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BOOLEAN OPERATIONS ON GRAPHS

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There are several operations on two graphs G_1 and G_2 which result in a graph G whose set of points is the cartesian product $V_1 \times V_2$, where V_k is the point set of G_k. These include the cartesian product (Sabidussi [11]), the composition (Harary [5], Sabidussi [10]), and the tensor product (Weichsel [14], McAndrew [8], Harary and Trauth [7], and Brualdi [2]). Some of the operations have been independently rediscovered several times. This has led to considerable ambiguity because of the use of different terminology and notation. It is hoped that our systematic nomenclature based on the usual boolean operations becomes standard.

These operations are important for constructing new classes of graphs which in turn may be useful for the recognition and decomposition of graphs and for the determination of structural properties of graphs in terms of their constituent subgraphs.

The boolean viewpoint introduced here has served to coordinate the definitions of all known operations and to suggest new ones. The algebraic representation of the adjacency matrix of a graph is most convenient in expressing each boolean operation in terms of its constituent graphs G_1 and G_2.

The purposes of this review article are (i) to develop new boolean operations on two graphs, (ii) to relate these to the various existing operations, (iii) to investigate some invariant properties of boolean operations, (iv) to demonstrate the way in which boolean operations are related to one another (v) to provide the conditions for the connectedness of graphs obtained by boolean operations, and (vi) to pose some unsolved problems relating to the automorphism group of such a composite graph.

Preliminaries.

A graph G consists of a finite set V of points and a set X of lines which is a subset of all unordered pairs of points. Our terminology and nota-

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tion will follow [6]. We name the points of G by the distinct labels $\{v_1, v_2, \ldots, v_p\}$ and call the result a *labeled graph*. Only labeled graphs are considered. Two distinct points v_i, v_j are said to be *adjacent*, written $v_i \operatorname{adj} v_j$, if line $\{v_i, v_j\} \in X$. For brevity we denote the line $\{v_i, v_j\}$ by $v_i v_j$.

The *adjacency matrix* $A = A(G) = [a_{ij}]$ of a graph G is the $p \times p$ matrix with entries

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \operatorname{adj} v_j\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Notice that each $a_{ii} = 0$ and A is symmetric. Let $J = J_p$ be the $p \times p$ matrix with every entry 1. As usual, let $I = I_p = [\delta_{ij}]$ be the $p \times p$ identity matrix. Addition, denoted $\oplus$, is taken modulo 2. For example, $A(K_p) = J_p \oplus I_p$ denotes the adjacency matrix of K_p, the complete graph with p points.

A graph $\bar{G}$ is the *complement* of G if it also has V as its set of points and for $i \neq j$, $v_i \operatorname{adj} v_j$ in $\bar{G}$ whenever v_i and v_j are not adjacent in G. Thus, the adjacency matrix of $\bar{G}$ is $\bar{A} = A(\bar{G}) = A \oplus J \oplus I$. Consequently, denoting $\bar{A} = [\bar{a}_{ij}]$, we have $\bar{a}_{ij} = a_{ij} \oplus 1 \oplus \delta_{ij}$.

Let $A = [a_{ij}]$, $p_1 \times p_1$, and $B = [b_{rs}]$, $p_2 \times p_2$, be binary matrices. Their *tensor product* $A * B$ is defined as the partitioned matrix $[a_{ij}B]$:

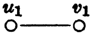
$$A * B = \begin{bmatrix} a_{11}B & a_{12}B & \ldots a_{1p_1}B \\ a_{21}B & a_{22}B & \ldots a_{2p_1}B \\ \ldots & \ldots & \ldots \\ a_{p_1 1}B & a_{p_1 2}B & \ldots a_{p_1 p_1}B \end{bmatrix}.$$

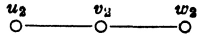
The tensor product, also known as the Kronecker product, is associative, distributive over $\oplus$, but not commutative.

1. Boolean operations.

We say that a *boolean operation* on an ordered pair of disjoint labeled graphs G_1 and G_2 results in a labeled graph $G = G_1 \circ G_2$ which has the cartesian product $V = V_1 \times V_2$ as its set of points. Of course the set X of lines of G is expressed in terms of the lines in X_1 and X_2, differently for each boolean operation.

