MATH. SCAND. 20 (1967), 79–88

ON THE RELATION $PQ - QP = -iI$

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We show that the characterization given by Dixmier [1] of the (self-adjoint) solutions of the equation $PQ - QP = -iI$, interpreted as by Weyl and v. Neumann, cannot be weakened in a certain manner which might have appeared plausible.

THEOREM 1. *In the complex Hilbert space* $\mathscr{H} = L^2(\boldsymbol{R})$ *there exist two self-adjoint operators* P, Q *and a dense subspace* Φ *of* $\mathscr{H}$ *such that*

$$\Phi \subset \operatorname{dom} P \cap \operatorname{dom} Q, \qquad P\Phi \subset \Phi, \qquad Q\Phi \subset \Phi ,$$

and that

a) $(Pf, Qg) - (Qf, Pg) = -i(f, g)$ *for every* $f, g \in \operatorname{dom} P \cap \operatorname{dom} Q$;
b) *the restrictions of* P *and* Q *to* Φ *are essentially self-adjoint*;
c) *the pair* (P, Q) *is not unitary equivalent to any direct sum of Schrödinger pairs* (p, q).

The example which we propose to construct in order to establish this result makes use of another example serving to prove the corresponding assertion concerning the question of commutativity of self-adjoint operators:

THEOREM 0. *In the complex Hilbert space* $\mathscr{H} = L^2(\boldsymbol{R})$ *there exist two self-adjoint operators* X, Y *and a dense subspace* Φ *of* $\mathscr{H}$ *such that*

$$\Phi \subset \operatorname{dom} X \cap \operatorname{dom} Y, \qquad X\Phi \subset \Phi, \qquad Y\Phi \subset \Phi ,$$

and that

a) $(Xf, Yg) - (Yf, Xg) = 0$ *for every* $f, g \in \operatorname{dom} X \cap \operatorname{dom} Y$;
b) *the restrictions of* X *and* Y *to* Φ *are essentially self-adjoint*;
c) X *and* Y *do not commute.*

This result was essentially established by Nelson [2, p. 606]. My own example is simpler than that of Nelson, and since the actual construction will be used in the proof of Theorem 1, I begin by presenting my example to Theorem 0.

Received October 19, 1966.

It should be noted that the property a) in Theorem 1 implies $XY-YX\subset -iI$, and hence $(XY-YX)\varphi=-i\varphi$ for every $\varphi\in\Phi$. The verification of this weaker form of the commutation relation in our example would have been simpler than the stated form a). Quite similar remarks apply to a) in Theorem 0; only the weaker form of a) is established in the quoted example of Nelson.

PROOF OF THEOREM 0. We denote by p_d and q_d the following operators on the space $\mathscr{D}'$ of all distributions on $\boldsymbol{R}$:

$$p_d f(x) = -i\,df(x)/dx, \qquad q_d f(x) = xf(x) .$$

By restriction of the graphs of these operators to $L^2\times L^2$ one obtains the self-adjoint Schrödinger operators p and q on $\mathscr{H}=L^2$. Next, put

$$(1) \qquad \omega = (2\pi)^{\frac{1}{2}}, \qquad X = e^{\omega q}, \qquad Y = e^{-\omega p} .$$

Then X and Y are self-adjoint (and $\geqq 0$). If we denote by X_d the operator on $\mathscr{D}'$ defined by

$$X_d f(x) = e^{\omega x} f(x) ,$$

then X is the restriction of X_d to L^2 in the sense explained above for p and q.

We define Φ as the subspace of $\mathscr{H}$ generated by the family of functions

$$x^n \exp(-rx^2+cx), \qquad n\in \boldsymbol{N},\ r\in \boldsymbol{R},\ r>0,\ c\in \boldsymbol{C} .$$

These bounded functions do belong to $\mathscr{H}$. Since Φ contains all the Hermite orthogonal functions, Φ is dense in $\mathscr{H}$.[1]

Defining the Fourier–Plancherel operator F on $\mathscr{H}$ by

$$(Ff)(x) = \omega^{-1} \lim_{a\to+\infty} \int_{-a}^{a} e^{-ixt} f(t)\,dt \quad \text{in } \mathscr{H} ,$$

we have

$$(2) \qquad F^{-1}qF = p, \qquad FqF^{-1} = -p, \qquad FXF^{-1} = Y .$$

It is easily checked that $\Phi\subset \operatorname{dom} q \cap \operatorname{dom} X$ and that Φ is invariant under q, X, F, F^{-1}, and also under multiplication of two of its elements. Consequently $\Phi\subset \operatorname{dom} p\cap \operatorname{dom} Y$, and Φ is invariant under p, Y, and under convolution of two of its elements.

