MATH. SCAND. 20 (1967), 65–78

L^p ESTIMATES FOR (PLURI-) SUBHARMONIC FUNCTIONS

LARS HÖRMANDER

1. Introduction.

The work of Carleson [2], [3] on interpolation by bounded analytic functions of one complex variable was based on estimates of the form

$$\int_D |f|^p \, d\mu \leqq C_p \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \, . \tag{1.1}$$

Here μ denotes a positive measure on the open unit disc D, and either f is a harmonic function and $1 < p < \infty$ or f is an analytic function and $0 < p < \infty$. Carleson found that (1.1) is then valid if and only if for all real θ and $r > 0$

$$\mu\{z;\ z \in D,\ |z - e^{i\theta}| \leqq r\} \leqq Cr \, , \tag{1.2}$$

where C is some other constant. The estimate (1.1) is closely related to the Hardy–Littlewood maximal theorem. Using arguments introduced by Smith [7] in generalizing that result to several dimensions, we shall prove some new estimates of maximal functions in section 2. As shown in section 3 they lead to a simple proof of Carleson's estimates and their extensions to harmonic functions of n variables. Thus if Ω is a bounded open set in $\boldsymbol{R}^n$ with C^2 boundary and if $1 < p < \infty$, we prove that

$$\int_\Omega |f|^p \, d\mu \leqq C_p \int_{\partial\Omega} |f|^p \, d\omega \tag{1.3}$$

for all harmonic functions f in Ω if and only if the positive measure μ on Ω has the property

$$\mu(B \cap \Omega) \leqq C' r^{n-1} \text{ if } B \text{ is any ball with center on } \partial\Omega \text{ and radius } r \, . \tag{1.4}$$

In particular, (1.3) can be applied to analytic functions of several complex variables if $\Omega \subset \boldsymbol{C}^k$. However, if f is required to be analytic, a larger class of measures μ can be used in (1.3). This is obvious when Ω

Received November 14, 1966.

is not a domain of holomorphy for clearly no restriction on μ except that the total mass be finite is needed near a boundary point of Ω which is in the interior of the hull of holomorphy. Also for domains of holomorphy a larger class of measures can be used in (1.3) when f is analytic than when f is merely harmonic. Indeed, in section 4 we shall obtain a necessary and sufficient condition for (1.3) to hold when Ω is a strictly pseudoconvex domain. This condition is analogous to (1.4) but strictly weaker. Instead of balls B it involves sets which are much larger in the directions of complex tangents to $\partial\Omega$ than in other directions. For the precise statement see section 4.

It is useful to consider positive subharmonic or plurisubharmonic functions throughout, for $|u|$ is subharmonic if u is harmonic, and $|f|^p = \exp(p \log|f|)$ is plurisubharmonic for every $p>0$ if f is analytic.

2. Estimates for maximal functions.

In this section we estimate mean values of an integrable function over certain families of sets. Applications to subharmonic and plurisubharmonic functions will be given in sections 3 and 4 respectively.

Let M be a locally compact topological space and m a positive measure on M. Set $M' = M \times \boldsymbol{R}_+$, where $\boldsymbol{R}_+$ is the open positive real axis. We assume that to every $X=(x,t) \in M'$ there is assigned a subset B_X of M so that the following hypotheses are fulfilled:

(i) B_X *is integrable with respect to* m.

(ii) *There exist constants* k *and* K *with* $1 < k \leqq K$ *such that*

$$s \leqq kt,\ B_{(x,s)} \cap B_{(y,t)} \neq \emptyset \ \Rightarrow\ B_{(x,s)} \subset B_{(y,Kt)}\,.$$

(iii) *There is a constant* C_1 *such that for the same* K *as in* (ii)

$$m(B_{(x,Kt)}) \leqq C_1 m(B_{(x,t)})\,.$$

EXAMPLE. Let M be a metric space and define $B_{(x,t)}$ to be the set of points in M with distance $\leqq t$ from x. Then (ii) is fulfilled if $s \leqq kt$ implies that $t+2s \leqq Kt$, that is, if $1+2k \leqq K$. Conditions (i) and (iii) are of course also fulfilled if $M=\boldsymbol{R}^n$ with the Euclidean metric and Lebesgue measure, $C_1 = K^n$, or if M is a compact Riemannian manifold with the Riemannian metric and element of integration. (Another example will occur in section 4.)

For every $f \in L^1_{\mathrm{loc}}(M)$ we define a "maximal function" by

$$\tilde{f}(X) = \sup_{B_Y \supset B_X} \int_{B_Y} |f|\, dm/m(B_Y), \qquad X \in M'\,. \tag{2.1}$$

Here we use the convention $0/0=0$.

