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INTERPOLATION IN NON-QUASI-ANALYTIC CLASSES OF INFINITELY DIFFERENTIABLE FUNCTIONS

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1. Introduction.

Let $B=\{B_\nu\}_0^\infty$ be a sequence of positive numbers satisfying

$$(1.1)\qquad \begin{cases} \text{(a)}\ B_0 = 1\ , \\ \text{(b)}\ \log B_\nu \text{ is a convex function of } \nu\ , \\ \text{(c)}\ \sum_1^\infty B_{\nu-1}/B_\nu < \infty\ . \end{cases}$$

Let $\mathscr{C}_B$ be the class of infinitely differentiable functions $f(x)$, defined on $(-\infty,\infty)$, for which there exists a constant $C=C(f)$ such that

$$\sup_x |f^{(\nu)}(x)| \leqq C^{\nu+1} B_\nu, \qquad \nu=0,1,2,\ldots\ .$$

The condition (1.1) (c) is equivalent to

$$(1.2)\qquad \int\limits_0^\infty t^{-2} \log \left(\sum_0^\infty t^{2\nu}/B_\nu^{\,2}\right) dt \ < \ \infty$$

(see Mandelbrojt [7]), and this implies, by Denjoy–Carleman's theorem, that $\mathscr{C}_B$ is non-quasi-analytic (see [7], [9]).

Furthermore let $A=\{A_\nu\}_0^\infty$ be a sequence of positive numbers, $A_0=1$, and denote by c_A the class of sequences $\gamma=\{\gamma_\nu\}_0^\infty$ for which there exists a constant $c=c(\gamma)$ such that

$$(1.3)\qquad |\gamma_\nu| \leqq c^{\nu+1} A_\nu, \qquad \nu=0,1,2,\ldots\ .$$

We shall consider the following

Interpolation problem. *What conditions, imposed on A and B, are necessary and sufficient for the existence of a function $f(x) \in \mathscr{C}_B$ with*

$$f^{(\nu)}(0) = \gamma_\nu, \qquad \nu=0,1,2,\ldots\ ,$$

for every $\gamma \in c_A$?

Bang [1, pp. 87–91] obtained the following sufficient condition by real variable methods.

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Theorem 1 (Bang). *Let*

$$r_n = \sum_{\nu=n+1}^{\infty} B_{\nu-1}/B_\nu, \qquad n=1,2,3,\ldots,$$

and

$$\check{B}_\nu = \prod_{n=1}^{\nu-1} r_n^{-1}, \qquad \nu=1,2,3,\ldots.$$

Then, if

$$A_\nu \leqq k^\nu \check{B}_\nu, \qquad \nu=1,2,3,\ldots,$$

for some constant k, the interpolation problem is soluble.

For example, if $B_\nu=(\nu!)^\alpha$, $\alpha>1$, Theorem 1 shows that it is possible to interpolate if

$$A_\nu \leqq k^\nu(\nu!)^{\alpha-1}, \qquad \nu=1,2,3,\ldots.$$

The following more general sufficient condition has been proved independently by Carleson [4], Ehrenpreis [6] and Mityagin [8]:

Theorem 2 (Carleson; Ehrenpreis; Mityagin). *Let*

$$h(t) = (1+t^2)^{\frac{1}{2}} \sup_{\nu\geqq 0}|t|^\nu/B_\nu, \qquad -\infty<t<\infty,$$

and

$$\log H(r) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{r}{r^2+t^2}\log h(t)\,dt, \qquad r\geqq 0,$$

(the integral converges by (1.2)*). Define* $\tilde{B}_\nu$ *by*

$$\tilde{B}_\nu = \sup_{r\geqq 0} r^{\nu+\frac{1}{2}}/H(r), \qquad \nu=1,2,3,\ldots.$$

Then, if

$$A_\nu \leqq k^\nu \tilde{B}_\nu, \qquad \nu=1,2,3,\ldots, \quad \text{some } k,$$

the interpolation problem is soluble.

It follows from Theorem 2 that if $B_\nu=(\nu!)^\alpha$, $\alpha>1$, it is possible to solve the interpolation problem with $A=B$ (see [4]). In this case an explicit construction has been given by Džanašija [5].