Perhaps the simplest boolean operation on graphs is the "conjunction" $G_1 \wedge G_2$ introduced by Weichsel [14] who called it the "Kronecker product". The operation was extended to directed graphs by McAndrew [8], Harary and Trauth [7], and Brualdi [2]. The *conjunction* $G = G_1 \wedge G_2$ is defined by specifying its set of lines. For any two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V = V_1 \times V_2$, the line uv is in X if $[u_1 v_1$ is in $X_1]$ *and* $[u_2 v_2$ is in $X_2]$. For example, if

$G_1 = K_2 =$ 

and $G_2 = K_{1,2} =$ 

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then $G = G_1 \wedge G_2$ is the labeled graph of Figure 1.

$G_1 \wedge G_2$:

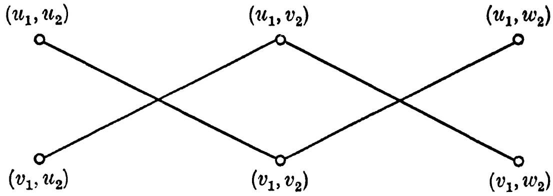


Fig. 1. The Conjunction

As Weichsel observed, the adjacency matrix of the conjunction $G = G_1 \wedge G_2$ is the tensor product

$$A(G_1 \wedge G_2) = A_1 * A_2 \tag{1}$$

of the adjacency matrices A_1 and A_2. We may illustrate (1) with the above graphs $G_1 = K_2$ and $G_2 = K_{1,2}$ by combining

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

to give

$$A(G_1 \wedge G_2) = A_1 * A_2 = \begin{bmatrix} 0 & A_2 \\ A_2 & 0 \end{bmatrix} = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right].$$

The adjacency matrices of other boolean operations may also be characterized by the notation already introduced. The *cartesian product* (see Sabidussi [11]) is that boolean operation $G = G_1 \times G_2$ in which for any two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$, the line uv is in X whenever $[u_1 = v_1$ *and* $u_2 v_2 \in X_2]$ *or* $[u_2 = v_2$ *and* $u_1 v_1 \in X_1]$. For example, with $G_1 = K_2$ and $G_2 = K_{1,2}$, the cartesian product $G = G_1 \times G_2$ is illustrated in Figure 2. We may also express $A(G_1 \times G_2)$, the adjacency matrix of the cartesian product, in terms of A_1 and A_2:

$$A(G_1 \times G_2) = (A_1 * I_{p_2}) \oplus (I_{p_1} * A_2)\,. \tag{2}$$

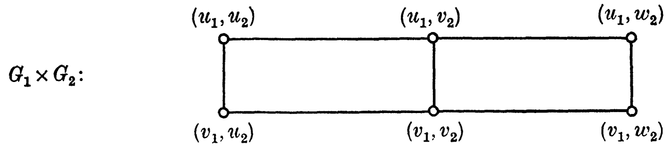


Fig. 2. The Cartesian Product

We note that Berge [1, p. 23] refers to the conjunction and the cartesian product as the "product" and "sum", respectively. Ore [9, p. 35–36] refers to them as the "cartesian product graph" and the "cartesian sum graph", respectively.

The *composition* $G = G_1[G_2]$ is a boolean operation which was introduced by Harary [4] and investigated by Sabidussi [10], [12], who called it the "lexicographic product". With $u = (u_1, u_2)$ and $v = (v_1, v_2)$ as before, $uv \in X$ whenever $[u_1 v_1 \in X_1]$ *or* $[u_1 = v_1$ *and* $u_2 v_2 \in X_2]$. Thus, with $G_1 = K_2$ and $G_2 = K_{1,2}$ we may illustrate both $G_1[G_2]$ and $G_2[G_1]$ in Figure 3. Again the adjacency matrix $A(G_1[G_2])$ of this boolean operation may