Our estimates of $\tilde{f}$ involve a non-negative subadditive function $\mu(E)$ defined for all subsets E of M', such that $\mu(E_j) \nearrow \mu(E)$ if $E_j \nearrow E$. For example, $\mu(E)$ may be the outer measure of E defined by a positive measure on M', or $\mu(E)$ may be an outer measure of the projection of E on M. The latter case with the outer measure defined by m leads to the original Hardy–Littlewood estimates. Following Carleson [3] we assume

(iv) *There is a constant C_2 such that*

$$\mu\{X ;\ X \in M', B_X \subset B_Y\} \leqq C_2 m(B_Y), \qquad Y \in M' .$$

THEOREM 2.1. *If the conditions* (i)–(iv) *are fulfilled, we have*

$$\mu\{X : \tilde{f}(X) > s\} \leqq C_1 C_2 s^{-1} \int |f|\, dm, \qquad f \in L^1(M),\ s > 0 . \tag{2.2}$$

The proof depends on the following variant of a covering lemma due to Aronszajn and Smith [1], used in a similar context by Smith [7].

LEMMA 2.2. *Let E be a subset of M' such that t is bounded when $(x,t) \in E$ and for no infinite sequence $X_j = (x_j, t_j) \in E$ the sets B_{X_j} are all disjoint. Then one can find finitely many points $X_j = (x_j, t_j) \in E$ such that the sets B_{X_j} are disjoint and*

$$E \subset \bigcup_j \{X ;\ B_X \subset B_{(x_j, Kt_j)}\} .$$

PROOF OF LEMMA 2.2. Let $T_1 = \sup\{t ;\ (x,t) \in E\}$, and choose $X_1 = (x_1, t_1) \in E$ with $kt_1 \geqq T_1$, which is possible since $k > 1$. If $X_1, \ldots, X_{j-1}$ have already been chosen, we let T_j be the supremum of t when $(x,t) \in E$ and $B_{(x,t)}$ does not intersect $B_{X_1}, \ldots, B_{X_{j-1}}$, if such points (x,t) exist. We then choose $X_j = (x_j, t_j)$ with $kt_j \geqq T_j$ so that $X_j \in E$ and B_{X_j} is disjoint with B_{X_i} when $i < j$. By hypothesis this construction must break off after a finite number of steps. For every $X \in E$ we then have $B_X \cap B_{X_j} \neq \emptyset$ for some j. If j is the smallest such index we have $t \leqq T_j$ if $X = (x,t)$. On the other hand, $kt_j \geqq T_j$ so that $t \leqq kt_j$. From condition (ii) it follows then that $B_X \subset B_{(x_j, Kt_j)}$, and this proves the lemma.

PROOF OF THEOREM 2.1. Let $0 < \varepsilon$, $0 < T$ and set

$$E_{\varepsilon,T} = \left\{X = (x,t) \in M' ;\ 0 < t \leqq T, \int_{B_X} |f|\, dm > s(\varepsilon + m(B_X))\right\}.$$

If $X_j \in E_{\varepsilon,T}$ and B_{X_j} are disjoint, we have

$$\sum_j s(\varepsilon+m(B_{X_j})) \leqq \sum_j \int_{B_{X_j}} |f|\,dm \leqq \int_M |f|\,dm\,. \tag{2.3}$$

Hence the sequence X_j must be finite, so we can apply Lemma 2.2 to $E_{\varepsilon,T}$. Writing $E'_{\varepsilon,T}=\{X\,;\,B_X\subset B_Y$ for some $Y\in E_{\varepsilon,T}\}$, we obtain

$$E'_{\varepsilon,T} \subset \bigcup_j \{X\,;\,B_X\subset B_{(x_j,Kt_j)}\}\,.$$

In view of the conditions (iv) and (iii) and the inequality (2.3) it follows that

$$\begin{aligned}\mu(E'_{\varepsilon,T}) &\leqq \sum_j \mu\{X\,;\,B_X\subset B_{(x_j,Kt_j)}\}\\ &\leqq C_2\sum_j m(B_{(x_j,Kt_j)})\\ &\leqq C_1C_2\sum_j m(B_{X_j}) \leqq C_1C_2s^{-1}\int_M |f|\,dm\,.\end{aligned}$$

When $\varepsilon\downarrow 0$ and $T\nearrow\infty$, the set $E'_{\varepsilon,T}$ increases to the set $\{X\,;\,\tilde f(X)>s\}$. This proves (2.2).

Example. When $M=\boldsymbol{R}^n$ and $B_{(x,t)}$ is the ball with radius t and center at x, we can take for K any number >3 and $C_1=K^n$. If m is Lebesgue measure and μ is the outer Lebesgue measure of the projection on M the inequality (2.2) is valid with C_1C_2 replaced by 3^n. More generally, (2.2) follows from (iv) with C_1 replaced by 3^n.

