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HOMOLOGICAL DIMENSIONS OF $\aleph_0$-COHERENT RINGS

CHR. U. JENSEN

In a previous paper [5] relationships between homological dimensions of rings with countably generated ideals have been considered. It is the purpose of this note to extend these results to a class of rings which include (properly) the rings with countably generated ideals and the so-called coherent rings (cf. Chase [4] and Bourbaki [2, p. 63]).

Let R be an associative, but not necessarily commutative ring with an identity element. A left R-module A is called *countably related*, if there exists a short exact sequence

$$0 \to K \to F \to A \to 0 ,$$

where F is a free R-module with a countable base and K is countably generated.

By a well-known argument ("diagramme du serpent") we have the following analogue of lemme 1 [2] Chap. I § 2.8.

Lemma 1. *If*

$$0 \to C \to B \to A \to 0$$

is any short exact sequence of left R-modules, where A is countably related and B countably generated, then C is countably generated.

The following theorem gives rise to natural generalization of left coherent rings.

Theorem 1. *For an associative ring R with an identity element the following conditions are equivalent:*

a) *Any finitely generated left ideal of R is countably related.*
a′) *Any countably generated left ideal of R is countably related.*
b) *Any finitely generated left R-module, which is a submodule of a free R-module is countably related.*
b′) *Any countably generated left R-module, which is a submodule of a free R-module is countably related.*
c) *The left annihilator of any element in R is countably generated and*

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the intersection of any two finitely generated left ideals of R is countably generated.

c') *The left annihilator of any element in R is countably generated and the intersection of any two countably generated left ideals of R is countably generated.*

PROOF. Since any countably generated module can be written as a countable ascending union of finitely generated submodules, the implications a $\Rightarrow$ a', b $\Rightarrow$ b' and c $\Rightarrow$ c' are easily verified. The converse implications are trivial, so we have the equivalences a $\Leftrightarrow$ a', b $\Leftrightarrow$ b' and c $\Leftrightarrow$ c'.

a') $\Rightarrow$ b). Let A be a finitely generated left R-module, contained in a free R-module F. We may obviously assume that F has a finite base. We shall proceed by induction on the number n of base elements of F. It is convenient to prove b) in a slightly stronger form, namely that any countably generated submodule of a finite free R-module is countably related. For $n=1$ this is exactly what a') says. To pass from $n-1$ to n let A be a countably generated submodule of a free R-module F with n base elements, $e_1,\ldots,e_n$. If F' is the free submodule of F with the $n-1$ base elements $e_1,\ldots,e_{n-1}$, we have a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C & \to & A & \to & B & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & F' & \to & F & \to & R & \to & 0\,. \end{array}$$

B is (isomorphic to) a left ideal of R and is therefore countably related. By lemma 1, C is countably generated. By the inductive assumption C is countably related. Resolving B and C countably and applying [3, chap. I, prop. 2.5], it readily follows that A is countably related.

b') $\Rightarrow$ c'). For any element s in R we have an exact sequence

$$(*)\qquad 0 \to \operatorname{Ann}(a) \to R \xrightarrow{a} Ra \to 0\,.$$

Ra is by assumption countably related, so lemma 1 shows that $\operatorname{Ann}(a)$ is countably generated.

For countably generated left ideals $\mathfrak{a}$ and $\mathfrak{b}$ we have a short exact sequence

$$0 \to \mathfrak{a}\cap\mathfrak{b} \to \mathfrak{a}\oplus\mathfrak{b} \to \mathfrak{a}+\mathfrak{b} \to 0\,.$$

Since $\mathfrak{a}\oplus\mathfrak{b}$ is countably generated and $\mathfrak{a}+\mathfrak{b}$ is countably related, lemma 1 implies that $\mathfrak{a}\cap\mathfrak{b}$ is countably generated.

c) $\Rightarrow$ a). Let $\mathfrak{a}$ be a left ideal in R generated by n elements. We shall prove by induction on n that $\mathfrak{a}$ is countably related. For $n=1$ this follows from the sequence (*). For $n>1$ let $\mathfrak{a}=\mathfrak{b}+Ra$, where $\mathfrak{b}$ is a left ideal generated by $n-1$ elements. By the inductive assumption $\mathfrak{b}$ is the homomorphic image of a free module F with a countably generated kernel. This means that in the following commutative diagram, where the maps are the obvious ones and the rows are exact

$$\begin{array}{ccccccccc} 0 & \to & K & \to & F\oplus R & \to & \mathfrak{b}+Ra & \to & 0 \\ & & \downarrow\psi & & \downarrow\varphi & & \| & & \\ 0 & \to & \mathfrak{b}\cap Ra & \to & \mathfrak{b}\oplus Ra & \to & \mathfrak{b}+Ra & \to & 0 \end{array}$$

$\operatorname{Ker}(\varphi)$ must be countably generated.

