MATH. SCAND. 20 (1967), 89–90

# ASYMPTOTES OF CONVEX BODIES

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For two subsets $A$ and $X$ of Euclidean $d$-space $E^d$, let

$$\delta(A,X) = \inf\{\|a-x\| : a \in A,\ x \in X\}.$$

The set $A$ is a *$j$-asymptote* of $X$ provided that $A$ is a $j$-dimensional flat with

$$A \cap X = \emptyset \quad \text{and} \quad \delta(A,X) = 0.$$

Thus $X$ admits a 0-asymptote if and only if $X$ fails to be closed, and $X$ admits a $j$-asymptote if and only if $X$'s orthogonal projection on some $(d-j)$-dimensional flat in $E^d$ fails to be closed. For each convex body $C$ in $E^d$ (closed convex set with nonempty interior) let $\alpha C$ denote the set of all integers $j$ such that $C$ admits a $j$-asymptote. If $\alpha C \neq \emptyset$ then (as noted in [2]) $\alpha C = \{j:\ 1 \leqq j \leqq d-1\}$ when $C$ has no boundary ray and $\alpha C = \{j:\ 1 \leqq j \leqq d-2\}$ when $C$ is a cone. Here we settle a problem raised in [2] by showing that every set of integers between 1 and $d-1$ can be realized as the set $\alpha C$ for suitably constructed convex bodies $C$ in $E^d$. The construction is adapted from [1].

THEOREM. *For each set $J \subset \{j:\ 1 \leqq j \leqq d-1\}$ there is a convex body $C$ in $E^d$ such that $C$ contains no line and $\alpha C = J$.*

PROOF. The assertion being obvious for $d \leqq 2$, we proceed by induction on $d$. Suppose the assertion known for $E^d$ and consider a set $J$ of integers between 1 and $d$. We want to produce a convex body $K$ in $E^{d+1}$ such that $K$ contains no line and $\alpha K = J$. Let $C$ be a convex body in $E^d$ such that $C$ contains no line and

$$\alpha C = \{h:\ 1 \leqq h \leqq d-1 \text{ and } h+1 \in J\}.$$

Choose an extreme point $p$ of $C$ (possible since $C$ contains no line) and let

$$X = C \text{ if } 1 \notin J, \qquad X = C \sim \{p\} \text{ if } 1 \in J.$$

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Received November 19, 1966.

This research was supported in part by the National Science Foundation, U.S.A. (NSF-GP-3579).

In each case $X$ is a convex $F_\sigma$ set and hence is the union of an increasing sequence $Y_1 \subset Y_2 \subset \ldots$ of compact convex sets such that $\|y\| \leqq i$ for all $y \in Y_i$. Let $E^d$ be embedded as usual in $E^{d+1}$, so that $E^{d+1} = E^d \oplus Rz$ for a line $Rz$ orthogonal to $E^d$. Finally, let

$$K = \operatorname{con} \bigcup_1^\infty (Y_i \oplus i^2 z) ,$$

so that $X$ is the orthogonal projection of $K$ on $E^d$. On p. 101 of [1] it is proved that $K$ is closed, whence of course $K$ is a convex body containing no line. Plainly

$$\alpha K \supset \{h+1 : h \in \alpha X\} ,$$

for $A \oplus Rz$ is an asymptote of $K$ in $E^{d+1}$ whenever $A$ is an asymptote of $X$ in $E^d$. From the choice of $C$ and from the care in defining $X$ when $1 \in J$ it follows that

$$\{h+1 : h \in \alpha X\} = J .$$

Thus to complete the proof it suffices to show that $\alpha K \subset J$, or equivalently that $j \in \alpha K$ implies $j-1 \in \alpha X$. Note first that no asymptote of $K$ is parallel to $E^d$, for $K$ is closed and lies in paraboloidal region

$$(*) \qquad Q = \{y \oplus rz : y \in E^d, \ r \geqq 0, \ \|y\| \leqq r^{\frac{1}{2}}\}$$

whose intersection with any translate of $E^d$ is compact.

Now consider an arbitrary $j$-asymptote $A$ of $K$. For each $r \in R$ let $A_r$ denote the $(j-1)$-flat $A \cap (E^d \oplus rz)$ and let $P_r$ denote the orthogonal projection of $A_r$ on $E^d$. Note that $A_r = A_0 + r(A_1 - A_0)$, whence $P_r = A_0 + r(P_1 - A_0)$ and

$$({*\atop *}) \qquad \delta(P_r, A_0) = r\delta(P_1, A_0) .$$

If $P_1 = A_0$ then $A = A_0 \oplus Rz$ and $A_0$ is plainly a $(j-1)$-asymptote of $X$. If $P_1 \neq A_0$ then $\delta(P_1, A_0) > 0$ and it follows from (*) and $({*\atop *})$ that

$$\begin{aligned} \delta(A_r, Q \cap (E^d \oplus [0, 4r]z)) &\geqq \delta(P_r, \{y \in E^d : \|y\| \leqq 2r^{\frac{1}{2}}\}) \\ &\geqq \delta(P_1, A_0)r - 2r^{\frac{1}{2}} - \delta(A_0, \{0\}) , \end{aligned}$$

whence $\lim_{r \to \infty} \delta(A_r, Q) = \infty$ and $A$ is not an asymptote of $K$. This completes the proof.

## REFERENCES

1. V. Klee, *Some characterizations of convex polyhedra*, Acta Math. 102 (1959), 79–107.
2. V. Klee, *Asymptotes and projections of convex sets*, Math. Scand. 8 (1960), 356–362.

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