If $m(B_Y)\neq 0$ and we apply (2.2) to the characteristic function f of B_Y, noting that $\tilde f(X)=1$ when $B_X\subset B_Y$, we conclude that (iv) must be valid with C_2 replaced by C_1C_2. Apart from the size of the constants, *the condition* (iv) *is thus essentially equivalent to* (2.2) *when* (i)–(iii) *are fulfilled.*

In some applications given by Carleson [2] it is important that one need not assume (iv) to be fulfilled for every $Y\in M'$. This is proved in the following theorem.

Theorem 2.3. *Assume that* (i)–(iii) *are valid, that* $m(B_Y)>0$ *for every* $Y\in M'$ *and that for* μ*-almost all* $Y=(y,t)$ *the condition* (iv) *is fulfilled with* Y *replaced by* (y,Kt). *Then* (iv) *is valid for every* $Y\in M'$ *with* C_2 *replaced by* C_1C_2.

Proof. Given Y we set $E=\{X\,;\,B_X\subset B_Y\}$ and denote by E' the set of all $X=(x,t)\in E$ such that (iv) is valid with Y replaced by (x,Kt). Then $E\setminus E'$ is a nullset with respect to μ, so we have $\mu(E)=\mu(E')$. Let $E'_{\varepsilon,T}$ be the set of all $X=(x,t)\in E'$ such that $m(B_X)>\varepsilon$ and $t<T$. If $X_j\in E'_{\varepsilon,T}$ and B_{X_j} are disjoint, it follows from the inequality

$$\sum m(B_{X_j}) \leqq m(B_Y)$$

and the fact that $m(B_{X_j}) > \varepsilon$ that there are only finitely many X_j. Hence Lemma 2.2 can be applied, and we obtain a finite number of points $X_j = (x_j, t_j) \in E'_{\varepsilon,T}$ such that

$$E'_{\varepsilon,T} \subset \bigcup_j \{X\,;\ B_X \subset B_{(x_j, Kt_j)}\}\,.$$

Hence

$$\begin{aligned}\mu(E'_{\varepsilon,T}) \leqq \sum \mu\{X\,;\ B_X \subset B_{(x_j,Kt_j)}\} &\leqq C_2 \sum m(B_{(x_j,Kt_j)}) \\ &\leqq C_1 C_2 \sum m(B_{X_j}) \leqq C_1 C_2 m(B_Y)\,.\end{aligned}$$

Since $E'_{\varepsilon,T} \nearrow E'$ when $\varepsilon \downarrow 0$ and $T \nearrow \infty$, it follows that $\mu(E') \leqq C_1 C_2 m(B_Y)$, which completes the proof.

Using Marcinkiewicz' interpolation theorem, which we prove in the simple case we need, we can derive an L^p estimate from (2.2).

Theorem 2.4. *Let the conditions* (i)–(iv) *be fulfilled. Then we have*

$$\int_0^\infty s^{p-1} g(s)\, ds \leqq C_1 C_2 (p/(p-1))^p \int |f|^p\, dm, \qquad f \in L^p(M),\ 1 < p < \infty\,, \tag{2.4}$$

where $g(s) = \mu\{X\,;\ \tilde{f}(X) > s\}$.

Very mild assumptions concerning the continuity of B_X as a function of X guarantee that f is semi-continuous from below. If μ is a measure in M' we can then rewrite (2.4) in the more useful form

$$\int |\tilde{f}(X)|^p\, d\mu \leqq C_1 C_2 p (p/(p-1))^p \int |f|^p\, dm, \qquad f \in L^p(M)\,. \tag{2.4$'$}$$

Proof of Theorem 2.4. Let ε be a number with $0 < \varepsilon < 1$ to be determined later, set $f_1 = f$ where $|f| < s\varepsilon$ and $f_1 = 0$ elsewhere, and define f_2 so that $f = f_1 + f_2$. Then we have $\tilde{f}_1 \leqq s\varepsilon$ so it follows that $\tilde{f}_2(X) > s(1-\varepsilon)$ if $\tilde{f}(X) > s$. By Theorem 2.1 we therefore have

$$g(s) \leqq C_1 C_2 (1-\varepsilon)^{-1} s^{-1} \int_{|f| \geqq s\varepsilon} |f|\, dm\,,$$

which implies that

$$\begin{aligned}\int_0^\infty s^{p-1} g(s)\, ds &\leqq C_1 C_2 (1-\varepsilon)^{-1} \iint_{0<s<|f|/\varepsilon} s^{p-2} |f|\, ds\, dm \\ &= C_1 C_2 (1-\varepsilon)^{-1} \varepsilon^{1-p} (p-1)^{-1} \int |f|^p\, dm\,.\end{aligned}$$

If we choose $\varepsilon = 1 - 1/p$, the theorem is proved.

