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ON SUBGROUPS OF THE FIRST KIND

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In his fundamental work on discontinuous groups C. L. Siegel [1] calls a discrete subgroup H of a locally compact group G which satisfies the second axiom of countability a subgroup of the first kind if the following conditions are satisfied. There is a fundamental domain F relative to H of finite measure which has only a finite number of neighbours and which is normal. A translate Fh of F by an element h in H is called a neighbour of F if $\overline{F}h \cap \overline{F}$ is not empty so that the condition that F has but finitely many neighbours means that $\overline{F}^{-1}\overline{F} \cap H$ is finite. That F is normal means that the covering of G by the family $\{Fh\colon h \in H\}$ is locally finite, that is, every point of G has a neighbourhood which meets Fh for only a finite number of $h \in H$. It is furthermore clear that if F is normal, then there exists an open set U such that $\overline{F} \subset U \subset FH_0$ where H_0 consists of those h in H for which Fh is a neighbour of F. On the other hand, the existence of such a set U together with the finiteness of H_0 implies normality of F.

We are interested here in the fact, pointed out by Siegel (see [1, p. 683]), that if G/H is compact, then H is of the first kind. For the compactness of G/H implies the existence of a fundamental domain F having compact closure and the conditions that H be of the first kind are clearly met by such an F.

We observe that the compactness of $\overline{F}$ implies rather more in respect of normality namely that an open set U satisfying the condition $\overline{F} \subset U \subset FH_0$ is realizable in the form $\overline{F}V$ where V is an open neighbourhood of the identity, e, having compact closure. For the assumptions on G provide that there exists a sequence (V_i) of open neighbourhoods of e in which $\overline{V}_i$, $i=1,2,\ldots$, is compact and $\cap V_i = e$. Since $\overline{F}^{-1}\overline{F}\,\overline{V}_i$ is compact and H is discrete, it follows that $\overline{F}^{-1}\overline{F}\,\overline{V}_i \cap H$ is a finite set and, choosing i sufficiently large, $\overline{F}^{-1}\overline{F}\,\overline{V}_i \cap H = H_0$. In regard to normality, we have, in these circumstances that for any x in G there is a neighbourhood V of e which is independent of x such that Vx is covered by a

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finite number of translates Fh, $h \in H$, this number being at most the number of elements in H_0.

The above observation leads us to make the following definition: A discrete subgroup H is *uniformly of the first kind* when there exists a fundamental domain F relative to H of finite measure for which $H_0 = \overline{F}^{-1}\overline{F} \cap H$ is finite and there exists an open neighbourhood V of e such that $\overline{F} \subset \overline{F}V \subset FH_0$. Stated in these terms what we have observed above is that if G/H is compact, then H is uniformly of the first kind. What we are now going to show is that the converse assertion holds. It is a special case of the following

THEOREM. *Let G be a locally compact σ-compact group with left invariant measure m. Let A be a subset of G of finite measure which is such that there exists a covering $\{A\lambda\colon \lambda \in \Lambda\}$ of G in which, for some compact neighbourhood V of e, $AV \cap A\lambda \neq \emptyset$ for only a finite subset of Λ. Then $\bar{A}$ is compact.*

PROOF. Let $\Lambda_0 = \Lambda \cap A^{-1}AV$ where V is as given. Denoting $A\Lambda_0$ by $\tilde{A}$, we have that $m(\tilde{A}) \leqq |\Lambda_0| m(A)$, where $|\Lambda_0|$ is the number of elements in Λ_0. It follows that $m(\tilde{A})$ is finite.

The hypotheses on G provide that G is the countable union of compact sets, say $G = \bigcup C_i$, in which we may assume that $C_i \subset C_{i+1}$ for $i = 1, 2, \ldots$.

Suppose that $\bar{A}$ is not compact. Then for each i, $\bar{A} - C_i V^{-1}$ is not empty since $C_i V^{-1}$ is compact. Let x be any point in $\bar{A} - C_i V^{-1}$, then $xV \cap C_i = \emptyset$. But $xV \subset \tilde{A}$ so that $\tilde{A}$ has a compact subset which by the left invariance of m has measure $m(V)$ outside C_i for each i. Since $m(V) > 0$, this contradicts the fact that $m(A)$ is finite. It follows that $\bar{A}$ is compact.

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