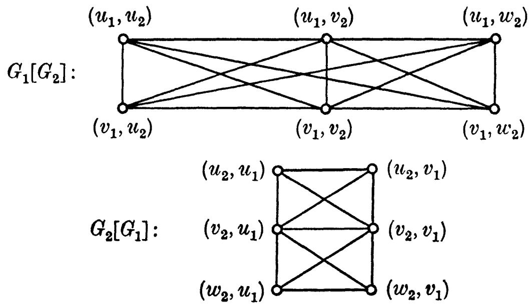


Fig. 3. The Composition

be expressed in terms of the adjacency matrices A_1 and A_2 of the labeled graphs G_1 and G_2,

(3a) $$A(G_1[G_2]) = (A_1 * J_{p_2}) \oplus (I_{p_1} * A_2)\,,$$

and we define $[G_1]G_2$ by its adjacency matrix

(3b) $$A([G_1]G_2) = (A_1 * I_{p_2}) \oplus (J_{p_1} * A_2)\,.$$

The matrix of (3b) is a convenient representation of $G_2[G_1]$; it is permutationally equivalent to $A(G_2[G_1])$.

2. New boolean operations.

It is natural to consider the graphs obtained by applying other conventional boolean operations from set theory.

The *symmetric difference* $G=G_1 \oplus G_2$ is defined as expected to be that boolean operation on G_1 and G_2 such that with $u=(u_1,u_2)$ and $v=(v_1,v_2)$, $uv \in X$ whenever

$$\textit{either } [u_1v_1 \in X_1] \quad \textit{or} \quad [u_2v_2 \in X_2] \quad \text{(but not both)} .$$

In this case the adjacency matrix is given by

$$(4) \qquad A(G_1 \oplus G_2) = (A_1 * J_{p_2}) \oplus (J_{p_1} * A_2) .$$

With $G_1=K_2$ and $G_2=K_{1,2}$,the graph $G_1 \oplus G_2$ is illustrated in Figure 4.

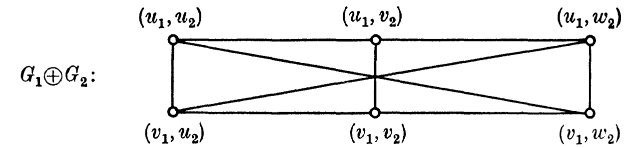


Fig. 4. The symmetric difference

The *disjunction* $G=G_1 \vee G_2$ has $uv \in X$ whenever $[u_1v_1 \in X_1]$ *or* $[u_2v_2 \in X_2]$ (or both, of course), so that

$$(5) \qquad A(G_1 \vee G_2) = (A_1 * J_{p_2}) \oplus (J_{p_1} * A_2) \oplus (A_1 * A_2) .$$

The disjunction $G=K_2 \vee K_{1,2}$ is illustrated in Figure 5.

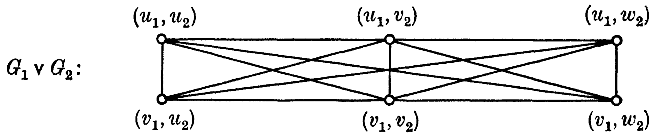


Fig. 5. The disjunction

It is a mere coincidence that $K_2 \vee K_{1,2}=K_2[K_{1,2}]$; for example, $K_{1,2} \vee K_2$ and $K_{1,2}[K_2]$ are not isomorphic.

The *rejection* $G=G_1 | G_2$ is that boolean operation defined by $uv \in X$ whenever $[u_1v_1 \notin X_1]$ *and* $[u_2v_2 \notin X_2]$, so that

$$(6) \qquad A(G_1 | G_2) = \bar{A}_1 * \bar{A}_2$$

where as before $\bar{A}_k=A_k \oplus I_{p_k} \oplus J_{p_k}$.