Applying relation (2), $FXF^{-1}=Y$, it is easily verified that

$$(Y\varphi)(x) = \varphi(x+i\omega) \quad \text{for } \varphi\in\Phi,\ x\in\boldsymbol{R} ,$$

[1] For the purpose of the present example it would suffice to consider the functions $\exp(-rx^2+cx)$ with r and c as above. The factor x^n is, however, included in view of the proof of Theorem 1.

and hence that

$$(3)\qquad (XY-YX)\varphi = 0 \quad \text{for } \varphi \in \Phi ,$$
$$(4)\qquad Y(\overline{Y\varphi}) = \bar{\varphi} \quad \text{for } \varphi \in \Phi .$$

LEMMA 1. *If $\varphi \in \Phi$ and $f \in \operatorname{dom} X$, then $\varphi f \in \operatorname{dom} X$, $\varphi * f \in \operatorname{dom} X$, and*

$$X(\varphi f) = \varphi\, Xf = X\varphi\, f ,$$
$$X(\varphi * f) = (X\varphi)*(Xf) .$$

If $\varphi \in \Phi$ and $f \in \operatorname{dom} Y$, then $\varphi f \in \operatorname{dom} Y$, and

$$Y(\varphi f) = Y\varphi\; Yf .$$

PROOF. The case $f \in \operatorname{dom} X$ is straightforward, and the case $f \in \operatorname{dom} Y$ is derived by Fourier transformation.

LEMMA 2. *If $\varphi \in \Phi$ and $f,g \in \operatorname{dom} X \cap \operatorname{dom} Y$, then*

$$\big(X(\varphi f),\, Y(\varphi g)\big) = \big(Y(\varphi f),\, X(\varphi g)\big) .$$

PROOF. Since X and Y are self-adjoint, it follows from (3), (4), and Lemma 1 that

$$\begin{aligned}
\big(X(\varphi f),\, Y(\varphi g)\big) &= (X\varphi\, f,\, Y\varphi\; Yg) \\
&= \big(Y(\overline{Y\varphi}\, X\varphi\, f),\, g\big) \\
&= (Y\overline{Y\varphi}\; YX\varphi\; Yf,\, g) \\
&= (\bar{\varphi}\, XY\varphi\; Yf,\, g) \\
&= \big(X(Y\varphi\; Yf),\, \varphi g\big) \\
&= \big(XY(\varphi f),\, \varphi g\big) \\
&= \big(Y(\varphi f),\, X(\varphi g)\big) .
\end{aligned}$$

In order next to prove statement a) of Theorem 0, we shall apply Lemmas 1 and 2, taking for φ the function $\varphi_\varepsilon \in \Phi$ defined for $\varepsilon > 0$ by

$$\varphi_\varepsilon(x) = \exp(-\varepsilon x^2) ,$$

and letting $\varepsilon \to 0$. Clearly

$$\lim_{\varepsilon \to 0} \varphi_\varepsilon(x) = 1, \qquad \lim_{\varepsilon \to 0} (Y\varphi_\varepsilon)(x) = 1$$

pointwise for $x \in \boldsymbol{R}$. Moreover the functions φ_ε are uniformly bounded (for small ε), and so are the functions $Y\varphi_\varepsilon$:

$$|(Y\varphi_\varepsilon)(x)| = |\exp(-\varepsilon(x+i\omega)^2)| \leqq \exp(\varepsilon\omega^2) .$$

According to Lebesgue's convergence theorem we infer that

$$\varphi_\varepsilon f \to f, \quad (Y\varphi_\varepsilon) f \to f \qquad \text{in } \mathscr{H}$$

for every $f \in \mathscr{H}$ as $\varepsilon \to 0$. It follows, therefore, from Lemma 1 that

$$X(\varphi_\varepsilon f) = \varphi_\varepsilon(Xf) \to Xf, \qquad Y(\varphi_\varepsilon f) = Y\varphi_\varepsilon Yf \to Yf$$

in $\mathscr{H}$ as $\varepsilon \to 0$, provided Xf, resp. Yf, is defined. Applying Lemma 2 with $\varphi = \varphi_\varepsilon$, and letting $\varepsilon \to 0$, we obtain statement a).