3. Estimates for subharmonic functions.

Let Ω be a bounded open subset of $\boldsymbol{R}^n$ with a C^2 boundary ω. (The results can easily be extended to the case of a Riemannian manifold with boundary.) By $S^p(\Omega)$, $1<p<\infty$, we denote the set of non-negative subharmonic functions u in Ω such that

$$\lim_{\varepsilon\to 0} \int u(y+\varepsilon N)^p \, d\omega(y) < \infty . \tag{3.1}$$

Here $N(y)$ denotes the interior unit normal of ω at y, and $d\omega$ is the surface element on ω. It follows from (3.1) (see e.g. Gårding–Hörmander [4]) that u can be defined on ω so that

$$\lim_{\varepsilon\to 0} \int |u(y+\varepsilon N(y)) - u(y)|^p \, d\omega(y) = 0 .$$

Note that if u is harmonic and $|u|$ satisfies (3.1), then $|u| \in S^p$.

THEOREM 3.1. *Let μ be a positive measure on Ω such that*

$$\int_{|x-y|<\delta} d\mu(x) \leqq C\delta^{n-1}, \qquad y\in\omega,\ \delta>0 . \tag{3.2}$$

Then there is another constant C' such that

$$\int_\Omega u^p \, d\mu \leqq C' p(p/(p-1))^p \int u^p \, d\omega, \qquad u\in S^p(\Omega),\ 1<p<\infty . \tag{3.3}$$

Conversely, (3.2) *is fulfilled if for some p the estimate* (3.3) *is valid when u is continuous in $\bar\Omega$, positive and harmonic in Ω.*

PROOF. For $(y,t)\in\omega\times\boldsymbol{R}_+$ we denote by $B_{(y,t)}$ the set of points in ω at distance $\leqq t$ from y (measured in $\boldsymbol{R}^n$), and we define

$$\tilde u(y,t) = \sup_{s\geqq t} \int_{B_{(y,s)}} u \, d\omega/\omega(B_{(y,s)}) .$$

If x is a point in Ω with distance d to ω, and x' is a point on ω where this shortest distance is attained, we have

$$u(x) \leqq C\tilde u(x',d) . \tag{3.4}$$

We postpone for a moment the proof of this quite well known fact in order to show first that it implies (3.3). Indeed, (3.3) is obvious if the support of μ is a compact subset of Ω, for on such a set we have

$$u^p(x) \leqq C \int u^p \, d\omega .$$

On the other hand, if μ has its support in a neighborhood K of ω which is so small that the map $K \ni x \to (x', d(x))$ is one to one and has a C^1 inverse, the estimate (3.3) is an immediate consequence of Theorem 2.4.

To prove (3.4) we introduce the Poisson kernel $P(x,y)$ of ω, where $x \in \Omega$ and $y \in \omega$. It follows for example from the solution of the Dirichlet problem by integral equations that

$$P(x,y) \leqq C d(x) |x-y|^{-n} .$$

Now we have

$$u(x) \leqq \int P(x,y) u(y)\, \partial\omega(y) \tag{3.5}$$

for by the definition of subharmonic functions we have the corresponding inequality where Ω is replaced by the set Ω_ε of points of distance $> \varepsilon$ from $\complement\Omega$, and this gives (3.5) when $\varepsilon \to 0$. The triangle inequality implies

$$d(x) + |x'-y| \leqq 2|x'-x| + |x-y| \leqq 3|x-y|, \qquad x \in \Omega,\ y \in \omega .$$

Hence for some C

$$u(x) \leqq C \int d(d+|x'-y|)^{-n}\, u(y)\, d\omega(y) .$$

The part of the integral where $|x'-y| < d$ can be estimated by

$$d^{1-n} \int_{B(x',d)} u(y)\, d\omega(y) \leqq C\tilde{u}(x',d) .$$

If k is an integer $\geqq 1$, the part of the integral where

$$2^{k-1} d \leqq |x'-y| \leqq 2^k d$$

can be estimated by

$$d^{1-n} 2^{n(1-k)} \int_{B(x',2^k d)} u(y)\, d\omega(y) \leqq C d^{1-n} 2^{n(1-k)} (2^k d)^{n-1} \tilde{u}(x',d) .$$

The right hand side can be simplified to $C' 2^{-k} \tilde{u}(x',d)$, and since $\sum 2^{-k} = 1$ we obtain (3.4).

Finally, to prove that (3.2) follows from (3.3) we only have to apply (3.3) to the Poisson integral of a continuous function on the boundary with values between 0 and 1 which is equal to 1 in $B_{(y,2\delta)}$ and 0 outside $B_{(y,3\delta)}$, for the Poisson integral will be bounded from below by a constant independent of y and δ when $|y-x| < \delta$.