The *complement* $\bar{G}=\overline{G_1 \circ G_2}$ of any boolean operation $G=G_1 \circ G_2$ has a line uv only if $uv \notin X$, the set of lines of G. Thus, the adjacency matrix of $\bar{G}$ satisfies the equality

$$(7) \qquad A(\overline{G_1 \circ G_2}) = A(G_1 \circ G_2) \oplus (I_{p_1} * I_{p_2}) \oplus (J_{p_1} * J_{p_2}) .$$

3. Some invariants of boolean operations.

In this section we determine the degree of each point of $G=G_1\circ G_2$ and the number of lines in G in terms of G_1 and G_2. The *degree* d_i of a point $u_i\in V$ is the number of lines incident with it. The degrees of the points of G are given in terms of its adjacency matrix $A=[a_{ij}]$ by $d_i=\sum_{j=1}^{p}a_{ij}$. Further, it is well known (by Euler) that the number q of lines of G satisfies the equation

$$q = \tfrac{1}{2}\sum_{i=1}^{p} d_i = \tfrac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p} a_{ij}\,. \tag{8}$$

The tensor product notation for boolean operations on graphs enables us to count easily the number of lines in each boolean operation $G=G_1\circ G_2$. Let

$$p_k = |V_k|,\quad q_k = |X_k|,\qquad k = 1,2\,,$$

and so $p=p_1p_2$. We illustrate the entries in Table 1 for the cartesian product $G=G_1\times G_2$. For convenience, we write $A_1=[a_{ij}]$, $A_2=[b_{rs}]$, $d_i=$ degree of v_i in G_i, and $e_r=$ degree of u_r in G_2. Using (2) we have

$$q = \tfrac{1}{2}\sum_{i=1}^{p_1}\sum_{r=1}^{p_2}\left[\sum_{j=1}^{p_1}\sum_{s=1}^{p_2}(a_{ij}\delta_{rs}\oplus\delta_{ij}b_{rs})\right]$$

$$= \tfrac{1}{2}\sum_{i=1}^{p_1}\sum_{r=1}^{p_2}\sum_{j=1}^{p_1}\sum_{s=1}^{p_2}(a_{ij}\delta_{rs}+\delta_{ij}b_{rs}-2a_{ij}\delta_{rs}\delta_{ij}b_{rs})\,.$$

But $a_{ij}\delta_{rs}\delta_{ij}b_{rs}=0$ for all i, j, r, s, hence

$$q = \sum_{i=1}^{p_1}\sum_{r=1}^{p_2}(d_i+e_r) = q_1p_2+q_2p_1\,, \tag{9}$$

by applying (8).

Table I

The degrees of points and the number of lines in boolean operations on graphs.

Name of Boolean Operation	Notation $G_1\circ G_2$	The number q of lines of G	The degree $d_{i,r}$ of a point $w=(v_i,u_r)$ of G
Conjunction	$G_1\wedge G_2$	$2q_1q_2$	d_ie_r
Cartesian product	$G_1\times G_2$	$q_1p_2+q_2p_1$	d_i+e_r
Composition	$G_1[G_2]$	$q_1p_2^2+q_2p_1$	$d_ip_2+e_r$
Symmetric difference	$G_1\oplus G_2$	$q_1p_2^2+q_2p_1^2-4q_1q_2$	$d_ip_2+e_rp_1-2d_ie_r$
Disjunction	$G_1\vee G_2$	$q_1p_2^2+q_2p_1^2-2q_1q_2$	$d_ip_2+e_rp_1-d_ie_r$
Rejection	$G_1\mid G_2$	$\binom{p}{2}-q_1p_2^2-q_2p_1^2+2q_1q_2$	$(p_1-d_i-1)(p_2-e_r-1)$

If we let $d_{i,r}$ be the degree of a point $w=(v_i,u_r)$, $v_i \in V_1$ and $u_r \in V_2$, in G, then for $G=G_1 \times G_2$ we have from (9)

$$d_{i,r} = d_i + e_r . \tag{10}$$

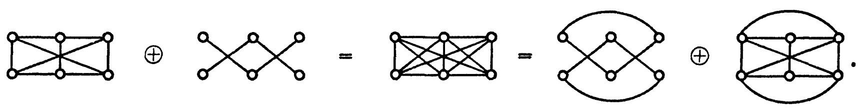
Table 1 gives the number of lines and the degrees of points for the boolean operations defined in Sections 1 and 2.