By complex variable methods, Carleson (unpublished) has shown, with the additional hypothesis that $\log A_\nu$ be a convex function of ν, that if the interpolation problem is soluble there exists an integer μ so that

$$A_\nu \leqq k^\nu \tilde{B}_{\nu+\mu}, \qquad \nu=1,2,3,\ldots, \quad \text{some } k.$$

If $\log B_\nu=O(\nu^2)$, it follows that $\tilde{B}_{\nu+\mu}\leqq \lambda^\nu \tilde{B}_\nu$ for some λ, and consequently the condition in Theorem 2 is necessary and sufficient in this case.

We shall treat here the interpolation problem using a method which is elementary and entirely different from those of the above-mentioned authors. We prove the following theorem.

THEOREM 3. *Let* $B=\{B_\nu\}_0^\infty$ *be a sequence of positive numbers satisfying the conditions* (1.1) *and suppose that*

$$(1.4)\qquad \{\nu^{\frac{1}{2}+\delta}B_{\nu-1}/B_\nu\}_1^\infty \text{ is decreasing for some } \delta>0\,.$$

Define

$$(1.5)\qquad k_B(t) = \sum_{\nu=0}^{\infty} t^{2\nu}/B_\nu{}^2, \qquad -\infty<t<\infty\,,$$

$$(1.6)\qquad \log K(r) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{r}{r^2+t^2}\log k_B(t)\,dt, \qquad r\geqq 0\,,$$

and

$$(1.7)\qquad \hat{B}_\nu = \sup_{r\geqq 0} r^{\nu+\frac{1}{2}}/K(r), \qquad \nu=0,1,2,\ldots\,.$$

Then the condition

$$(1.8)\qquad A_\nu \leqq k^{\nu+1}\hat{B}_\nu, \qquad \nu=0,1,2,\ldots, \quad \textit{some } k\,,$$

is necessary for the interpolation problem to be soluble and sufficient for the interpolation from an arbitrary $\gamma\in c_A$ *to be possible by a function* $f(x)\in\mathscr{C}_{B'}$, *where* $B'=\{(B_\nu B_{\nu+1})^{\frac{1}{2}}\}_0^\infty$.

REMARK 1. In the proof of Theorem 3 we first show (Lemma 1) that the class $\mathscr{C}_B$ can be defined by a sequence $B^*=\{B_\nu{}^*\}_0^\infty$ of positive numbers satisfying the conditions (1.1) and such that the corresponding function $k_{B^*}(t)$ (see (1.5)) has the representation

$$(1.9)\qquad k_{B^*}(t) = \prod_{k=1}^{\infty}(1+t^2r_k{}^{-2}), \qquad r_k>0, \quad k=1,2,3,\ldots\,.$$

Using (1.4) this is easy to prove, and that is the reason why the assumption (1.4) has been added. We observe that it follows, from (1.1) (b) and (c), that $\{B_{\nu-1}/B_\nu\}_1^\infty$, is decreasing and $\nu\, B_{\nu-1}/B_\nu\to 0$ as $\nu\to\infty$. Hence condition (1.4) is not very restrictive; it is fulfilled for all "regular" sequences satisfying (1.1), e.g.

$$(1.10)\qquad \{(\nu\,!)^\alpha\}_1^\infty, \quad \alpha>1\,; \qquad \{(\nu(\log\nu)^\beta)^\nu\}_1^\infty, \quad \beta>1\,.$$

REMARK 2. If $\log B_\nu=O(\nu^2)$, the sequences B and B' define the same class $\mathscr{C}_B$ ([1, p. 22]). Consequently Theorem 3 gives a necessary and sufficient condition for the interpolation to be possible in this case. This clearly applies, for instance, to the sequences (1.10).

I wish to express my deep gratitude to Professor Lennart Carleson for introducing me to this problem and for all his valuable guidance during the preparation of this paper.