In view of Theorem 2.3 the hypothesis (3.2) can be seemingly relaxed:

THEOREM 3.1′. *Assume that in a neighborhood of ω we have for μ-almost all $x \in \Omega$*

(3.2)′
$$\int_{|x-y|<4d(x)} d\mu(y) \leqq C d(x)^{n-1},$$

where $d(x)$ is the distance from x to ω. Then (3.2) *and* (3.3) *are valid for some other constants.*

The simple verification may be left to the reader. Instead we shall show that Theorem 3.1′ contains the following result of Carleson [2] (see also Shapiro and Shields [6]):

THEOREM 3.2. *Let z_j, $j=1,2,\ldots$, be a sequence of different points in the open unit disc in $\mathbf{C}$, and assume that*

(3.6)
$$\prod_{j;j\neq k} |z_j - z_k|/|1-z_j\bar{z}_k| > c > 0, \qquad k=1,2,\ldots,$$

for some constant c. Then we have

(3.7)
$$\sum (1-|z_j|^2)\,|f(z_j)|^p \leqq C\int |f(e^{i\theta})|^p\, d\theta, \qquad f\in H^p,\ p>0,$$

where H^p is the Hardy class.

PROOF. Since $\log|f|$ is subharmonic, we have $|f|^{\frac{1}{2}p} \in S^2$, so it is sufficient to prove that the measure μ having the mass $1-|z_j|^2$ at z_j for every j and no mass elsewhere satisfies the hypothesis (3.2)′ with Ω equal to the unit disc. To do se we note (cf. Carleson [2]) that

$$1 - |z-z_j|^2|1-\bar{z}_j z|^{-2} = (1-|z|^2)(1-|z_j|^2)\,|1-\bar{z}_j z|^{-2}.$$

Since an inequality $e^{-a} \leqq \prod(1-\alpha_j)$ where $0\leqq \alpha_j < 1$ implies that $\sum \alpha_j \leqq a$, it follows from (3.6) that

$$\sum_{j;j\neq k} (1-|z_j|^2)(1-|z_k|^2)\,|1-z_j\bar{z}_k|^{-2} \leqq \log c^{-2}, \qquad k=1,2,\ldots.$$

If $|\zeta - z| < 4(1-|z|)$, we have

$$|1-\bar{z}\zeta| \leqq 1-|z|^2 + |z||\zeta - z| \leqq 6(1-|z|).$$

Hence

$$\sum_{j;\,|z_j-z_k|<4(1-|z_k|)} (1-|z_j|^2) \leqq (2+36\log c^{-2})(1-|z_k|),$$

which proves (3.2)′ and so (3.7).

We refer the reader to Carleson [2] or Shapiro and Shields [6] for the proof that (3.7) with $p=1$ implies the interpolation theorem that for any bounded sequence w_j there is a function $f\in H^\infty$ with $f(z_j)=w_j$, $j=1,2,\ldots$.

4. Estimates for plurisubharmonic functions.

Let Ω be a bounded open set in C^n with a C^2 boundary ω. We assume that ω is strictly pseudo-convex, that is, if $\varphi \in C^2(C^n)$ is a function which is <0 in Ω and >0 in $\complement\bar{\Omega}$ and has no critical point on ω

$$\sum_{j,k=1}^{n} \partial^2\varphi(z)/\partial z_j \partial \bar{z}_k \, t_j \bar{t}_k > 0 \quad \text{if} \quad z \in \omega, \ 0 \neq t \in C^n, \ \sum_1^n t_j \partial\varphi/\partial z_j = 0 .$$

Let PS^p, $1<p<\infty$, denote the space of non-negative plurisubharmonic functions u in Ω belonging to the class S^p defined in section 3. Boundary values $\in L^p(\omega)$ of functions in PS^p are then well defined.

For every boundary point x of Ω the tangent plane π_x of ω, which is of real codimension 1, contains a unique complex hyperplane $\pi_x{}^c$. For $t>0$ we denote by $A_{(x,t)}$ the set of points at distance $\leqq t$ from the ball in that plane with center at x and radius $t^{\frac{1}{2}}$, and we set $B_{(x,t)} = A_{(x,t)} \cap \omega$. If we compute the area of $B_{(x,t)}$ for small t using the projection of ω on π_x as local parametrization of ω near x, we find that the area of $B_{(x,t)}$ can be estimated from above and from below by a constant times $(t^{\frac{1}{2}})^{2(n-1)}t = t^n$, thus

$$C_1 t^n \leqq \omega(B_{(x,t)}) \leqq C_2 t^n \tag{4.1}$$

when t is less than some constant T which we can choose so large that $A_{(x,T)} \supset \omega$ for every $x \in \omega$.