4. Relations between boolean operations.

The first observation we make on relations between boolean operations follows immediately from (7): the symmetric difference of any two boolean operations on the same two graphs is the symmetric difference of their complements, i.e., for any two boolean operations o and o' and any two graphs G_1 and G_2,

$$A(G_1 \circ G_2) \oplus A(G_1 \circ' G_2) = A(\overline{G_1 \circ G_2}) \oplus (\overline{G_1 \circ' G_2}) . \tag{11}$$

Equation (11) may be illustrated by taking $G_1=K_2$, $G_2=K_{1,2}$, and the operations of symmetric difference and conjunction, so that



This illustration suggest that

$$A(G_1 \vee G_2) = A(G_1 \oplus G_2) \oplus A(G_1 \wedge G_2) , \tag{12}$$

which is, in fact, easily verified.

The boolean operation $\overline{\bar{G}_1 \vee \bar{G}_2}$ was introduced by Berge [1, p. 38], and independently introduced by Teh and Yap [13], the latter calling it the "γ-product". Expanding $A(\overline{\bar{G}_1 \vee \bar{G}_2})$ using (7) and (5), and comparing the result with (1) and (2), we find

$$A(\overline{\bar{G}_1 \vee \bar{G}_2}) = A(G_1 \times G_2) \oplus A(G_1 \wedge G_2) . \tag{13}$$

We list several other relations between boolean operations, which are obtained similarly:

$$A(G_1[G_2]) \oplus A([G_1]G_2) = A(G_1 \times G_2) \oplus A(G_1 \oplus G_2) . \tag{14}$$

$$A(G_1 \times G_2) = A(\bar{G}_1 \times G_2) \oplus A(G_1 \times \bar{G}_2) \oplus A(\bar{G}_1 \times \bar{G}_2) . \tag{15}$$

$$A(\overline{\bar{G}_1 \wedge \bar{G}_2}) = A(G_1 \times G_2) \oplus A(K_{p_1} \times K_{p_2}) . \tag{16}$$

Of course other identities could be listed, but equations (11)–(16) are adequate to reveal the ease with which one can manipulate boolean operations. In the next statement, the labeling of the graphs must be kept in mind.

THEOREM 1. *$A(G_1[G_2])=A([G_1]G_2)$ if and only if both G_1 and G_2 are complete or both totally disconnected.*

PROOF. We first demonstrate the necessity. By hypothesis,

$$A(G_1[G_2]) = A([G_1]G_2) .$$

Therefore by equations (3a) and (3b), we must have

$$(A_1 * J_{p_2}) \oplus (J_{p_1} * A_2) = (A_1 * I_{p_2}) \oplus (I_{p_1} * A_2) .$$

With

$$A_1 = [a_{ij}], \quad I_{p_1} = [\delta_{ij}], \qquad i,j = 1,2,\ldots,p_1 ,$$

and

$$A_2 = [b_{rs}], \quad I_{p_2} = [\delta_{rs}], \qquad r,s = 1,2,\ldots,p_2 ,$$

the equation which must be satisfied by the entries is

$$a_{ij} \oplus b_{rs} = a_{ij}\delta_{rs} \oplus \delta_{ij}b_{rs} ,$$

or equivalently

$$a_{ij}(1 \oplus \delta_{rs}) = b_{rs}(1 \oplus \delta_{ij}) .$$

Thus, $a_{ij}=b_{rs}$ for all $i \neq j$, $r \neq s$ since $\delta_{ij}=\delta_{rs}=0$. This proves the necessity. It is very easy to verify that the equation

$$a_{ij}(1 \oplus \delta_{rs}) = b_{rs}(1 \oplus \delta_{ij})$$

is satisfied when G_1 and G_2 are both complete or totally disconnected, thus proving the sufficiency and completing the proof of the theorem.

COMMENT. The conditions of Theorem 1 should not be construed as an answer to the more general question of isomorphism. For example, if $G_1=G_2$, then $G_1[G_2]$ and $G_2[G_1]$ are isomorphic, but

$$A(G_1[G_2]) \neq A([G_2]G_2)$$

because of the labeling. For any two graphs G_1 and G_2, it is only known that $G_1[G_2] \cong G_2[G_1]$ when G_1 and G_2 are both complete, both totally disconnected, or isomorphic.