2. An extremal problem.

We start by considering the following

EXTREMAL PROBLEM. *Let* $\{m_\nu\}_0^n$ *be a given sequence of positive numbers with* $m_0 = 1$. *Let* p *be a fixed integer satisfying* $0 \leqq p \leqq n-1$ *and consider the functional*

$$F(f) = \sum_{\nu=0}^{n} m_\nu^{-2} \int_0^\infty [f^{(\nu)}(x)]^2 \, dx$$

for the class $C^{(p)}$ *of all those* n *times differentiable functions defined on* $[0,\infty)$ *for which*

$$f^{(\nu)}(0) = \begin{cases} 1 & \text{for } \nu = p, \\ 0 & \text{for } \nu \neq p, \end{cases} \qquad 0 \leqq \nu \leqq n-1 ,$$

and

$$f^{(\nu)}(x) \in L^2[0,\infty), \qquad \nu = 0,1,\ldots,n .$$

The problem is to find

$$\mu^{(p)} = \inf_{f \in C^{(p)}} F(f) .$$

(*For the case* $p=0$, *compare* [9].)

As in [9] we can show that $\mu^{(p)} = F(\varphi_p)$, where $\varphi_p(x)$ is the unique solution in $C^{(p)}$ of the differential equation

$$\sum_{\nu=0}^{n} (-1)^\nu m_\nu^{-2} D^{2\nu} y = 0 . \tag{2.1}$$

Partial integrations give, for $\nu \geqq 1$,

$$\int_0^\infty [\varphi_p^{(\nu)}(x)]^2 \, dx = \sum_{j=1}^{\nu} (-1)^j \varphi_p^{(\nu-j)}(0)\varphi_p^{(\nu+j-1)}(0) + (-1)^\nu \int_0^\infty \varphi_p(x)\varphi_p^{(2\nu)}(x) \, dx .$$

Hence

$$\mu^{(p)} = \sum_{\nu=1}^{n} m_\nu^{-2} \sum_{j=1}^{\nu} (-1)^j \varphi_p^{(\nu-j)}(0)\varphi_p^{(\nu+j-1)}(0) + \\ + \int_0^\infty \left(\sum_{\nu=0}^{n} (-1)^\nu m_\nu^{-2} \varphi_p^{(2\nu)}(x) \right) \varphi_p(x) \, dx$$

$$= \sum_{\nu=p+1}^{n} (-1)^{\nu-p} m_\nu^{-2} \varphi_p^{(2\nu-p-1)}(0) = \sum_{\nu=[\frac{1}{2}(n+p)]+1}^{n} (-1)^{\nu-p} m_\nu^{-2} \varphi_p^{(2\nu-p-1)}(0) .$$

In the calculation of $\mu^{(p)}$ it is sufficient (compare [9]) to consider the case where the characteristic equation

$$\sum_{\nu=0}^{n} (-1)^{\nu} m_{\nu}^{-2} z^{2\nu} = 0 \tag{2.2}$$

of the differential equation (2.1) has only simple roots; let them be $\pm r_k$, $\operatorname{Re} r_k > 0$, $k=1,2,\ldots,n$. Then

$$\varphi_p(x) = \sum_{k=1}^{n} c_k e^{-r_k x} ,$$

where

$$\sum_{k=1}^{n} c_k r_k^{\nu} = \begin{cases} (-1)^p & \text{for } \nu=p , \\ 0 & \text{for } \nu \neq p , \end{cases} \qquad 0 \leqq \nu \leqq n-1 . \tag{2.3}$$

Hence

$$\mu^{(p)} = \sum_{k=1}^{n} c_k P_k , \tag{2.4}$$

where

$$P_k = \sum_{\nu=[\frac{1}{2}(n+p)]+1}^{n} (-1)^{\nu+1} m_{\nu}^{-2} r_k^{2\nu-p-1}, \qquad k=1,2,\ldots,n .$$

Now let $P(z)$ be the polynomial of degree $\leqq n-1$ which takes the values P_k at the points $z=r_k$, $k=1,2,\ldots,n$, that is

$$P(z) = \sum_{j=0}^{n-1} A_j z^j ,$$

where

$$P(r_k) = \sum_{j=0}^{n-1} A_j r_k^j = P_k, \qquad k=1,2,\ldots,n .$$

From (2.4) and (2.3) we obtain

$$\mu^{(p)} = \sum_{j=0}^{n-1} A_j \sum_{k=1}^{n} c_k r_k^j = (-1)^p A_p . \tag{2.5}$$