The definitions made are essentially *invariant under an analytic change of variables*. For the first order part of the Taylor expansion at x of an analytic isomorphism will map $A_{(x,t)}$ linearly on a set which is contained in the set $A'_{(x',Kt)}$ defined at the image x' of x relative to the image Ω' of Ω, if K is sufficiently large and t is small. The remainder term in the Taylor expansion will not move the image of a point in $A_{(x,t)}$ more than by a constant times $(t^{\frac{1}{2}})^2$. Hence one can choose K' so that the image of $A_{(x,t)}$ is contained in $A'_{(x',K't)}$ for small t and arbitrary $x \in \omega$. Since the roles of Ω and Ω' can be interchanged, an opposite inclusion is valid for another K'. In view of this remark it is in fact easy to extend the results of this section to manifolds.

LEMMA 4.1. *There is a constant K such that if $s \leqq 2t$ and $x, y \in \omega$, the condition $B_{(x,s)} \cap B_{(y,t)} \neq \emptyset$ implies $B_{(x,s)} \subset B_{(y,Kt)}$.*

PROOF. We may assume that $t \leqq T$, for $B_{(y,Kt)} = \omega$ for every $K \geqq 1$ otherwise. The hypotheses imply that

$$|x-y| \leqq t^{\frac{1}{2}} + t + s^{\frac{1}{2}} + s \leqq 3(T^{\frac{1}{2}}+1)t^{\frac{1}{2}} .$$

Hence the unit complex normals of the planes $\pi_x{}^c$ and $\pi_y{}^c$ when suitably normalized differ by a vector which is $O(t^{\frac{1}{2}})$. The maximal distance from points in $A_{(x,s)}$ to π_y^c is therefore $O(t)+O(t^{\frac{1}{2}}s^{\frac{1}{2}})=O(t)$. This implies the statement of the lemma.

In view of Lemma 4.1 the hypothesis (ii) in section 2 is fulfilled by the sets $B_{(x,t)}$, and (4.1) shows that the condition (iii) is fulfilled. Let $\tilde{u}(x,t)$ be the coresponding maximal function on $\omega \times \boldsymbol{R}_+$.

Lemma 4.2. *Let ω be strictly pseudo-convex. For $x \in \Omega$ we denote by $d(x)$ the distance from x to ω and by x' a point in ω where the distance is attained. Then there is a constant C such that*

$$u(x) \leqq C\tilde{u}(x',d(x)), \qquad u \in PS^p(\Omega),\ x \in \Omega . \tag{4.2}$$

Proof. The statement is trivial when x is in a compact subset of Ω (cf. proof of Theorem 3.1) and it follows from (3.4) when $n=1$. It is therefore sufficient to prove the lemma when $d(x)$ is small and $n>1$. We do so assuming first that Ω is strictly convex at x', that $x'=0$ and that $\pi_{x'}$ is the plane $x_{2n}=\operatorname{Im} z_n=0$. In a neighborhood of 0 the set $\bar{\Omega}$ is then defined by

$$x_{2n} = \operatorname{Im} z_n \geqq \varphi(x_1,\ldots,x_{2n-1}) ,$$

where φ is equal to a positive definite quadratic form A in $x'=(x_1,\ldots,x_{2n-1})$ apart from an error which is $o(|x'|^2)$.

Set $V=\{\zeta \in \boldsymbol{C};\ |\operatorname{Re}\zeta|<\operatorname{Im}\zeta\}$. When $z_n \in V$ and $|z_n|$ is sufficiently small, the set where $\varphi(x')\leqq \operatorname{Im} z_n$ closely approximates an ellipsoid in $\boldsymbol{R}^{2n-2}$ with center at the origin and half axes proportional to $(\operatorname{Im} z_n)^{\frac{1}{2}}$. Indeed, if we put $x_j=(\operatorname{Im} z_n)^{\frac{1}{2}} y_j$ when $j=1,\ldots,2n-2$, the equation can be written

$$\varphi(y_1(\operatorname{Im} z_n)^{\frac{1}{2}},\ldots,y_{2n-2}(\operatorname{Im} z_n)^{\frac{1}{2}},\ \operatorname{Re} z_n))/\operatorname{Im} z_n \leqq 1 ,$$

and the left hand side converges to $A(y_1,\ldots,y_{2n-2},0)$ in the C^2 topology as a function of $y_1,\ldots,y_{2n-2}$ when $z_n \to 0$. For sufficiently small $z_n \in V$ we obtain as a trivial case of (3.4)

$$u(0,\ldots,0,z_n) \leqq C \int_{\varphi(x')=x_{2n}} u(x'',z_n)\, d\sigma(x'') , \tag{4.3}$$

where $x''=(x_1,\ldots,x_{2n-2})$, the projection of x'' on the unit sphere in $\boldsymbol{R}^{2n-2}$ is denoted by $\sigma(x'')$ and $d\sigma$ is the element of area on that sphere. With $\varrho=|x''|$ the element of area on ω is bounded from below by

$$dx''\, dx_{2n-1} = \varrho^{2n-3}\, d\varrho\, d\sigma(x'')\, dx_{2n-1} .$$

When $z_n \in V$ and $|z_n|$ is sufficiently small we have seen that the equation