COROLLARY 1a. *The cartesian product and symmetric difference of two graphs are equal if and only if both are complete or both totally disconnected.*

This follows immediately by applying Theorem 1 to equation (14).

5. Connectedness of boolean operations.

We introduce some additional definitions. A *path* in G is a sequence of distinct points in which each consecutive pair of points is a line of G. A *cycle* is obtained when the end points of a path, with at least three points, are joined by a line. The *length* of a path or cycle is the number of lines in it. An *odd path* or *cycle* has odd length.

The points u and v are *connected* in G if there is some path, denoted u—v, joining u and v. A graph is *connected* if there is a path joining every pair of points. If G is not connected then clearly G may be partitioned into maximal connected subgraphs. These disjoint subgraphs are the *components* of G. A component is trivial if it consists of a single isolated point.

THEOREM 2. (Weichsel [14].) *The conjunction* $G=G_1 \wedge G_2$ *is connected if and only if* G_1 *or* G_2 *has an odd cycle.*

Clearly this theorem may be readily rephrased to handle the connectedness of the rejection operation; see equation (6).

THEOREM 3 (Harary and Trauth [7]). *The cartesian product* $G=G_1 \times G_2$ *is connected if and only if* G_1 *and* G_2 *are both connected.*

The next three lemmas will help provide a connectedness criterion (Theorem 4; we thank D. L. Richards for helpful discussions on this theorem and its lemmas) for the symmetric difference $G_1 \oplus G_2$.

LEMMA 4a. *The symmetric difference* $G=G_1 \oplus G_2$ *is connected if* G_1 *or* G_2 *is connected.*

PROOF. Consider G_1 connected and let $r=(r_1,r_2)$ and $w=(w_1,w_2)$ be any two points of G. Let $r_1,s_1,t_1,u_1,\ldots,v_1,w_1$ be a r_1—w_1 path in G_1. If $r_2 w_2 \in X_2$, then

$$(r_1,r_2),\ (r_1,w_2),\ (s_1,w_2),\ (s_1,r_2),\ldots,(w_1,r_2),\ (w_1,w_2)$$

is a sequence of points which forms a $r-w$ path in G since consecutive pairs of points are adjacent. On the other hand, if $r_2 w_2 \notin X_2$, then for r_1—w_1 odd,

$$(r_1,r_2),\ (s_1,r_2),\ (t_1,w_2),\ (u_1,w_2),\ldots,(v_1,w_2),\ (w_1,w_2)$$

is a r—w path, and for r_1—w_1 even,

$$(r_1,r_2),\ (s_1,w_2),\ (t_1,r_2),\ (u_1,w_2),\ldots,(v_1,r_2),\ (w_1,w_2)$$

is a r—w path in G, and the proof is complete.

LEMMA 4b. *If G_1 and G_2 each contain at least one line, then the symmetric difference $G = G_1 \oplus G_2$ has exactly one nontrivial component.*

OUTLINE OF PROOF. Assuming the contrary, suppose that G has two nontrivial components G' and G''. Then there exist a pair of adjacent points $u' = (u_1', u_2')$ and $v' = (v_1', v_2')$ in G', and also a pair of adjacent points $u'' = (u_1'', u_2'')$ and $v'' = (v_1'', v_2'')$ in G''. By definition of symmetric difference, there are several possible ways for adjacencies in G_1 and G_2 to imply that u' adj v' in G' and u'' adj v'' in G''. In fact, one can verify that there are exactly 16 such possibilities. By exhaustion, it can be shown that each of these cases implies the existence in G of a path joining u' or v' with u'' or v'', contrary to the assumption that G' and G'' are distinct components of G.

LEMMA 4c. *The isolates of $G = G_1 \oplus G_2$ consist of ordered pairs of isolates of G_1 and G_2.*

PROOF. Let $u = (u_1, u_2)$ be an isolate of G. If either u_1 or u_2 is not an isolate then there exist points v_1 and v_2 such that either $u_1 v_1 \in X_1$ or $u_2 v_2 \in X_2$. But then either $(u_1, u_2)(v_1, v_2)$ or $(u_1, u_2)(u_1, v_2)$ is a line of G, and u can not be an isolate.