Clearly the equation

$$\sum_{\nu=0}^{n} (-1)^{\nu} m_{\nu}^{-2} z^{2n-2\nu} = 0$$

has the roots $\pm \varrho_k$, $k=1,2,\ldots,n$, where $\varrho_k = r_k^{-1}$. Let α_j, $j=0,1,\ldots,n$, be the elementary symmetric functions of the variables ϱ_k, $k=1,2,\ldots,n$:

$$\alpha_0 = 1, \qquad \alpha_j = \sum \varrho_{k_1} \varrho_{k_2} \cdots \varrho_{k_j}, \quad j=1,2,\ldots,n ,$$

where the summation extends over all indices with

$$1 \leqq k_1 < k_2 < \ldots < k_j \leqq n .$$

From

$$\sum_{\nu=0}^{n} (-1)^{\nu} m_{\nu}^{-2} z^{2n-2\nu} = \sum_{i=0}^{n} (-1)^i \alpha_i z^{n-i} \sum_{j=0}^{n} \alpha_j z^{n-j}$$

we obtain the following relations between the m_ν, $\nu=0,1,\ldots,n$, and the α_j, $j=0,1,\ldots,n$:

$$(2.6)\qquad (-1)^\nu m_\nu^{-2} = \sum_{i+j=2\nu} (-1)^i \alpha_i\alpha_j, \qquad \nu=0,1,\ldots,n\,.$$

To construct the polynomial $P(z)$ we now put

$$Q(z) = \sum_{i=0}^{n} (-1)^i \alpha_i z^i \sum_{j=p+1}^{n} \alpha_j z^{j-p-1} + \sum_{\nu=[\frac{1}{2}(n+p)]+1}^{n} (-1)^{\nu+1} m_\nu^{-2} z^{2\nu-p-1}\,.$$

If $i+j-p-1=2\nu-p\geqq n$ we have $i\geqq p+1$, and then the coefficient of $z^{2\nu-p}$ in $Q(z)$ is

$$\sum_{i+j=2\nu+1} (-1)^i \alpha_i\alpha_j = 0\,.$$

By (2.6), the coefficient of $z^{2\nu-p-1}$ also vanishes if $2\nu-p-1\geqq n$. Hence the degree of $Q(z)$ is $\leqq n-1$, and since

$$(2.7)\qquad \sum_{i=0}^{n} (-1)^i \alpha_i z^i = \prod_{k=1}^{n} (1-\varrho_k z)$$

we have

$$Q(r_k) = P_k, \qquad k=1,2,\ldots,n\,.$$

Then $Q(z)\equiv P(z)$, and we obtain

$$A_p = \sum_{\substack{i+j=2p+1\\ j\geqq p+1}} (-1)^i \alpha_i\alpha_j$$

and so, from (2.5),

$$(2.8)\qquad \mu^{(p)} = \sum_{k=0}^{\min(p,\,n-p-1)} (-1)^k \alpha_{p-k}\alpha_{p+1+k}\,.$$

3. Lemmas.

For the proof of Theorem 3 we need some simple lemmas.

Lemma 1. *Let B be a sequence with the properties* (1.1) *and* (1.4). *Then there exists a sequence B^* satisfying* (1.1) *such that $\mathscr{C}_{B^*}=\mathscr{C}_B$ and*

$$(3.1)\qquad k_{B^*}(t) = \sum_{\nu=0}^{\infty} t^{2\nu}/B_\nu^{*2} = \prod_{k=1}^{\infty} (1+t^2 r_k^{-2})\,,$$

where $r_k>0$, $k=1,2,3,\ldots$.

Proof. Put

$$r_k = B_k/B_{k-1}, \qquad k=1,2,3,\ldots\,,$$

and then define the numbers $B_\nu^*>0$, $\nu=0,1,2,\ldots$, by (3.1). Then by Cauchy's estimates

$$(3.2) \qquad \frac{1}{B_\nu^{*2}} \leqq \min_{r>0} \frac{k_{B^*}(r)}{r^{2\nu}} \leqq \left(\frac{B_{\nu-1}}{B_\nu}\right)^{2\nu} k_{B^*}\left(\frac{B_\nu}{B_{\nu-1}}\right).$$