A routine diagram chasing shows that ψ is surjective and $\operatorname{Ker}(\psi)\cong\operatorname{Ker}(\varphi)$. Since $\mathfrak{b}\cap Ra$ and $\operatorname{Ker}(\varphi)$ are countably generated this implies that K is countably generated, that is $\mathfrak{a}=\mathfrak{b}+Ra$ is countably related.

DEFINITION. A ring R with an identity element is called *left $\aleph_0$-coherent,* if it satisfies one and hence all of the conditions listed in theorem 1. R is called *right $\aleph_0$-coherent* if its opposite ring is left $\aleph_0$-coherent.

The class of left $\aleph_0$-coherent rings includes all rings with countably generated left ideals ([5, lemma 1]), and all left coherent rings, i.e. for instance left semi-hereditary rings and rings of polynomials in any set of variables with coefficients in a field.

An example of a commutative integral domain which is $\aleph_0$-coherent but not coherent and not having countably generated ideals is the following. Let X and Y be indeterminates over a countable field K;

$$R = K[X, X^2Y, X^3Y^2, \ldots, X^{t+1}Y^t, \ldots]$$

is a non-coherent ring with countably many elements. Any polynomial ring over R in uncountably many variables is $\aleph_0$-coherent, but does not have countably generated ideals.

In the non-commutative case a ring may be left $\aleph_0$-coherent without being right $\aleph_0$-coherent (cf. Small [7]).

The following lemma may be proved verbatim as lemma 2 in [5]. (The slightly more general and more convenient formulation has been suggested by Kaplansky.)

LEMMA 2. *For any countably related flat left R-module A we have* $\operatorname{l.dh}_R(A)\leqq 1$.

We are now able to prove

THEOREM 2. *Let R be a left $\aleph_0$-coherent ring for which any left ideal is generated by $\aleph_n$ elements. Then*

$$\mathrm{l.gl.dim}\, R \leqq \mathrm{w.gl.dim}\, R + n + 1\ .$$

PROOF. By a theorem of Auslander it suffices to prove

$$\mathrm{l.dh}_R(\mathfrak{a}) \leqq \mathrm{w.gl.dim}\, R + n$$

for any left ideal $\mathfrak{a}$ of R. It may be assumed that $d = \mathrm{w.gl.dim}\, R$ is finite, since otherwise there is nothing to prove. We shall prove slightly more than is needed, namely that

$$\mathrm{l.dh}_R(\mathfrak{a}) \leqq d + r$$

for any left ideal generated by $\aleph_r$ elements. This will be done by induction on r.

For $r=0$ repeated use of condition b′) of theorem 1 implies that there is an exact sequence

$$(**) \qquad 0 \to K \to F_{d-2} \to \ldots \to F_1 \to F_0 \to \mathfrak{a} \to 0$$

where $F_0, \ldots, F_{d-2}$ are free modules with countable bases and K is countably related.

By use of the iterated connecting homomorphism for the torsion functors we have for any right R-module M

$$\mathrm{Tor}_1^R(M,K) \cong \mathrm{Tor}_d^R(M,\mathfrak{a})\ .$$

Since $\mathrm{w.l.dh}_R(R/\mathfrak{a}) \leqq d$ we have

$$\mathrm{Tor}_d^R(M,\mathfrak{a}) \cong \mathrm{Tor}_{d+1}^R(M,R/\mathfrak{a}) = 0\ .$$

Hence K is a countably related flat left R-module and is therefore by lemma 2 a quotient of two projective R-modules

$$0 \to P_d \to P_{d-1} \to K \to 0\ .$$

Inserting this in (**) we get

$$0 \to P_d \to P_{d-1} \to F_{d-2} \to \ldots \to F_0 \to \mathfrak{a} \to 0$$

which shows that $\mathrm{l.dh}_R(\mathfrak{a}) \leqq d$.

To pass from $r-1$ to r we use that the well ordered set $Z(\aleph_{r-1})$ consisting of all ordinal numbers of cardinality $\aleph_{r-1}$, has the cardinal number $\aleph_r$. This means that for any left ideal $\mathfrak{a}$ generated by $\aleph_r$ elements there is a system of generators $\{a_\alpha\}$ where the index set $\{\alpha\}$ consists of all ordinal numbers of cardinality $\aleph_{r-1}$.

