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AN ASYMPTOTIC FORMULA FOR THE DERIVATIVES OF ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE

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1.

Let f be a non-negative function and assume that f and $\log f$ both belong to $L(-\pi,\pi)$. By k we denote the regular analytic function in $|z|<1$, which satisfies the conditions

1) $\lim_{r\to 1-0} \operatorname{Re}\{k(re^{i\theta})\} = \log f(\theta)$ a.e.,
2) $k(0)$ real.

Let

$$D(f;z) = \exp\{\tfrac{1}{2}k(z)\}.$$

The orthogonal polynomials Φ_n associated with f are uniquely defined by

a) $(2\pi)^{-1}\int_{-\pi}^{\pi} |\Phi_n(f;e^{i\theta})|^2 f(\theta)\,d\theta = 1$,
b) $\int_{-\pi}^{\pi} \Phi_n(f;e^{i\theta})\,\overline{\pi(e^{i\theta})}\,f(\theta)\,d\theta = 0$ for every polynomial π of degree at most $n-1$,
c) $\varkappa_n(f) > 0$, where $\varkappa_n(f)$ is the coefficient of z^n in $\Phi_n(f;z)$.

In this notation G. Szegö [3, p. 296] deduced the asymptotic formula

$$\Phi_n(f;e^{i\nu}) = e^{in\nu}\{D(f;e^{i\nu})\}^{-1} + o(1) \tag{1}$$

for certain functions f. Later G. Freud [1] proved the same formula under simplified conditions on f.

The aim of this note is to derive similar formulas for the derivatives of Φ_n under Freud's conditions. Our method is an extension of Freud's. In the sequel we use the notations

$$s_n(f;a,z) = \sum_{\nu=0}^{n} \overline{\Phi_\nu(f;a)}\,\Phi_\nu(f;z),$$

$$t_{n,q}(f;a) = \sum_{\nu=0}^{n} |\Phi_\nu^{(q)}(f;a)|^2$$

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and

$$r_{n,q}(f; a,z) = \sum_{\nu=0}^{n} \overline{\Phi_\nu(f; a)}\, \Phi_\nu^{(q)}(f; z) .$$

2.

THEOREM. *Let f of period 2π be of bounded variation, and suppose that there are constants M and m so that $0 < m \leqq f(\theta) \leqq M$ for all θ. Let p be a non-negative integer.*

If γ is a number such that

$$\int_{-\pi}^{\pi} \log f(\theta) \cot \tfrac{1}{2}(\gamma - \theta)\, d\theta \tag{2}$$

exists as a principal Cauchy value, then, for $z = e^{i\gamma}$, it follows that

$$\Phi_n^{(p)}(f; z) = n^p z^{n-p} \overline{\{D(f; z)\}}^{-1} + o(n^p) . \tag{3}$$

REMARK. That $\Phi_n^{(p)}(f; e^{i\theta}) = O(n^p)$ uniformly in θ and n follows immediately by Bernstein's inequality and the fact [1, p. 286] that $\Phi_n(f; e^{i\theta}) = O(1)$ uniformly in θ and n.

The proof of the theorem is given in sections 3–7.

3.

First we derive a suitable estimate of the difference between $\Phi_n^{(p)}(f; z)$ and $\Phi_n^{(p)}(g; z)$, when f and g satisfy the conditions of the theorem. In order to do this we use the well-known formula [3, p. 288]

$$\varkappa_n(f)\, \overline{\Phi}_n(f; z) = z^n s_n(f; 0, z^{-1}) . \tag{4}$$

Differentiating both sides with respect to z we get

$$\varkappa_n(f)\, \overline{\Phi}_n{}'(f; z) = n z^{n-1} s_n(f; 0, z^{-1}) - z^{n-2} r_{n,1}(f; 0, z^{-1}) ,$$

which implies the following inequality for $|z| = 1$

$$|\varkappa_n(f)\, \Phi_n'(f; z) - \varkappa_n(g)\, \Phi_n'(g; z)| \\ \leqq n\, |r_{n,0}(f; 0,z) - r_{n,0}(g; 0,z)| + |r_{n,1}(f; 0,z) - r_{n,1}(g; 0,z)| .$$