By (1.1) (b) the sequence $\{B_{k-1}/B_k\}_1^\infty$ is decreasing. Hence

$$(3.3) \qquad \prod_{k=1}^{\nu} \left(1+\frac{B_{k-1}^2}{B_k^2}\frac{B_\nu^2}{B_{\nu-1}^2}\right) \leqq 2^\nu \left(\prod_{k=1}^{\nu} \frac{B_{k-1}B_\nu}{B_k B_{\nu-1}}\right)^2 = \frac{2^\nu B_\nu^{2\nu}}{B_{\nu-1}^{2\nu}}\frac{1}{B_\nu^2}.$$

It follows from (3.2), (3.3), and (1.4) that

$$\begin{aligned}(B_\nu^2/B_\nu^{*2})^{1/\nu} &\leqq 2\exp\left(\nu^{-1}(B_\nu^2/B_{\nu-1}^2)\sum_{k=\nu+1}^{\infty}(B_{k-1}/B_k)^2\right)\\ &\leqq 2\exp\left(\nu^{2\delta}\sum_{k=\nu+1}^{\infty}k^{-(1+2\delta)}\right)\\ &\leqq 2\exp(\tfrac{1}{2}\delta^{-1}),\end{aligned}$$

and therefore

$$B_\nu \leqq (2^{\frac{1}{2}}e^{1/4\delta})^\nu B_\nu^*, \qquad \nu=1,2,3,\ldots .$$

But $B_\nu^* < B_\nu$, $\nu=1,2,3,\ldots$, and so $\mathscr{C}_{B^*}=\mathscr{C}_B$ ([1, p. 12]).

Since $k_{B^*}(it^{\frac{1}{2}})$ has only real zeros we have (see Boas [3, p. 24])

$$B_\nu^{*2} < (\nu/(\nu+1))^{\frac{1}{2}} B_{\nu-1}^* B_{\nu+1}^*;$$

hence B^* fulfils (1.1) (b). Obviously B^* also satisfies (1.1) (c).

Lemma 1 permits us to assume that

$$k_B(t) = \prod_{k=1}^{\infty}(1+{}^2r_k^{-2}),$$

where $r_k>0$, $k=1,2,3,\ldots$. For $n=1,2,3,\ldots$, we put

$$k_{B,n}(t) = \prod_{k=1}^{n}(1+t^2r_k^{-2}) = \sum_{\nu=0}^{n} t^{2\nu}/B_{\nu,n}^2 .$$

Then

$$(3.4) \qquad B_{0,n} = B_0 = 1, \qquad n=1,2,3,\ldots,$$

and for $n\geqq\nu$, $\nu=1,2,3,\ldots$,

$$(3.5) \qquad B_{\nu,\nu} > B_{\nu,\nu+1} > \ldots > B_{\nu,n} > \ldots > B_\nu,$$

$$(3.6) \qquad \lim_{n\to\infty} B_{\nu,n} = B_\nu .$$

Further, setting

$$(3.7) \qquad \hat{B}_{\nu,n} = \sup_{r\geqq 0} r^{\nu+\frac{1}{2}}/K_n(r), \qquad n>\nu,$$

where

$$K_n(r) = \exp\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{r}{r^2+t^2}\log k_{B,n}(t)\,dt\right),$$

we have obviously (compare (1.7)) for $n>\nu$, $\nu=0,1,2,\ldots$,

$$\hat{B}_{\nu,\nu+1} \geqq \hat{B}_{\nu,\nu+2} \geqq \ldots \geqq \hat{B}_{\nu,n} \geqq \ldots \geqq \hat{B}_{\nu}, \tag{3.8}$$

$$\lim_{n\to\infty} \hat{B}_{\nu,n} = \hat{B}_{\nu}. \tag{3.9}$$

Now let us fix n and use the simplified notation (compare Section 2)

$$\begin{cases} m_\nu = B_{\nu,n}, & \nu=0,1,\ldots,n, \\ \hat{m}_\nu = \hat{B}_{\nu,n}, & \nu=0,1,\ldots,n-1. \end{cases} \tag{3.10}$$

We also use the rest of the notation from our discussion of the extremal problem in Section 2.