$x_{2n} = \varphi(x') = \varphi(\varrho\sigma, x_{2n-1})$ permits us to use σ, x_{2n-1} and x_{2n} as parameters instead of σ, x_{2n-1} and ϱ. For fixed σ, x_{2n-1} we have

$$1 = \partial\varrho/\partial x_{2n}\, \varphi_\varrho \leqq C\, \partial\varrho/\partial x_{2n}\, \varrho\ ,$$

where the last inequality follows from the fact that $|x_{2n-1}| \leqq x_{2n} \leqq C\varrho^2$. For small t we obtain with positive constants C, C', C''

$$\begin{aligned}\int\limits_{x_{2n}\leqq t} u\,d\omega &\geqq C \int\limits_{x_{2n}\leqq t,\, z_n\in V} u\varrho^{2n-4}\, dx_{2n}\, d\sigma(x'')\, dx_{2n-1} \\ &\geqq C' \int\limits_{x_{2n}\leqq t,\, z_n\in V} u{x_{2n}}^{n-2}\, dx_{2n}\, d\sigma(x'')\, dx_{2n-1} \\ &\geqq C'' \int\limits_{x_{2n}\leqq t,\, z_n\in V} u(0,\ldots,0,z_n){x_{2n}}^{n-2}\, dx_{2n-1}\, dx_{2n}\ .\end{aligned}$$

The last inequality is a consequence of (4.3). Since $u(0,\ldots,0,z_n)$ is subharmonic we can estimate $u(0,\ldots,0,\frac{1}{2}it)$ by the mean value of $u(0,\ldots,0,z_n)$ over the disc $|z_n - \frac{1}{2}it| < \frac{1}{4}t$, which gives the estimate

$$u(0,\ldots,0,\tfrac{1}{2}it) \leqq Ct^{-n} \int\limits_{x_{2n}<t} u\, d\omega\ .$$

Since the constant only depends on lower bounds for the second order derivatives of φ near 0 and upper bounds for the modulus of continuity of the second order derivatives of φ, this proves that (4.2) is valid for all x in a neighborhood of a point on ω where ω is strictly convex. Now the assumption that ω is strongly pseudo-convex means that in a neighborhood of any point on ω one can make an analytic change of variables mapping ω to a strictly convex surface. Since our preceding arguments were purely local, this completes the proof of Lemma 4.2.

Combining Lemmas 4.1 and 4.2 with Theorem 2.4 we have now proved the first half of the following theorem.

THEOREM 4.3. *Let μ be a positive measure in a bounded open set $\Omega \subset \mathbf{C}^n$ with a C^2 strictly pseudo-convex boundary ω. Assume that there is a constant C such that*

$$\int\limits_{A(x,t)} d\mu \leqq Ct^n, \qquad x \in \omega,\ t > 0\ . \tag{4.4}$$

Then there is a constant C' such that

$$\int\limits_{\Omega} u^p\, d\mu \leqq C' p(p/(p-1))^p \int u^p\, d\omega, \qquad u \in PS^p(\Omega),\ 1 < p < \infty\ . \tag{4.5}$$

If f is analytic in Ω we have with $C''=8C'$

$$\int_{\Omega} |f|^p \, d\mu \leqq C'' \int |f|^p \, d\omega, \qquad p>0, \tag{4.6}$$

if f belongs to the Hardy class H^p. Conversely, if (4.6) *is valid for some p, it follows that* (4.4) *must be fulfilled.*

Since (4.6) follows by applying (4.5) to $u=|f|^{\frac{1}{2}p}$, with p replaced by 2 in (4.5), it only remains to prove the last statement. It is closely related to the results on the boundary behavior of Bergman's kernel function proved in section 3.5 of Hörmander [5], although we are concerned with the Szegö kernel rather than the Bergman kernel here.

For a fixed point $x \in \omega$ let $(\zeta_1,\ldots,\zeta_n)$ be an analytic coordinate system in a neighborhood U of x such that the coordinates of x are 0 and Ω is defined in U by an inequality $\operatorname{Im}\zeta_n > \varphi(\xi')$, where $\xi'=(\operatorname{Re}\zeta_1, \operatorname{Im}\zeta_1, \ldots, \operatorname{Re}\zeta_n)$ and φ satisfies the same hypotheses as in the proof of Lemma 4.2. Let $\chi \in C_0^\infty(U)$ be equal to 1 near x, and set with a positive integer k to be chosen later