Differentiating (4) p times we get for $|z| = 1$ that

$$|\varkappa_n(f)\, \Phi_n^{(p)}(f; z) - \varkappa_n(g)\, \Phi_n^{(p)}(g; z)| \\ \leqq \sum_{\nu=0}^{n} C_{p,\nu} |r_{n,\nu}(f; 0,z) - r_{n,\nu}(g; 0,z)| , \tag{5}$$

where

$$C_{p,\nu} = \begin{cases} \binom{p}{\nu}(n-\nu)(n-\nu-1)\dots(n-p+1) & \text{if } \nu \leqq p-1 \,, \\ 1 & \text{if } \nu = p \,. \end{cases}$$

Later on g^{-1} will be replaced by suitably selected trigonometrical polynomials.

4.

We now consider the following problem. Let λ, μ and a be complex numbers, and let f be positive and integrable on the unit circle. Determine the supremum of

$$|\lambda\varrho(0)+\mu\varrho^{(q)}(a)|^2, \quad q \text{ integer } \geqq 1 \,,$$

when ϱ ranges over the set of all polynomials of degree n which satisfy the condition

$$(2\pi)^{-1}\int_{-\pi}^{\pi} f(\theta)\,|\varrho(e^{i\theta})|^2\,d\theta \;=\; 1 \,.$$

We shall apply a method of G. Szegö [3, p. 303]. Let

$$\varrho(z) \;=\; u_0\Phi_0(f;z)+u_1\Phi_1(f;z)+\dots+u_n\Phi_n(f;z)\,.$$

Then $\sum_{\nu=0}^{n}|u_\nu|^2=1$, and according to Cauchy's inequality we get

$$\begin{aligned} |\lambda\varrho(0)+\mu\varrho^{(q)}(a)|^2 &= \left|\sum_{\nu=0}^{n} u_\nu\{\lambda\Phi_\nu(f;0)+\mu\Phi_\nu^{(q)}(f;a)\}\right|^2 \\ &\leqq \sum_{\nu=0}^{n}|\lambda\Phi_\nu(f;0)+\mu\Phi_\nu^{(q)}(f;a)|^2 \\ &= |\lambda|^2 s_n(f;0,0)+2\operatorname{Re}\{\bar{\lambda}\mu r_{n,q}(f;0,a)\}+|\mu|^2 t_{n,q}(f;a)\,. \end{aligned}$$

This expression is the desired maximum, since it is attained for

$$\varrho(z) \;=\; \varepsilon\left(\sum_{\nu=0}^{n}|\lambda\Phi_\nu(f;0)+\mu\Phi_\nu^{(q)}(f;a)|^2\right)^{-\frac{1}{2}}\sum_{\nu=0}^{n}\overline{\{\lambda\Phi_\nu(f;0)+\mu\Phi_\nu^{(q)}(f;a)\}}\,\Phi_\nu(f;z)\,,$$

where $|\varepsilon|=1$. If

$$0 < m \leqq f_1(\theta) \leqq f(\theta) \leqq f_2(\theta), \qquad |\theta|\leqq\pi\,,$$

then the preceding result shows that

$$\begin{aligned} &|\lambda|^2 s_n(f_1;0,0)+2\operatorname{Re}\{\bar{\lambda}\mu r_{n,q}(f_1;0,a)\}+|\mu|^2 t_{n,q}(f_1;a) \\ &\quad\geqq |\lambda|^2 s_n(f;0,0)+2\operatorname{Re}\{\bar{\lambda}\mu r_{n,q}(f;0,a)\}+|\mu|^2 t_{n,q}(f;a) \qquad (6) \\ &\quad\geqq |\lambda|^2 s_n(f_2;0,0)+2\operatorname{Re}\{\bar{\lambda}\mu r_{n,q}(f_2;0,a)\}+|\mu|^2 t_{n,q}(f_2;a)\,. \end{aligned}$$

For $\lambda=0$ and $\mu=1$ this reduces to

$$t_{n,q}(f_1;a) \geqq t_{n,q}(f;a) \geqq t_{n,q}(f_2;a)\,. \tag{7}$$

From (6) we get the inequality

$$|\lambda|^2[s_n(f;0,0)-s_n(f_2;0,0)]+2\,\mathrm{Re}\{\bar{\lambda}\mu(r_{n,q}(f;0,a)-r_{n,q}(f_2;0,a))\}+ \\ +|\mu|^2[t_{n,q}(f;a)-t_{n,q}(f_2;a)] \geqq 0\,.$$