LEMMA 2.

$$\hat{m}_\nu = \sup_{r\geqq 0} \frac{r^{\nu+\frac{1}{2}}}{\sum_0^n \alpha_i r^i}, \qquad \nu=0,1,\ldots,n-1.$$

PROOF. If

$$p_n(z) = c_0 \prod_{\nu=1}^{n} (z-c_\nu),$$

where

$$\operatorname{Im} c_\nu > 0, \qquad \nu=0,1,\ldots,n;$$

then, for $\operatorname{Im} z>0$ ([10, p. 135]),

$$\log|p_n(\bar{z})| = \frac{\operatorname{Im} z}{\pi} \int_{-\infty}^{\infty} \frac{\log|p_n(t)|}{|t-z|^2}\, dt.$$

For real t,

$$k_{B,n}(t) = \left| m_n^{-1} \prod_{\nu=1}^{n} (t+ir_\nu) \right|^2.$$

Putting $z=ir$, $r>0$, and $c_0=m_n^{-1}$, $c_\nu=i\bar{r}_\nu$, $1\leqq\nu\leqq n$, we find

$$\log K_n(r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{r}{r^2+t^2} \log k_{B,n}(t)\, dt$$

$$= \log\left| m_n^{-1} \prod_{\nu=1}^{n} (r+r_\nu) \right| = \log \prod_{\nu=1}^{n} (1+\varrho_\nu r).$$

Lemma 2 now follows from (3.7), (3.10) and (2.7).

LEMMA 3.

$$\frac{\alpha_{k-1}}{\alpha_k} < \frac{k}{k+1} \frac{\alpha_k}{\alpha_{k+1}}, \qquad k=1,2,\ldots,n-1.$$

PROOF. Let

$$\prod_{\nu=1}^{n}(x+\varrho_\nu) = \sum_{k=0}^{n}\binom{n}{k} p_k x^{n-k},$$

that is,

$$\alpha_k = \binom{n}{k} p_k, \qquad k=0,1,\ldots,n .$$

Then the lemma follows immediately from Newton's inequality (see e.g. [2, p. 11])

$$p_{k-1}p_{k+1} \leqq p_k^2, \qquad k=1,2,\ldots,n-1 .$$

LEMMA 4.

$$\alpha_\nu\alpha_{\nu+1} \leqq \hat{m}_\nu^{-2} < 2\cdot 4^{\nu+1}\alpha_\nu\alpha_{\nu+1}, \qquad \nu=0,1,\ldots,n-1 .$$

PROOF. The left hand inequality follows from

$$\hat{m}_\nu^2 = \sup_{r\geqq 0}\frac{r^{2\nu+1}}{(\sum_0^n \alpha_i r^i)^2} \leqq \sup_{r\geqq 0}\frac{r^\nu}{\sum_0^n \alpha_i r^i}\sup_{r\geqq 0}\frac{r^{\nu+1}}{\sum_0^n \alpha_i r^i} \leqq \frac{1}{\alpha_\nu\alpha_{\nu+1}} .$$

Next, take $r=\frac{1}{2}\alpha_\nu/\alpha_{\nu+1}$ in Lemma 2 to obtain

$$\hat{m}_\nu^{-2} = \inf_{r>0} r^{-1}\left(\sum_{i=0}^{n}\alpha_i r^{i-\nu}\right)^2 \leqq 2(\alpha_{\nu+1}/\alpha_\nu)\left(\sum_{i=0}^{n}2^{\nu-i}\frac{\alpha_i\alpha_{\nu+1}^{\nu-i}}{\alpha_\nu^{\nu-i}}\right)^2 .$$

By Lemma 3,

$$\frac{\alpha_{k-1}}{\alpha_k} < \frac{\alpha_k}{\alpha_{k+1}}, \qquad k=1,2,\ldots,n-1 , \tag{3.11}$$

and repeated use of this inequality yields

$$\frac{\alpha_i\alpha_{\nu+1}^{\nu-i}}{\alpha_\nu^{\nu-i}} \leqq \alpha_\nu, \qquad i=0,1,\ldots,n .$$

Hence

$$\frac{1}{\hat{m}_\nu^2} \leqq \frac{2\alpha_{\nu+1}}{\alpha_\nu}\alpha_\nu^2\left(\sum_{i=0}^{n}2^{\nu-i}\right)^2 < 2\cdot 4^{\nu+1}\alpha_\nu\alpha_{\nu+1} ,$$

as required.