THEOREM 4. *Let G_1 and G_2 be nontrivial graphs. If neither G_1 nor G_2 is totally disconnected, then their symmetric difference is connected if and only if G_1 and G_2 do not both contain isolates. If one of G_1 or G_2 is totally disconnected, then $G_1 \oplus G_2$ is connected if and only if the other is connected.*

PROOF. For the first part of the theorem, if $G = G_1 \oplus G_2$ is connected then by Lemma 4c, G_1 and G_2 do not both have isolates. On the other hand if G_1 and G_2 do not both have isolates then by Lemma 4c, G has no isolates and by Lemma 4b, G is connected. The second part of the theorem is a restatement of Lemma 4a.

THEOREM 5. *The disjunction of two graphs is connected if and only if their symmetric difference is connected.*

PROOF. Since $G_1 \vee G_2$ must have at least all the lines in $G_1 \oplus G_2$, the theorem is proved if the isolates of $G_1 \vee G_2$ are precisely the isolates of $G_1 \oplus G_2$. This must be true since if $u = (u_1, u_2)$ is an isolate of $G_1 \vee G_2$ then certainly it is an isolate of $G_1 \oplus G_2$. On the other hand if $u = (u_1, u_2)$ is an isolate of $G_1 \oplus G_2$ then by Lemma 4c there exist no points v_1, v_2 such that $u_1 v_1 \in X_1$ *or* $u_2 v_2 \in X_2$. Thus u is an isolate of $G_1 \vee G_2$ also.

THEOREM 6. *The composition $G_1[G_2]$ is connected if and only if G_1 is connected.*

The theorem follows at once from the definition of composition.

It is clear that the theorems of this section easily provide criteria for the connectedness of compound boolean operations.

6. The group of a boolean operation.

The *group of a graph G* is the collection of all automorphisms of G, so that it is a permutation group acting on V. Sabidussi [11] gave a necessary and sufficient condition for the group of the cartesian product of two graphs to be the "cartesian product" (see Harary [4]) of their groups. Sabidussi [10] also settled the question of when the group of the composition of two graphs is the "composition" of their groups (see Harary [5] or Hall [3, p. 81]; this is also known as the "wreath product" of two permutation groups).

It would be interesting to develop appropriate operations on permutation groups and provide criteria to tell when the group of a boolean operation on two graphs (including conjunction, disjunction, symmetric difference, and rejection) is given by the respective composite permutation group.

REFERENCES

1. C. Berge, *The theory of graphs and its applications*, London, 1962.
2. R. Brualdi, *Kronecker products of fully indecomposable matrices and of ultrastrong digraphs*, J. Combinatorial Theory 2 (1967), to appear.
3. M. Hall, *The theory of groups*, New York, 1959.
4. F. Harary, *On the number of bicolored graphs*, Pacific J. Math 8 (1958), 743–755.
5. F. Harary, *On the group of the composition of two graphs*, Duke Math. J. 26 (1959), 29–34.
6. F. Harary, *A seminar on graph theory*, New York, 1967.
7. F. Harary and C. A. Trauth, Jr., *Connectedness of products of two directed graphs*, SIAM Journal, 14 (1966), 250–254.
8. M. H. McAndrew, *On the product of directed graphs*, Proc. Amer. Math. Soc. 14 (1963), 600–606.
9. O. Ore, *Theory of graphs* (Amer. Math. Soc. Colloq. Publ. 38), Providence, 1962.
10. G. Sabidussi, *The composition of graphs*, Duke Math. J. 26 (1959), 693–696.
11. G. Sabidussi, *Graph multiplication*, Math. Z. 72 (1960), 446–457.
12. G. Sabidussi, *The lexicographic product of graphs*, Duke Math. J. 28 (1961), 573-578.
13. H. H. Teh and H. D. Yap, *Some construction problems of homogeneous graphs*, Bull. Math. Soc. Nanyang Univ. (1964), 164–196.
14. P. M. Weichsel, *The Kronecker product of graphs*, Proc. Amer. Math. Soc. 13 (1963), 47–52.

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