In verifying statement b) it suffices to consider the case of X and to show that $(X|\Phi)^* \subset X$. Suppose accordingly that $(X|\Phi)^* g = h$. For every $\varphi \in \Phi$ and $f \in \Phi$ we have $\varphi f \in \Phi$, and hence $(X(\varphi f), g) = (\varphi f, h)$. Since $X\varphi \in \Phi$, $X\varphi$ is bounded, and hence

$$(X(\varphi f), g) = (X\varphi\, f, g) = (f, \overline{X\varphi}\, g) = (f, \bar{\varphi} h)\,.$$

Since Φ is dense in $\mathscr{H}$, it follows that

$$\overline{X\varphi}\, g = \bar{\varphi} h\,.$$

Recall that $\overline{X\varphi}\, g = \overline{X_d \varphi}\, g = \bar{\varphi}\, X_d g$ (where $X_d g(x) = e^{\omega x} g(x)$). Choosing $\varphi \in \Phi$ so that $\varphi(x) \neq 0$ for all $x \in \boldsymbol{R}$, we conclude that $X_d g = h$. Since $g, h \in L^2$, this means that $Xg = h$.

Finally, statement c) follows from the well-known fact that the Schrödinger operators $q = \omega^{-1} \log X$ and $p = -\omega^{-1} \log Y$ do not commute.

Note that the couple X, Y is irreducible (i.e., X and Y do not commute simultaneously with any self-adjoint projection operator $\neq O, I$). This follows from the corresponding property of the couple (q, p).

Proof of Theorem 1. We denote by $P_d = p_d + X_d$ the operator on $\mathscr{D}'$ defined by

$$P_d f(x) = -i\, df(x)/dx + e^{\omega x} f(x)$$

with $\omega = (2\pi)^{\frac{1}{2}}$. The operator P on $\mathscr{H} = L^2$ is now defined by restricting the graph of P_d to $L^2 \times L^2$. Next we put

$$Q = FPF^{-1}\,, \tag{5}$$

F being the Fourier–Plancherel operator. The subspace Φ is defined as in the proof of Theorem 0. In terms of the operators X and Y, defined in (1), we have the inclusions

$$P \supset p + X, \qquad Q \supset q + Y\,, \tag{6}$$

of which the latter follows from the former on account of (2). Note that $\Phi \subset \operatorname{dom} p \cap \operatorname{dom} X \subset \operatorname{dom} P$, and that Φ is invariant under p and X, hence under P. By Fourier transformation similar assertions are derived with Q in place of P.

It will follow from the proof of statement b) which we give below that

P is the closure of its restriction to Φ, and hence also of its restriction to $\operatorname{dom}(p+X)$, that is,

$$P = (p+X)^{**}.$$

We propose now to show that P (and hence Q) is self-adjoint and unitary equivalent to p. Denote by $U=u(q)$ the unitary operator of multiplication by the function

$$u(x) = \exp(i\omega^{-1} e^{\omega x}) \tag{7}$$

of modulus 1. Since $i\,du/dx + e^{\omega x}u(x) \equiv 0$, one easily obtains

$$P = U^{-1}pU. \tag{8}$$

Next we make pertinent additions to Lemmas 1 and 2 above.

LEMMA 1a. *If $\varphi \in \Phi$ and $f \in \operatorname{dom} P$, then φf and $\varphi * f$ are in* $\operatorname{dom} p \cap \operatorname{dom} X \subset \operatorname{dom} P$, *and*

$$P(\varphi f) = p(\varphi f) + X(\varphi f) = p\varphi\, f + \varphi P f,$$
$$P(\varphi * f) = (p\varphi - pX\varphi) * f + (X\varphi) * (Pf).$$

If $\varphi \in \Phi$ and $f \in \operatorname{dom} Q$, then $\varphi f \in \operatorname{dom} q \cap \operatorname{dom} Y \subset \operatorname{dom} Q$, and

$$Q(\varphi f) = q(\varphi f) + Y(\varphi f) = (q\varphi - qY\varphi) f + Y\varphi\, Qf.$$

PROOF. The assertions concerning $P(\varphi f)$ follow from the elementary identity (for $f \in \mathscr{D}'$)

$$P_d(\varphi f) = p_d(\varphi f) + X_d(\varphi f) = p\varphi\, f + \varphi P_d f$$

because $X_d(\varphi f) = X\varphi\, f \in L^2$ when $f \in L^2$. (Recall that $X\varphi$ is bounded.)

