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EXCISION IN SINGULAR THEORY

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The purpose of this note is to give a proof of the excision theorem in singular homology based on the acyclic model theorem. This theorem is the formalization of a simple inductive argument in homological algebra, and thus our proof should be considered elementary. The proof does away with most of the calculations connected with the barycentric subdivisions, and seems to be more conceptual than the traditional proofs. It also works for homology with local coefficients. The theorem we want to prove is the following.

Excision theorem. *Let X be a topological space, Γ a local system on X and $\mathcal{U}$ a collection of subsets whose interiors form a covering of X. Then the inclusion*

$$\Delta(\mathcal{U};\Gamma) \subset \Delta(X;\Gamma)$$

is a chain equivalence.

Here $\Delta(\mathcal{U};\Gamma)$ means the subcomplex of $\Delta(X;\Gamma)$ consisting of chains with simplexes small of order $\mathcal{U}$. This is consistent with the notation of Spanier [1]. All our notations will be taken from this book. Also a *local system* means a local system of R-modules, where R is some fixed commutative ring with unit.

1.

For every natural number $m \geqq 0$ and for every topological space X a natural augmentation-preserving chain map

$$sd^m\colon\ \Delta(X) \to \Delta(X)$$

is defined ([1, Ch. 4.4]). By the acyclic model theorem, since the singular chain functor Δ is free and acyclic on the *standard models* $\{\Delta^q\}_{q\geqq 0}$, sd^m is chain homotopic to $1_{\Delta(X)}$ by some natural chain deformation D^m. We now have the following result.

Lemma 1. *Let X be some standard model and let $\mathcal{U}$ be a collection of*

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subsets whose interiors form a covering of X. For any singular simplex $\sigma \in \Delta(X)$ *there is an* $m \geqq 0$ *such that* $sd^m \sigma \in \Delta(\mathscr{U})$.

The proof of this is easy (see [1, 4.4.13]). Note that since X is compact it suffices to consider finite coverings $\mathscr{U}$. Using this lemma we prove

LEMMA 2. *Let X be some standard model and let* $\mathscr{U}$ *be a collection of subsets whose interiors form a covering of X. Then the inclusion*

$$\Delta(\mathscr{U}) \subset \Delta(X)$$

is a chain equivalence, that is, $\tilde{\Delta}(\mathscr{U})$ *is acyclic.*

PROOF. It is sufficient to show that every singular q-chain of $\Delta(\mathscr{U})$ which is a boundary of $\Delta(X)$, also is a boundary of $\Delta(\mathscr{U})$. Let $c_q \in \Delta(\mathscr{U})$ be a singular q-chain such that $c_q = \partial c'_{q+1}$ for a $q+1$-chain $c'_{q+1} \in \Delta(X)$. The chain c'_{q+1} is a finite linear combination of singular $q+1$-simplexes, and by lemma 1 there is a natural number $m \geqq 0$ such that $sd^m c'_{q+1} \in \Delta(\mathscr{U})$. Because the chain deformation D^m: $sd^m \cong 1_{\Delta(X)}$ is natural,

$$D^m(\Delta(\mathscr{U})) \subset \Delta(\mathscr{U}) \quad \text{and} \quad D^m c_q \in \Delta(\mathscr{U})\,.$$

Put

$$c_{q+1} = sd^m c'_{q+1} - D^m c_q\,.$$

Then $c_{q+1} \in \Delta(\mathscr{U})$ and $\partial c_{q+1} = c_q$, and the lemma is proved.

2.

Consider all pairs $(X,\mathscr{U})$ where X is a topological space and $\mathscr{U}$ is a family of subsets of X such that $\operatorname{int}\mathscr{U} = \{\operatorname{int} U\}_{U\in\mathscr{U}}$ is an open covering of X. A map f: $(X,\mathscr{U}) \to (Y,\mathscr{V})$ of such pairs is by definition a continuous map f: $X \to Y$ such that $\mathscr{U} = f^{-1}(\mathscr{V})$. Clearly there is a category whose objects are pairs $(X,\mathscr{U})$ and whose morphisms are maps f: $(X,\mathscr{U}) \to (Y,\mathscr{V})$. On this category define the two functors

$$(X,\mathscr{U}) \mapsto \Delta(X)$$

and

$$(X,\mathscr{U}) \mapsto \Delta(\mathscr{U})\,,$$

and denote these by Δ' and Δ'' respectively. Then Δ' and Δ'' take values in the category of augmented chain complexes. Let $\mathscr{M}$ be the collection of all pairs $(\Delta^q, \mathscr{V})$ for $q \geqq 0$ and $\mathscr{V}$ varies over all coverings of Δ^q such that $\operatorname{int}\mathscr{V}$ is a covering. If we can show that Δ' and Δ'' are both free and acyclic with respect to the models $\mathscr{M}$, it follows from the theorem of acyclic models that there are natural chain maps from one functor to the other, unique up to natural homotopy, and that any such is a chain equivalence. In particular the inclusion map $\Delta(\mathscr{U}) \subset \Delta(X)$ defines a natural chain map $\Delta'' \to \Delta'$, and so is a chain equivalence.