$$f_t(z) = \chi(z)\big(\zeta_n(z)+it\big)^{-k} - g_t(z), \qquad t>0 .$$

The first term is defined as 0 outside U, and g_t shall be chosen so that f_t is analytic in Ω, that is,

$$\bar\partial g_t = \big(\zeta_n(z)+it\big)^{-k}\bar\partial\chi = h_t \quad \text{in} \quad \Omega, \tag{4.7}$$

where the last equality is a definition. Since $\operatorname{Im}\zeta_n$ has a positive lower bound in $\bar\Omega \cap \operatorname{supp}\bar\partial\chi$, we can find a strictly pseudo-convex open set $\tilde\Omega \supset \bar\Omega$ such that h_t is uniformly bounded in $\tilde\Omega$ when $t>0$. Then it follows that the equation (4.7) has a solution g_t with uniformly bounded norm in $L^2(\tilde\Omega)$ when $t \to 0$. (See e.g. [5], section 2.2.) Since $\bar\partial g_t=0$ in a fixed neighborhood of the origin, it follows that g_t is uniformly bounded in a neighborhood of the origin. Furthermore, the analytic function f_t is also uniformly bounded in the complement in Ω of any neighborhood of the origin. We now apply (4.6) to f_t. This gives for small t

$$\begin{aligned}(2t)^{-pk} \int_{|\zeta_n(z)|<t} d\mu &\leqq C\int |f_t|^p \, d\omega \\ &\leqq C_1 + C_2 \int_{U\cap\omega} \big(|\operatorname{Re}\zeta_n| + t + |\xi'|^2\big)^{-kp} \, d\omega \\ &\leqq C_1 + C_3 \int \big(|\xi_{2n-1}| + t + |\xi''|^2\big)^{-kp} \, d\xi'' d\xi_{2n-1} ,\end{aligned}$$

where $\xi'' = (\xi_1, \ldots, \xi_{2n-2})$. If $kp > 1$ we can integrate with respect to ξ_{2n-1} first and rewrite the right hand side in the form

$$C_1 + C_4 \int (t + |\xi''|^2)^{1-kp}\, d\xi'' = C_1 + C_4 t^{n-1+1-kp} \int (1 + |\xi''|^2)^{1-kp}\, d\xi'' .$$

If $2(kp-1) > 2(n-1)$, that is, if $kp > n$, the integral converges and we obtain

$$\int_{|\zeta_n(z)|<t} d\mu \leqq C_5 t^n .$$

Since $|\zeta_n(z)| < Ct$ for some C when $z \in A_{(x,t)}$, this completes the proof of (4.4) for a fixed x. However, it is clear that the estimates are uniform in x so we have in fact proved (4.4) completely.

We shall finally give an example which shows that the condition (4.4) is strictly weaker than the condition (3.2). To do so we choose for $j = 1, 2, \ldots$ a point $z_j \in \Omega$ so that the boundary distance $d(z_j) = 2^{-j}$ for large j. Let μ be the measure with mass 2^{-aj} at z_j and no mass elsewhere, a being a fixed positive number. Then (4.4) is fulfilled if and only if $a \geqq n$. Indeed, we have $d(z) \leqq Ct$ for all $z \in A_{(x,t)}$, which implies that

$$\int_{A_{(x,t)}} d\mu \leqq \sum_{2^{-j} \leqq Ct} 2^{-aj} \leqq C' t^a ;$$

on the other hand, if $t = 2^{-j}$ and x is the point on ω closest to z_j, we have

$$\int_{A_{(x,t)}} d\mu \geqq 2^{-aj} = t^a .$$

Similarly it follows that (3.2) is fulfilled if and only if $a \geqq 2n-1$. Except when $n = 1$, the case studied by Carleson, the two conditions are therefore not equivalent.

REFERENCES

1. N. Aronszajn and K. T. Smith, *Functional spaces and functional completion*, Ann. Inst. Fourier (Grenoble) 6 (1955–56), 125–185.
2. L. Carleson, *An interpolation problem for bounded analytic functions*, Amer. J. Math. 80 (1958), 921–930.
3. L. Carleson, *Interpolation by bounded analytic functions and the corona problem*, Ann. of Math. (2) 76 (1962), 547–559.
4. L. Gårding and L. Hörmander, *Strongly subharmonic functions*, Math. Scand. 15 (1964), 93–96.
5. L. Hörmander, *L^2 estimates and existence theorems for the $\bar\partial$ operator*, Acta Math. 113 (1965), 89–152.

6. H. S. Shapiro and A. Shields, *On some interpolation problems for analytic functions*, Amer. J. Math. 83 (1961), 513–532.
7. K. T. Smith, *A generalization of an inequality of Hardy and Littlewood*, Canad. J. Math. 8 (1956), 157–170.

INSTITUTE FOR ADVANCED STUDY, PRINCETON, N. J., U. S. A.