In order to establish the assertions concerning $P(\varphi * f)$ we note that, for $f \in \operatorname{dom} P$, the distribution $p_d f = Pf - X_d f$ is a (locally integrable) function, and so f is (locally) absolutely continuous, and $p_d f = -if'$. Hence we obtain for $\varphi \in \Phi$, $f \in \operatorname{dom} P$:

$$\begin{aligned}
[X_d(\varphi * f)](x) &= e^{\omega x} \int_{-\infty}^{\infty} \varphi(x-t) f(t)\, dt \\
&= \int_{-\infty}^{\infty} [e^{\omega(x-t)} \varphi(x-t)][e^{\omega t} f(t)]\, dt \\
&= \int_{-\infty}^{\infty} (X\varphi)(x-t)\, [(Pf)(t) + if'(t)]\, dt \\
&= [(X\varphi) * (Pf)](x) + i \int_{-\infty}^{\infty} (X\varphi)(x-t)\, f'(t)\, dt.
\end{aligned}$$

By partial integration we get

$$\int_a^b (X\varphi)(x-t)\,f'(t)\,dt = \int_a^b (X\varphi)'(x-t)\,f(t)\,dt + [(X\varphi)(x-t)\,f(t)]_a^b\,.$$

Since $(X\varphi)' \in \Phi \subset L^2$, and $f \in L^2$, we infer, letting $a \to -\infty$ and $b \to +\infty$ through values such that $f(a)$ and $f(b)$ remain bounded,

$$\int_{-\infty}^{\infty} (X\varphi)(x-t)\,f'(t)\,dt = \int_{-\infty}^{\infty} (X\varphi)'(x-t)\,f(t)\,dt\,,$$

because

$$(X\varphi)(x-t) \to 0 \quad \text{as } |t| \to \infty\,.$$

Inserting this in the above evaluation, we obtain

$$X_d(\varphi * f) = (X\varphi) * (Pf) - (pX\varphi) * f\,.$$

Both terms on the right belong to $\mathscr{H} = L^2$, and consequently $\varphi * f \in \operatorname{dom} X$. But $\varphi * f \in \operatorname{dom} p$ (for every $f \in \mathscr{H}$), and thus

$$\varphi * f \in \operatorname{dom} p \cap \operatorname{dom} X \subset \operatorname{dom} P$$

in view of (6). Moreover,

$$\begin{aligned} P(\varphi * f) &= p(\varphi * f) + X(\varphi * f) \\ &= (p\varphi - pX\varphi) * f + (X\varphi) * (Pf)\,. \end{aligned}$$

Finally the assertions concerning $Q(\varphi f)$ are derived from those regarding $P(\varphi * f)$ by Fourier transformation, cf. (5).

Lemma 2a. *If $\varphi \in \Phi$ and $f, g \in \operatorname{dom} P \cap \operatorname{dom} Q$, then*

$$(P(\varphi f), Q(\varphi g)) - (Q(\varphi f), P(\varphi g)) = -i\,(\varphi f, \varphi g)\,.$$

Proof. It is well known and elementary that

$$(pf, qg) - (qf, pg) = -i(f, g)$$

for $f, g \in \operatorname{dom} p \cap \operatorname{dom} q$. Clearly,

$$(Xf, qg) - (qf, Xg) = 0$$

for $f, g \in \operatorname{dom} X \cap \operatorname{dom} q$. By Fourier transformation this implies that

$$(Yf, pg) - (pf, Yg) = 0$$

for $f, g \in \operatorname{dom} Y \cap \operatorname{dom} p$. According to a) of Theorem 0

$$(Xf, Yg) - (Yf, Xg) = 0$$

for $f,g \in \mathrm{dom} X \cap \mathrm{dom} Y$. In view of (6) we obtain, by adding the above four relations,

$$(Pf,Qg) - (Qf,Pg) = -i(f,g)\,,$$

provided f and g belong to the domain of p, X, q and Y (and hence of P and Q). Replacing f,g by $\varphi f, \varphi g$, now with $f,g \in \mathrm{dom} P \cap \mathrm{dom} Q$, we arrive at the assertion of Lemma 2a on account of Lemma 1a.