LEMMA 5.

$$\frac{2}{p+2}\alpha_p\alpha_{p+1} \leqq \mu^{(p)} \leqq \alpha_p\alpha_{p+1}, \qquad p = 0,1,\ldots,n-1 .$$

PROOF. The right hand inequality follows immediately from (2.8) and (3.11). For $1\leqq p\leqq n-2$,

$$\mu^{(p)} \geqq \alpha_p\alpha_{p+1}-\alpha_{p-1}\alpha_{p+2} ,$$

and, from Lemma 3, we have

$$\alpha_{p-1}\alpha_{p+2} < \frac{p}{p+2}\alpha_p\alpha_{p+1} ,$$

which yields the left hand inequality. The cases $p=0$ and $p=n-1$ are trivial.

LEMMA 6. *Let f be a function with a continuous derivative on $[0,\infty)$ for which*

$$\int_0^\infty |f(x)|^2\,dx = M^2 > 0 \quad and \quad \int_0^\infty |f'(x)|^2\,dx = N^2 .$$

Then

$$\sup_{x\geq 0}|f(x)| < 2(MN)^{\frac{1}{2}} .$$

PROOF. Suppose $a>0$ is arbitrary. For $0\leqq x\leqq a^2$, we have

$$|f(x)-f(0)|^2 = \left|\int_0^x f'(t)\,dt\right|^2 \leqq a^2N^2 .$$

Hence

$$|f(0)| \leqq aN + \min_{0\leqq x\leqq a^2}|f(x)| < aN + M/a .$$

Choosing a to make the right hand side a minimum, we obtain

$$|f(0)| < (MN)^{\frac{1}{2}} .$$

By means of the translation $x\to x+x_0$, $x_0>0$, it follows immediately that $|f(x_0)|$ satisfies the same inequality, and this proves the lemma.— Lemma 6 also follows as a special case of a theorem of Nagy; see Beckenbach–Bellman [2, p. 167].

4. Proof of Theorem 3; the sufficiency.

We now proceed to the proof of the sufficiency of the condition (1.8) in Theorem 3. Fix n, and let

$$\Phi_n(x) = \sum_{p=0}^{n-1}\gamma_p\varphi_p(x) ,$$

where $\varphi_p(x)$ is the extremal function of the extremal problem in Section 2. It follows from

$$\Phi_n^{(\nu)}(x) = \sum_{p=0}^{n-1}\gamma_p\varphi_p^{(\nu)}(x)$$

that

$$\Phi_n^{(\nu)}(0) = \gamma_\nu, \qquad \nu=0,1,\ldots,n-1 ,$$

and

$$|\Phi_n^{(\nu)}(x)| \leqq \sum_{p=0}^{n-1} c^{p+1} A_p |\varphi_p^{(\nu)}(x)| .$$

Since $F(\varphi_p) = \mu^{(p)}$, we have

$$\int_0^\infty |\varphi_p^{(\nu)}(x)|^2 \, dx < \mu^{(p)} m_\nu^2, \qquad \nu = 0, 1, \ldots, n ,$$

and hence, by Lemma 6,

$$\sup_{x \geqq 0} |\varphi_p^{(\nu)}(x)| < 2 (\mu^{(p)} m_\nu m_{\nu+1})^{\frac{1}{2}}, \qquad \nu = 0, 1, \ldots, n-1 .$$

Then

$$\sup_{x \geqq 0} |\Phi_n^{(\nu)}(x)| < 2 (m_\nu m_{\nu+1})^{\frac{1}{2}} \sum_{p=0}^{n-1} c^{p+1} A_p (\mu^{(p)})^{\frac{1}{2}}, \qquad \nu = 0, 1, \ldots, n-1 .$$

Obviously we may assume

$$A_p \leqq (\tfrac{1}{2} c^{-1})^{p+1} \hat{B}_p, \qquad p = 0, 1, 2, \ldots ,$$

instead of (1.8). But $\hat{B}_p \leqq \hat{m}_p$ and, by Lemmas 4 and 5,

$$\hat{m}_p^{\,2} \mu^{(p)} \leqq \frac{\mu^{(p)}}{\alpha_p \alpha_{p+1}} \leqq 1 .$$