LEMMA 3. *Δ' and Δ'' are both free and acyclic on the models $\mathcal{M}$.*

PROOF. For $(X,\mathcal{U})$ an object in our category and $\sigma \in \Delta_q'(X,\mathcal{U})$ an arbitrary singular q-simplex there is *one and only one* covering, namely $\mathcal{V} = \sigma^{-1}(\mathcal{U})$, of Δ^q such that σ defines a morphism

$$\sigma\colon\ (\Delta^q,\mathcal{V}) \to (X,\mathcal{U})\,.$$

By this remark we see that the family

$$\{\xi_q \in \Delta_q'(\Delta^q,\mathcal{V})\}\,,$$

where ξ_q is the identity map $\Delta^q \subset \Delta^q$ and $\mathcal{V}$ varies over all coverings of Δ^q such that $\operatorname{int}\mathcal{V}$ also is a covering, is a basis for Δ_q'. Hence Δ' is free with models $\mathcal{M}$. By the same remark we also see that the family

$$\{\xi_q \in \Delta_q''(\Delta^q,\mathcal{V})\}\,,$$

where ξ_q is the identity map $\Delta^q \subset \Delta^q$ and $\mathcal{V}$ varies over all coverings of Δ^q such that $\{\Delta^q\} \in \mathcal{V}$, is a basis for Δ_q'', and so Δ'' is free with models $\mathcal{M}$.

It is trivial that Δ' is acyclic on the models $\mathcal{M}$, and by lemma 2 it follows that also Δ'' is acyclic. This completes the proof of the excision theorem for constant coefficients.

3.

For the general case we have to modify our construction of section 2.

There is a category whose objects are triples $(X,\mathcal{U},\Gamma)$, where X is a space, $\mathcal{U}$ a family of sets whose interiors cover X and Γ a local system on X, and whose morphisms

$$f\colon\ (X,\mathcal{U}',\Gamma') \to (Y,\mathcal{U},\Gamma)$$

are continuous maps $f\colon X \to Y$ for which

$$f^{-1}\mathcal{U} = \mathcal{U}', \qquad f^{-1}\Gamma = \Gamma'\,.$$

Define functors $\bar{\Delta}'$, $\bar{\Delta}''$ by

$$(X,\mathcal{U},\Gamma) \mapsto \Delta(X,\Gamma)\,,$$
$$(X,\mathcal{U},\Gamma) \mapsto \Delta(\mathcal{U},\Gamma)\,.$$

We show that any natural chain map

$$\tau\colon\ \Delta' \to \Delta'' \qquad (\tau'\colon\ \Delta'' \to \Delta')$$

can be lifted to a natural chain map

$$\bar{\tau}\colon\ \bar{\Delta}' \to \bar{\Delta}'' \qquad (\bar{\tau}'\colon\ \bar{\Delta}'' \to \bar{\Delta}')\,.$$

In fact, if $\tau(\xi_q)=\sum n_\varrho \varrho$ with $\varrho\colon \Delta^q \to \Delta^q$, let

$$\Gamma_\varrho{}'\colon\ \Gamma'(v_0) \to \Gamma'(\varrho(v_0))$$

be the isomorphism obtained by applying Γ' on Δ^q to the unique path class in Δ^q from v_0 to $\varrho(v_0)$. Then, for $\alpha \in \Gamma'(v_0)$ define

$$\bar{\tau}(\alpha\xi_q) = \sum n_\varrho \Gamma_\varrho{}'(\alpha)\varrho .$$

Now extend this definition by naturality to arbitrary simplexes $\alpha\sigma \in \Delta(X,\mathscr{U},\Gamma)$ and then by linearity to arbitrary chains, that is, define

$$\bar{\tau}(\alpha\sigma) = \bar{\Delta}''(\sigma)\big(\bar{\tau}(\alpha\xi_q)\big) .$$

This is meaningfull since σ give rise to a *unique* morphism to $(X,\mathscr{U},\Gamma)$, namely

$$\sigma\colon\ (\Delta^q,\sigma^{-1}\mathscr{U},\, \sigma^{-1}\Gamma) \to (X,\mathscr{U},\Gamma) .$$

In the same way any natural chain map $\tau'\colon \Delta'' \to \Delta'$ and chain deformations

$$D\colon\ \tau'\circ\tau \cong 1_{\Delta'}, \qquad D'\colon\ \tau\circ\tau' \cong 1_{\Delta''}$$

extend to a natural chain map $\bar{\tau}'\colon \bar{\Delta}'' \to \bar{\Delta}'$ and deformations

$$\bar{D}\colon\ \bar{\tau}'\circ\bar{\tau} \cong 1_{\bar{\Delta}'}, \qquad \bar{D}'\colon\ \bar{\tau}\circ\bar{\tau}' \cong 1_{\bar{\Delta}''} .$$

REFERENCE

1. E. H. Spanier, *Algebraic topology*, New York, 1966.

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