In order to prove statement a) of Theorem 1, we shall apply Lemmas 1a and 2a, taking for φ the function $\varphi_\varepsilon \in \Phi$ defined for $\varepsilon > 0$ by $\varphi_\varepsilon(x) = \exp(-\varepsilon x^2)$, and making $\varepsilon \to 0$ (cf. the corresponding part of the proof of Theorem 0). Just like the families (φ_ε) and $(Y\varphi_\varepsilon)$, the families $(p\varphi_\varepsilon)$ and $(q\varphi_\varepsilon - qY\varphi_\varepsilon)$ are uniformly bounded for small ε, as we shall now see. Writing $\varepsilon x^2 = t > 0$ and $\varepsilon(2i\omega x - \omega^2) = s$, we obtain

$$|p\varphi_\varepsilon(x)| = 2\varepsilon |x| \exp(-\varepsilon x^2) = 2\varepsilon^{\frac12} t^{\frac12} e^{-t}\,,$$

$$(q\varphi_\varepsilon - qY\varphi_\varepsilon)(x) = (1-e^{-s})\, x \exp(-\varepsilon x^2)\,. \tag{9}$$

Let $\varepsilon\omega^2 \leqq \frac12$, and suppose first that $2\varepsilon\omega|x| \leqq \frac12$. Then $|s| \leqq 1$, hence

$$|1-e^{-s}| \leqq e|s| \leqq e\varepsilon(2\omega|x| + \omega^2)\,,$$

and

$$|(q\varphi_\varepsilon - qY\varphi_\varepsilon)(x)| \leqq e^{1-t}(2\omega t + \omega^2 \varepsilon^{\frac12} t^{\frac12})\,,$$

which is uniformly bounded. In the remaining case $2\varepsilon\omega|x| > \frac12$ we have

$$\varepsilon x^2 = \varepsilon |x|^2 > |x|/(4\omega), \qquad |e^{-s}| = \exp \varepsilon\omega^2\,,$$

and so by (9)

$$|(q\varphi_\varepsilon - qY\varphi_\varepsilon)(x)| \leqq \big(1 + \exp(\varepsilon\omega^2)\big)\, |x| \exp\big(-|x|/(4\omega)\big)\,,$$

which is likewise uniformly bounded in x for small ε.

Combining these results on the uniform boundedness of the functions $p\varphi_\varepsilon$ and $q\varphi_\varepsilon - qY\varphi_\varepsilon$ with the fact that these functions tend pointwise to 0 as $\varepsilon \to 0$, we conclude that

$$(p\varphi_\varepsilon)f \to 0, \quad (q\varphi_\varepsilon - qY\varphi_\varepsilon)f \to 0 \qquad \text{in } \mathscr{H}$$

for every $f \in \mathscr{H}$ as $\varepsilon \to 0$. It was shown similary in the proof of Theorem 0 that $\varphi_\varepsilon f \to f$ and $(Y\varphi_\varepsilon)f \to f$ in $\mathscr{H}$ as $\varepsilon \to 0$. Applying Lemma 1a, we now obtain

$$P(\varphi_\varepsilon f) \to Pf, \quad Q(\varphi_\varepsilon f) \to Qf \qquad \text{in } \mathscr{H}$$

as $\varepsilon \to 0$, provided Pf, resp. Qf, is defined. Consequently, the assertion a) of Theorem 1 follows from Lemma 2a with $\varphi = \varphi_\varepsilon$, $\varepsilon \to 0$.

In verifying statement b) it suffices to consider the case of P and to show that $(P\,|\,\Phi)^* \subset P$. Suppose accordingly that $(P\,|\,\Phi)^* g = h$. For every

$\varphi \in \Phi$ and $f \in \Phi$ we have $\varphi f \in \Phi$, and hence $(P(\varphi f),g) = (\varphi f,h)$. Since f and pf are bounded, we get

$$\begin{aligned}(P(\varphi f),g) &= (p(\varphi f) + X(\varphi f), g) \\ &= (p\varphi\, f + \varphi\, pf + \varphi\, Xf, g) \\ &= (p\varphi, \bar{f}g) + (\varphi, \overline{pf}\, g) + (\varphi, \overline{Xf}\, g)\,.\end{aligned}$$

Since this equals $(\varphi f,h) = (\varphi,\bar{f}h)$ for all $\varphi \in \Phi$, and since $(p\,|\,\Phi)^\star = p$,[2] it follows that $\bar{f}g \in \operatorname{dom} p$, and

$$p(\bar{f}g) = -\overline{pf}\, g - \overline{Xf}\, g + \bar{f}h\,. \tag{10}$$

Taking $f \in \Phi$ to be real, and noting that $\overline{pf} = -p\bar{f}$, we obtain, evaluating both members of (10) as distributions,

$$pf\, g + f\, p_d g = pf\, g - f\, X_d g + fh\,.$$

Choosing, for example, $f(x) = \exp(-x^2) \neq 0$, we conclude after division by f that

$$P_d g = p_d g + X_d g = h\,.$$

Since $g,h \in L^2$, this means that $Pg = h$.