This gives the estimate

$$\sup_{x \geqq 0} |\Phi_n^{(\nu)}(x)| < 2 (m_\nu m_{\nu+1})^{\frac{1}{2}} \sum_{p=0}^{n-1} (\tfrac{1}{2})^{p+1}, \qquad 0 \leqq \nu \leqq n-1 ,$$

and so

$$\sup_{x \geqq 0} |\Phi_n^{(\nu)}(x)| < 2 (B_{\nu,n} B_{\nu+1,n})^{\frac{1}{2}}, \qquad 0 \leqq \nu \leqq n-1 . \tag{4.1}$$

By (3.5) this means that for fixed $\nu \geqq 0$ the sequence $\{\Phi_n^{(\nu)}(x)\}$ is uniformly bounded and equicontinuous. Then there exists a subsequence $\{\Phi_{n_k}(x)\}$ converging to an infinitely differentiable function $\Phi(x)$ and such that $\Phi_{n_k}^{(\nu)}(x) \to \Phi^{(\nu)}(x)$, $\nu = 0, 1, 2, \ldots$, uniformly on every compact subinterval $[0, a]$ (see e.g. Mandelbrojt [7]; compare also [9]). Passing to the limit in (4.1) we obtain

$$\sup_{x \geqq 0} |\Phi^{(\nu)}(x)| \leqq 2 (B_\nu B_{\nu+1})^{\frac{1}{2}}, \qquad \nu = 0, 1, 2, \ldots .$$

Let $\Phi_1(x)$ be the solution we obtain in the same way interpolating from the sequence $\{(-1)^\nu \gamma_\nu\}_0^\infty$. Define

$$\Phi(x) = \Phi_1(-x), \qquad x \leqq 0 .$$

Then $\Phi(x)$ will be infinitely differentiable on $(-\infty, \infty)$, and the derivatives satisfy

$$\left.\begin{aligned} \Phi^{(\nu)}(0) &= \gamma_\nu, \\ \sup_x |\Phi^{(\nu)}(x)| &\leqq 2 (B_\nu B_{\nu+1})^{\frac{1}{2}}, \end{aligned}\right\} \qquad \nu = 0, 1, 2, \ldots .$$

This proves the sufficiency part of Theorem 3.

5. Proof of Theorem 3; the necessity.

For the proof of the necessity of condition (1.8) in Theorem 3 we fix a non-negative integer p and consider the sequence γ in c_A for which

$$\gamma_\nu = \begin{cases} 0 & \text{if } \nu \neq p, \\ A_p & \text{if } \nu = p. \end{cases}$$

By assumption, $\mathscr{C}_B$ contains a function $f_p(x)$ satisfying

$$f_p^{(\nu)}(0) = \gamma_\nu, \qquad \nu = 0, 1, 2, \ldots .$$

Since $\mathscr{C}_B$ is non-quasi-analytic we may assume that $f_p(x) \equiv 0$ for $x \geqq a > 0$. Using the notation from Section 2 we have, for $n > p$,

$$F(f_p) \geqq A_p^2 \mu^{(p)} .$$

Elementary considerations show that in the inequalities

$$\sup_x |f_p^{(\nu)}(x)| \leqq C^{\nu+1} B_\nu, \qquad \nu = 0, 1, 2, \ldots ,$$

we can choose C independent of p. Furthermore, we may obviously assume that $C < 1$. Then, since $B_\nu \leqq m_\nu$ for $0 \leqq \nu \leqq n$,

$$F(f_p) = \sum_{\nu=0}^{n} m_\nu^{-2} \int_0^a |f_p^{(\nu)}(x)|^2 \, dx \leqq \frac{aC^2}{1-C^2}.$$

Lemmas 5 and 4 yield

$$\frac{1}{\mu^{(p)}} \leqq \frac{p+2}{2\alpha_p \alpha_{p+1}} < (p+2)\, 4^{p+1} \hat{m}_p^{\,2} \leqq 4^{2p+2} \hat{m}_p^{\,2} .$$

Hence for some constant k, independent of n and p,

$$A_p \leqq k^{p+1} \hat{B}_{p,n}, \qquad n > p .$$

Letting $n \to \infty$, we obtain

$$A_p \leqq k^{p+1} \hat{B}_p, \qquad p = 0, 1, 2, \ldots .$$

This completes the proof of Theorem 3.

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