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RINGS WITH PRIMARY IDEALS AS MAXIMAL IDEALS

M. SATYANARAYANA

1. Introduction.

If every primary ideal is a maximal ideal in a commutative ring R with identity, then we say that R is a P-ring. Evidently in P-rings primary, prime and maximal ideals coincide. Commutative von Neumann regular rings and in particular a direct sum of finite number of fields are P-rings. In general it can be shown that a P-ring is a subdirect sum of fields (trivial regular rings) and furthermore if a P-ring is Noetherian, it is a finite direct sum of fields. This characterization enables us to prove that rings mentioned in section 3 are P-rings and hence direct sums of a finite number of fields.

2. *P*-rings.

THEOREM. *If R is a P-ring then R is a subdirect sum of fields. In addition, if R is Noetherian, then R is a finite direct sum of fields.*

PROOF. Since, in P-rings, prime and maximal ideals coincide, the intersection of all primary ideals is (0) by virtue of a result due to Krull mentioned in [4, p. 492]. Hence the intersection of all maximal ideals, i.e. the Jacobson radical of R, is (0). This implies that R is a subdirect sum of fields.

If R is a Noetherian P-ring, then R satisfies the descending chain condition since prime ideals are maximal. But R has zero Jacobson radical. Hence R is a finite direct sum of fields.

If the defining property of a P-ring is replaced by the property that every proper-primary ideal is maximal, then the above theorem need not be true, as the example "R = the integers modulo 4", shows.

There exist P-rings which are not Noetherian. For example, the complete direct product of infinitely many copies of a (commutative) field is a non-Noetherian P-ring.

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3. Applications.

We shall first establish the main consequence of the theorem in section 2, in order to deduce theorem 3.2 of this section.

THEOREM (3.1). *Let R be a commutative ring with identity in which every maximal ideal is generated by an idempotent. Then R is a direct sum of a finite number of fields.*

PROOF. We begin by proving that R is a P-ring. Let A be any primary ideal in R which is not maximal. Then A is included in a proper maximal ideal, say eR, where e is an idempotent different from 0 and 1. Evidently $e \notin A$. But $e(1-e)=0$. This implies $(1-e)^n \in A$ for some positive integer n, since A is a primary ideal. Hence $(1-e) \in eR$. Thus $1 \in eR$, a contradiction.

Since R is a P-ring, prime ideals are maximal. Therefore every prime ideal is finitely generated (principal) by virtue of the hypothesis. This implies that R is Noetherian by the application of Cohen's result [2, Theorem 2]. Hence it follows from 2.1 that R is a direct sum of a finite number of fields.

THEOREM (3.2). *Let R be a commutative ring with identity. Then the following are equivalent:*

(1) *R is a finite direct sum of fields.*
(2) *Every maximal ideal is generated by an idempotent.*
(3) *Every maximal ideal is a direct summand of R.*
(4) *Every maximal ideal is R-projective as a right R-module and is principally generated by a zero-divisor.*
(5) *Every proper maximal ideal is R-injective as a right R-module.*
(6) *R has no nilpotents and every proper maximal ideal has a non-zero annhilator.*

PROOF. Since (1) implies everyone of the other stated conditions, we have by 3.1, (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). Hence it suffices to show that each one of the conditions (4), (5) and (6) separately imply (2) or (3).

(4) $\Rightarrow$ (2): Let M be an arbitrary maximal ideal and let $M=xR$, x being a zero-divisor. Then x^r, the annihilator of x is non-zero. Consider the exact sequence of R-modules,

$$0 \longrightarrow x^r \xrightarrow{i} R \xrightarrow{j} xR \longrightarrow 0,$$

where "i" is an inclusion mapping and $j: a \to xa$, $a \in R$. Since xR is projective, the above exact sequence splits. Then x^r is a direct summand of R. This implies $x^r = eR$, e being an idempotent $\neq 0$ and 1, since R has an identity. Now $0 = xe$. Therefore

$$x = x(1-e) \quad \text{and} \quad xR \subset (1-e)R .$$

This in turn implies that $xR = (1-e)R$ since xR is a maximal ideal. Thus every maximal ideal is generated by an idempotent.

(5) $\Rightarrow$ (2): Let M be a proper maximal ideal. Since M is R-injective by hypothesis, M is a direct summand of R [1, prop. 3.4] and hence (2) follows.

(6) $\Rightarrow$ (3): If M is a proper maximal ideal and if M^* is its non-zero annihilator, then $M \cap M^* = 0$. For, take any $x \in M \cap M^*$. Then $x^2 = 0$, hence $x = 0$. This implies $R = M \oplus M^*$ and hence (3).

REMARK (3.3). Comparing the condition (4) of theorem 3.2, with an important theorem of Kaplansky [3, theorem 2.3] we observe that the projective nature of all maximal ideals does all that an ascending chain condition can, and thus make the ring a principal ideal ring. In addition we obtained a nice structure of the ring.

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S. V. UNIVERSITY, TIRUPATI, INDIA
AND
BOWLING GREEN STATE UNIVERSITY, OHIO, U.S.A.