It remains to establish the assertion c). If the pair (P,Q) were (simultaneously) unitary equivalent to a direct sum of m copies of the Schrödinger pair (p,q), then the cardinal m would have to be 1 because p has a simple spectrum, and so has P since it is unitary equivalent to p by (8). The only possibility is, therefore, that there exist a unitary operator W such that

$$P = W^{-1}pW, \qquad Q = W^{-1}qW\,.$$

Combining this with the previous relation (8) we find that the unitary operator $V = WU^{-1}$ must commute with p. Since p, or $-p$, has a simple spectrum, there is a function v with $|v| = 1$ such that

$$V = v(-p), \qquad W = VU = v(-p)u(q)\,.$$

In a similar manner it follows from the relations

$$Q = W^{-1}qW = W^{-1}FpF^{-1}W$$

and (see (5))

$$Q = FPF^{-1} = FU^{-1}pUF^{-1}$$

that the unitary operator

$$F^{-1}WFU^{-1} = v(q)u(p)u(q)^{-1}$$

[2] This follows by Fourier transformation from the relation $(q|\Phi)^\star = q$, which in turn is established just like the analogous relation for the operator X in the proof of Theorem 0.

commutes with p, and hence with $e^{i\lambda p}$ for every $\lambda \in \boldsymbol{R}$. Using twice the identity

$$e^{i\lambda p} f(q) = f(q+\lambda I) e^{i\lambda p} ,$$

which holds for any bounded measurable function f on $\boldsymbol{R}$, one therefore obtains

$$v(q+\lambda I) u(p) u(q+\lambda I)^{-1} = v(q) u(p) u(q)^{-1} .$$

Introducing the abbreviations

$$u_\lambda(x) = u(x+\lambda)/u(x), \qquad v_\lambda(x) = v(x+\lambda)/v(x) , \tag{11}$$

this leads to $v_\lambda(q) u(p) = u(p) u_\lambda(q)$, hence by Fourier transformation $F \ldots F^{-1}$ (cf. (2))

$$v_\lambda(-p) u(q) = u(q) u_\lambda(-p) .$$

Being identical with $v_\lambda(-p)$, the operator $u(q) u_\lambda(-p) u(q)^{-1}$ commutes with $e^{i\mu p}$ for every $\mu \in \boldsymbol{R}$. Proceeding as above, we infer that

$$u(q+\mu I) u_\lambda(-p) u(q+\mu I)^{-1} = u(q) u_\lambda(-p) u(q)^{-1} ,$$

$$u_\mu(q) u_\lambda(-p) = u_\lambda(-p) u_\mu(q) \tag{12}$$

for all $\lambda, \mu \in \boldsymbol{R}$. Inserting the expression (7) for u, we get from (11) and (1),

$$\begin{aligned} u_\lambda(x) &= \exp(i\varrho e^{\omega x}), \qquad \varrho = \omega^{-1}(e^{\omega\lambda}-1) , \\ u_\mu(x) &= \exp(i\sigma e^{\omega x}), \qquad \sigma = \omega^{-1}(e^{\omega\mu}-1) , \\ u_\mu(q) &= \exp(i\sigma e^{\omega q}) = \exp(i\sigma X) , \\ u_\lambda(-p) &= \exp(i\varrho e^{-\omega p}) = \exp(i\varrho Y) , \end{aligned}$$

and consequently from (12)

$$e^{i\sigma X} e^{i\varrho Y} = e^{i\varrho Y} e^{i\sigma X} \tag{13}$$

for all ϱ, σ in the range of the function $\omega^{-1}(e^{\omega\lambda}-1)$ of the real variable λ, in particular for all $\varrho, \sigma \in [0, +\infty[$. Since the inverse $e^{-i\sigma X}$ of $e^{i\sigma X}$ likewise commutes with $e^{i\varrho Y}$, and hence with $e^{-i\varrho Y}$, (13) actually holds for *all* real ϱ, σ. But this means that the self-adjoint operators $X = e^{\omega q}$ and $Y = e^{-\omega p}$ commute, and hence that q and p would have to commute, which is actually not the case. The hypothesis that the pair (P,Q) be unitary equivalent to a direct sum of Schrödinger pairs (p,q) is therefore false. This completes the proof of Theorem 1.

The author has not been able to decide whether the pair (P,Q) constructed in the present paper is irreducible.

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