From a consideration of the discriminant of the left-hand side and an application of (7) it follows that

$$|r_{n,q}(f;0,a)-r_{n,q}(f_2;0,a)|^2 \\ \leqq \{s_n(f;0,0)-s_n(f_2;0,0)\}\{t_{n,q}(f_1;a)-t_{n,q}(f_2;a)\}\,.$$

Since $s_n(f;0,0)=(\varkappa_n(f))^2$ (cf. e.g. [3, p. 288]) we get

$$|r_{n,q}(f;0,a)-r_{n,q}(f_2;0,a)|^2 \\ \leqq \{(\varkappa_n(f))^2-(\varkappa_n(f_2))^2\}\{t_{n,q}(f_1;a)-t_{n,q}(f_2;a)\}\,. \tag{8}$$

5.

We next want an estimate of the right side in (8) where f_1 and f_2 are replaced by reciprocals of suitable trigonometrical polynomials. If f satisfies the conditions of the theorem, $F(\theta)=f^{-1}(\theta)$ is clearly of bounded variation. Thus it follows from theorem 1 in [2] that there exists a sequence of trigonometrical polynomials (W_n) (where W_n is of degree at most n) such that

$$2m^{-1} \geqq W_n(\theta) \geqq F(\theta) = f^{-1}(\theta)$$

and

$$\int_{-\pi}^{\pi} |W_n(\theta)-F(\theta)|\,d\theta = O(n^{-1})\,.$$

Condition (2) in our theorem implies that F is continuous at γ. Hence the remark on theorem 1 in [2, p. 281] shows that to every positive number ε there exist numbers δ_1 and N_1 such that

$$|W_n(\theta)-F(\gamma)| < \varepsilon \tag{9}$$

for all θ and n satisfying $|\theta-\gamma|<\delta_1$ and $n>N_1$.

Similarly another sequence of trigonometrical polynomials (w_n) exists satisfying

$$F(\theta) \geqq w_n(\theta) \geqq \tfrac{1}{2}M^{-1} > 0$$

and

$$\int_{-\pi}^{\pi} |w_n(\theta)-F(\theta)|\,d\theta = O(n^{-1})\,.$$

To every positive number ε there exist numbers δ_2 and N_2 such that

$$|w_n(\theta)-F(\gamma)| < \varepsilon \tag{10}$$

for all θ and n satisfying $|\theta-\gamma|<\delta_2$ and $n>N_2$. Furthermore we know according to [1, p. 287] that

$$[\varkappa_n(f)]^2-[\varkappa_n(w_n^{-1})]^2 = O(n^{-1}) . \tag{11}$$

In order to get an estimate of

$$t_{n,q}(W_n^{-1}; e^{i\gamma})-t_{n,q}(w_n^{-1}; e^{i\gamma}) ,$$

which is the second factor of the right side of (8) with f_1 and f_2 replaced by W_n^{-1} and w_n^{-1} respectively, we proceed as follows.

With the chosen ε and with $\delta=\min(\delta_1,\delta_2)$ and $N=\max(N_1,N_2)$, we consider two trigonometrical polynomials χ and ψ satisfying the conditions (12) and (13):

$$\begin{array}{lll}
\text{(12a)} & F(\gamma)+2\varepsilon \geqq \chi(\theta), & \theta\in(\gamma-\tfrac12\delta,\ \gamma+\tfrac12\delta) ,\\
\text{(12b)} & F(\gamma)+\varepsilon \leqq \chi(\theta), & \theta\in(\gamma-\delta,\ \gamma+\delta) ,\\
\text{(12c)} & 3m^{-1} \geqq \chi(\theta) \geqq 2m^{-1}, & \theta\notin(\gamma-\delta,\ \gamma+\delta) ,\\
\text{(13a)} & F(\gamma)-2\varepsilon \leqq \psi(\theta), & \theta\in(\gamma-\tfrac12\delta,\ \gamma+\tfrac12\delta) ,\\
\text{(13b)} & F(\gamma)-\varepsilon \geqq \psi(\theta), & \theta\in(\gamma-\delta,\ \gamma+\delta) ,\\
\text{(13c)} & \tfrac13 M^{-1} \leqq \psi(\theta) \leqq \tfrac12 M^{-1}, & \theta\notin(\gamma-\delta,\ \gamma+\delta) .
\end{array}$$