For any α let A_α be the ideal generated by all a_β for which $\beta < \alpha$ and let $\bar{A}_\alpha$ be the ideal generated by all a_β for which $\beta \leqq \alpha$. A_α and $\bar{A}_\alpha$ are then generated by $\aleph_{r-1}$ elements; the inductive assumption implies that

$$\mathrm{l.dh}_R(A_\alpha) \leqq d+r-1 \quad \text{and} \quad \mathrm{l.dh}_R(\bar{A}_\alpha) \leqq d+r-1 .$$

From the exact sequence

$$0 \to A_\alpha \to \bar{A}_\alpha \to \bar{A}_\alpha/A_\alpha \to 0$$

we immediately conclude that

$$\mathrm{l.dh}_R(\bar{A}_\alpha/A_\alpha) \leqq d+r .$$

Applying proposition 3 in Auslander [1] we infer that

$$\mathrm{l.dh}_R\Big(\bigcup_\alpha A_\alpha\Big) = \mathrm{l.dh}_R(\mathfrak{a}) \leqq d+r$$

and the theorem is proved.

The following corollary is a consequence of the left-right symmetry of the weak global dimension.

COROLLARY. *Let R be left and right $\aleph_0$-coherent. If any left ideal and any right ideal of R can be generated by $\aleph_n$ elements, then*

$$|\mathrm{l.gl.dim}\,R - \mathrm{r.gl.dim}\,R| \leqq n+1 .$$

In particular, this inequality holds if $\mathrm{Card}(R) \leqq \aleph_n$.

For $n>0$ theorem 2 and its corollary are not very informative, unless we assume the continuum hypothesis or even the generalized continuum hypothesis. However, if we grant the continuum hypothesis, the following examples show that the above results do yield some explicit information.

1) R is the complete direct product of countably many fields. Here R is regular, so $\mathrm{w.gl.dim}\,R = 0$ and hence $\mathrm{gl.dim}\,R \leqq 2$; on the other hand R contains non-countably generated ideals, which by well-known results of Kaplansky cannot be projective R-modules. Consequently $\mathrm{gl.dim}\,R = 2$.

2) R is the complete direct product of countably many principal ideal domains R_i for which $\mathrm{Card}(R_i) \leqq \aleph$ and no R_i is a field. R is semi-hereditary, so

$$\mathrm{w.gl.dim}\,R = 1$$

and $\mathrm{gl.dim}\,R \leqq 3$. Let $R_i x_i \neq (0)$ be a maximal ideal of R_i. Then $x = \{x_i\}$ is neither a unit nor zero-divisor in R, and R/Rx is the complete direct product of the fields $R_i/R_i x_i$; by 1) there is a module M over R/Rx such that $\mathrm{dh}_{R/Rx}(M) = 2$. By a well-known change of rings theorem (cf. Kaplansky [6, th. 1.3]) we have $\mathrm{dh}_R(M) = 3$. Thus

$$\mathrm{gl.dim}\,R = 3 .$$

To take an explicit example let R be the ring of all sequences of rational integers; here we have

$$\text{w.gl.dim}\, R = 1 \quad \text{and} \quad \text{gl.dim}\, R = 3\,.$$

Similarly one finds for the ring of all bounded sequences of integers that again $\text{w.gl.dim}\, R = 1$ and $\text{gl.dim}\, R = 3$.

3) R is the complete direct product of countably many copies of the polynomial ring $K[X_1,\ldots,X_n]$, where K is a field of cardinality $\leqq \aleph$. By repeated use of the above argument in 2) we see that $\text{gl.dim}\, R \geqq n+2$. If $n=1$ or 2, R is coherent and $\text{w.gl.dim}\, R = n$; hence $\text{gl.dim}\, R = n+2$. Similarly, if R is the complete direct product of countably many regular local rings of dimension 2 and of cardinality $\leqq \aleph$, one finds $\text{w.gl.dim}\, R = 2$ and $\text{gl.dim}\, R = 4$.

4) For a valuation ring R we have $\text{w.gl.dim}\, R = 1$. If R is Noetherian, $\text{gl.dim}\, R = 1$; if R is not Noetherian but has countably generated ideals, $\text{gl.dim}\, R = 2$; finally, if R contains non-countably generated ideals it has been proved (unpublished) by Kaplansky that $\text{gl.dim}\, R \geqq 3$. Therefore if the ideals of R are $\aleph$-generated but not all of them $\aleph_0$-generated we have (assuming the continuum hypothesis) that $\text{gl.dim}\, R = 3$. It might be worth while noting that this statement does not hold for Prüfer rings. In fact, let R be a principal ideal domain for which the quotient field K is not a countably generated R-module. The subring S of $K[[X]]$ consisting of all formal power series whose constant terms are elements of R, is a Prüfer ring containing non-countably generated ideals, but $\text{gl.dim}\, S = 2$.

Remark. The examples 1)–4) show that the inequality of theorem 2 is best possible for $n=0$ and 1. For higher cardinal numbers the situation is messy, since it involves settheoretical difficulties.

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UNIVERSITY OF COPENHAGEN, DENMARK