

By (24), and with $n_\nu/N = m_\nu$, which is an integer if ν is large,

$$\left(\|F \circ (h_{\alpha_{n_\nu}}^N)^{m_\nu}\|^*\right)^{1/m_\nu} \to 1 \quad \text{as } \nu \to \infty .$$

For every fixed N, the set of elements of the form h_α^N obviously form a compact sub-set of B. Hence the conditions in Definition 2 are fulfilled for the set V with respect to E. Thus V has the property $A(F)$ with respect to E, and since V was arbitrary, the same is true for the point $z = ae^{i\alpha}$.

Theorem 3 is then a direct consequence of Lemma 3.

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