Because of (9) and (10)

$$\chi(\theta) \geqq W_n(\theta) \geqq w_n(\theta) \geqq \psi(\theta), \qquad n>N .$$

By (7) it then follows that

$$t_{n,q}(\chi^{-1}; e^{i\gamma}) \geqq t_{n,q}(W_n^{-1}; e^{i\gamma}) \geqq t_{n,q}(w_n^{-1}; e^{i\gamma}) \geqq t_{n,q}(\psi^{-1}; e^{i\gamma}) . \tag{14}$$

Since ψ is a positive trigonometrical polynomial, $D(\psi^{-1}; z)$ is a polynomial of the same type [3, p. 287]. We denote this polynomial by $h(\psi^{-1}; z)$ and by [3, p. 287] we have

$$\Phi_n(\psi^{-1}; z) = z^n\,\overline{h}(\psi^{-1}; z^{-1}) \tag{15}$$

for all $n \geqq$ degree of ψ. Differentiating both sides q times with respect to $z=e^{i\theta}$ we get

$$\Phi_n^{(q)}(\psi^{-1}; z) = n^q z^{n-q}\,\overline{h}(\psi^{-1}; z^{-1})+O(n^{q-1}) .$$

Since $\psi(\theta)=|h(\psi^{-1}; e^{i\theta})|^2$, we get $(\theta=\gamma)$

$$|\Phi_n^{(q)}(\psi^{-1}; e^{i\gamma})|^2 = n^{2q}\psi(\gamma)+O(n^{2q-1})$$

and hence

$$t_{n,q}(\psi^{-1};\, e^{i\gamma}) = \psi(\gamma)\sum_{0}^{n} \nu^{2q} + O(n^{2q}) = \psi(\gamma)\{n^{2q+1}/(2q+1) + O(n^{2q})\}\,.$$

Similarly we get

$$t_{n,q}(\chi^{-1};\, e^{i\gamma}) = \chi(\gamma)\{n^{2q+1}/(2q+1) + O(n^{2q})\}\,.$$

Introducing the last two estimates in (14) we get by aid of (12a) and (13a) that

$$\begin{aligned}(f^{-1}(\gamma)+3\varepsilon)n^{2q+1}/(2q+1) &\geqq t_{n,q}(W_n{}^{-1};\, e^{i\gamma})\\ &\geqq t_{n,q}(w_n{}^{-1};\, e^{i\gamma}) \geqq (f^{-1}(\gamma)-3\varepsilon)n^{2q+1}/(2q+1)\end{aligned}$$

if n is sufficiently large. This yields

$$t_{n,q}(W_n{}^{-1};\, e^{i\gamma}) - t_{n,q}(w_n{}^{-1};\, e^{i\gamma}) = o(n^{2q+1})\,.$$

By (8) and (11) we find that

$$r_{n,q}(f;\, 0, e^{i\gamma}) - r_{n,q}(w_n{}^{-1};\, 0, e^{i\gamma}) = o(n^q)\,.$$

In (5) we replace g^{-1} by w_n and z by $e^{i\gamma}$. Since according to [1]

$$s_n(f;\, 0, e^{i\gamma}) - s_n(w_n{}^{-1};\, 0, e^{i\gamma}) = o(1)\,,$$

we see that every term in the sum (5) is $o(n^p)$. It follows that

$$\varkappa_n(f)\,\Phi_n^{(p)}(f;\, e^{i\gamma}) - \varkappa_n(w_n^{-1})\,\Phi_n^{(p)}(w_n^{-1};\, e^{i\gamma}) = o(n^p)\,. \tag{16}$$

6.

Next we want to prove that

$$\Phi_n^{(p)}(w_n^{-1};\, e^{i\gamma}) = n^p e^{i(n-p)\gamma}\,\overline{\{D(w_n{}^{-1};\, e^{i\gamma})\}}^{-1} + o(n^p)\,. \tag{17}$$

The formula

$$\Phi_n(w_n{}^{-1};\, e^{i\theta}) = e^{in\theta}\,\overline{\{D(w_n{}^{-1};\, e^{i\theta})\}}^{-1}$$

follows in the same way as (15) from [3, p. 287]. Differentiation of this equation with respect to θ gives

$$\begin{aligned}\frac{d^p}{d\theta^p}\Phi_n(w_n{}^{-1};\, e^{i\theta}) &= (in)^p e^{in\theta}\,\overline{\{D(w_n{}^{-1};\, e^{i\theta})\}}^{-1} +\\ &\quad + \sum_{\nu=1}^{p}\binom{p}{\nu}(in)^{p-\nu}e^{in\theta}\frac{d^\nu}{d\theta^\nu}\{D(w_n{}^{-1};\, e^{i\theta})\, w_n(\theta)\}\,,\end{aligned} \tag{18}$$

since $\overline{\{D(w_n{}^{-1};\, e^{i\theta})\}}^{-1} = D(w_n{}^{-1};\, e^{i\theta})\, w_n(\theta)$.

In the sequel we shall use the following lemma.

LEMMA. *Let (T_n) be a sequence of trigonometric polynomials of degree n. If there are constant ε, δ and M such that*

$$|T_n(\theta)| \leqq M \quad \textit{for all } \theta ,$$
$$|T_n(\theta)| \leqq \varepsilon \quad \textit{for } |\theta| \leqq \delta ,$$

then

$$|T_n'(\theta)| \leqq 8n\varepsilon \quad \textit{for } |\theta| \leqq \tfrac{1}{2}\delta \textit{ and } n > N ,$$

where N depends on ε, δ and M.

PROOF. For $|\theta| \leqq \frac{1}{2}\delta$ our assumptions imply

$$\frac{T_n(\theta+h)-T_n(\theta-h)}{\sin h} \leqq \begin{cases} 4|h|^{-1}\varepsilon & \text{for } |h| \leqq \frac{1}{2}\delta , \\ 8M\delta^{-1} & \text{for } |h| > \frac{1}{2}\delta . \end{cases}$$

The expression

$$u_n(h) = [T_n(\theta+h)-T_n(\theta-h)][\sin h]^{-1}$$

is a trigonometrical polynomial in h of degree at most n. By Bernstein's inequality, $|u_n'| \leqq n \max |u_n|$, it follows that in an interval of length n^{-1} with center at a maximum point of $|u_n|$, the absolute value $|u_n|$ is not less than half the maximum. By aid of this observation we get that

$$\left|\frac{T_n(\theta+h)-T_n(\theta-h)}{\sin h}\right| \leqq \max(16n\varepsilon, 8M\delta^{-1}) \quad \text{for } |\theta| \leqq \tfrac{1}{2}\delta ,$$

which for n sufficiently large is $\leqq 16n\varepsilon$. This implies that

$$|T_n'(\theta)| \leqq 8\cdot n\cdot \varepsilon \quad \text{for } |\theta| \leqq \tfrac{1}{2}\delta \text{ and } n > N ,$$

where N only depends on ε, δ and M. The proof of the lemma is finished.

The definition of $\{w_n\}$ shows that $\{|w_n(\theta)|\}$ is bounded and furthermore we know according to (9) that to every positive ε there exist numbers δ and N_0 such that

$$|w_n(\theta)-F(\gamma)| < \varepsilon$$

when $|\theta-\gamma| \leqq \delta$ and $n > N_0$. Thus by our lemma

$$|w_n'(\theta)| \leqq 8n\varepsilon$$

when $|\theta-\gamma| \leqq \frac{1}{2}\delta$ and n is sufficiently large. We find by a q-fold iteration that

(19) $$|w_n^{(q)}(\theta)| < (8n)^q\varepsilon \quad \text{for } |\theta-\gamma| \leqq 2^{-q}\delta .$$

From the definition of D we obtain

(20) $$\frac{d}{d\theta} D(w_n^{-1}; e^{i\theta}) = -\tfrac{1}{2}D(w_n^{-1}; e^{i\theta}) \cdot \left\{\frac{w_n'(\theta)}{w_n(\theta)} + \frac{i}{2\pi}\int_{-\pi}^{\pi} \frac{d}{dt}[\log w_n(t)] \cot \tfrac{1}{2}(\theta-t)\, dt\right\}.$$

We know that there exists a constant $c > 0$ such that $c^{-1} < w_n(\theta) < c$ for all sufficiently large n. Thus, it follows from

$$|D(w_n;\, e^{i\theta})|^2 = w_n(\theta)$$

that $D(w_n;\, e^{i\theta})$ satisfies a similar inequality.

To obtain an estimate of the integral in the last term in (20) for $|\theta-\gamma| \leqq \frac{1}{4}\delta$ and sufficiently large n we proceed as follows. Putting $n^{-1}\varepsilon^{\frac{1}{4}} = \eta$ and combining the estimates of the proof of the lemma with (19), we find for one part of the integral in (20) that

$$\left| \int_{\theta-\eta}^{\theta+\eta} \frac{w_n'(t)}{w_n(t)} \cot \tfrac{1}{2}(\theta-t)\, dt \right| = \left| \int_{\theta-\eta}^{\theta+\eta} \left(\frac{w_n'(t)}{w_n(t)} - \frac{w_n'(\theta)}{w_n(\theta)} \right) \cot \tfrac{1}{2}(\theta-t)\, dt \right|$$

$$\leqq \int_{\theta-\eta}^{\theta+\eta} \left| \frac{w_n'(t)-w_n'(\theta)}{w_n(t)} \cot \tfrac{1}{2}(\theta-t) \right| dt + \int_{\theta-\eta}^{\theta+\eta} \left| \frac{w_n'(\theta)}{w_n(\theta)} \frac{w_n(\theta)-w_n(t)}{w_n(t)} \cot \tfrac{1}{2}(\theta-t) \right| dt$$

$$\leqq 2M\; 16n\; 8n\varepsilon\; 2n^{-1}\varepsilon^{-\frac{1}{4}} + 4M^2\; 8n\; 16n\varepsilon^2\; 2n^{-1}\varepsilon^{-\frac{1}{4}} = \varepsilon^{\frac{1}{2}} O(n) .$$

For the integral on the rest of the circle we have the estimate

$$\int_{\theta+\eta}^{2\pi+\theta-\eta} \frac{d}{dt}[\log w_n(t)] \cot \tfrac{1}{2}(\theta-t)\, dt$$

$$= [\log w_n(t) \cot \tfrac{1}{2}(\theta-t)]_{\theta+\eta}^{2\pi+\theta-\eta} + \tfrac{1}{2} \int_{\theta+\eta}^{2\pi+\theta-\eta} \log w_n(t) [\sin \tfrac{1}{2}(\theta-t)]^{-2}\, dt$$

$$= \varepsilon^{\frac{1}{2}} O(n) .$$

Thus

$$\left| \frac{d}{d\theta} D(w_n^{-1};\, e^{i\theta}) \right| \leqq \varepsilon^{\frac{1}{2}} O(n) \quad \text{for } |\theta-\gamma| \leqq \tfrac{1}{4}\delta .$$

If we differentiate (20) $q-1$ times and make repeated use of our lemma as above, we find that

$$\left| \frac{d^q}{d\theta^q} D(w_n^{-1};\, e^{i\theta}) \right| \leqq \varepsilon^{\frac{1}{2}} O(n^q) \quad \text{for } |\theta-\gamma| \leqq 2^{-q-1}\delta$$

and hence

$$\frac{d^q}{d\theta^q} D(w_n^{-1};\, e^{i\gamma}) = o(n^q) . \tag{21}$$

Then (17) follows by applying (19) and (21) to (18).

7.

Finally using the following two results [1, p. 287, 289]:

$$0 < \mu \leqq \varkappa_n(w_n{}^{-1}) \leqq \lambda \,, \tag{22}$$

where μ and λ are constants not depending on n, and

$$\lim_{n\to\infty} D(w_n{}^{-1}; e^{i\gamma}) = D(f; e^{i\gamma}) \,, \tag{23}$$

we obtain the desired result in the following way. Applying (11) and (22) to (16), we get

$$\Phi_n^{(p)}(f; e^{i\gamma}) = \Phi_n^{(p)}(w_n^{-1}; e^{i\gamma}) + o(n^p) \,,$$

and we find from (17) that this can be written in the form

$$\Phi_n^{(p)}(f; e^{i\gamma}) = n^p e^{i(n-p)\gamma} \{\overline{D(w_n^{-1}; e^{i\gamma})}\}^{-1} + o(n^p) \,.$$

Now the statement (3) follows by applying (23). The theorem is proved.

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