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# EXTREMELY AMENABLE SEMIGROUPS II

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## Introduction.

We continue in this paper the study of extremely amenable semigroups begun in [10]. We shall freely make use of the notations of [10].

Let $S$ be a semigroup. Throughout we use multiplicative notation for products of elements of $S$. If $f \in m(S)$ and $a \in S$, let

$$f_a(s) = f(as), \qquad f^a(s) = f(sa)$$

for any $s \in S$. Define $r_a, l_a \colon m(S) \to m(S)$ by $l_a f = f_a$, $r_a f = f^a$.

If $f \in m(S)$, let $K(f)$ denote the set of reals $c$ for which there is some net in $\{r_a f;\, a \in S\}$ which converges pointwise to the constant function $c \cdot 1_S$. ($1_S$ is the constant one function on $S$, sometimes denoted by 1.) We call $S$ *extremely right stationary* if $K(f) \neq \emptyset$ for each $f \in m(S)$. This definition is analogous to the definition of right stationary semigroups given in Mitchell [17].

Let $A \subset m(S)$ be a uniformly closed left invariant (that is, $f_x \in A$ for any $f \in A$ and $x \in S$) subalgebra of $m(S)$ with $1 \in A$. A linear functional $\varphi$ on $A$ is a mean if $\varphi(f) \geqq 0$ for any $f \geqq 0$, $f \in A$ and $\varphi(1) = 1$. We say that the subalgebra $A$ is extremely left amenable (ELA) if there is a multiplicative left invariant mean on $A$. $S$ will be an ELA semigroup in case $m(S)$ is ELA. For any uniformly closed left invariant subalgebra $A$ of $m(S)$ with $1 \in A$, we denote by $H_A$ the ideal of all $h \in A$ which have a representation

$$h = \sum_1^n f_j(g_j - l_{a_j} g_j)$$

for some $f_j, g_j \in A$ and $a_j \in S$, $n = 1, 2, \ldots$. By $K_A$ we denote the linear subspace of all $h \in A$ with

$$h = \sum_1^n (g_j - l_{a_j} g_j)$$

for some $g_j \in A$ and $a_j \in S$, $n = 1, 2, \ldots$. In case $A = m(S)$ we write $H = H_A$ and $K = K_A$.

We have the following characterisation of ELA semigroups:

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THEOREM A. *Let $S$ be a semigroup. The following conditions* 1.–6. *on $S$ are equivalent*:

1. *$S$ is* ELA.
2. *$S$ is extremely right stationary and in this case, for any $f \in m(S)$,*

$$K(f) = \{\varphi(f);\ \varphi \text{ a multiplicative left invariant mean}\}$$

(*in analogy to Mitchell's result in* [17]).[1]

3. *For each $h \in H$ there is some $s \in S$ with $h(s) = 0$ (there still may be some $h$ in the uniform closure of $H$ with $h(s) \neq 0$ for each $s \in S$ even though $S$ is* ELA).
4. *$S$ is left amenable and for* each *left invariant mean $\mu$ on $m(S)$,*

$$\mu(fg_x) = \mu(fg)$$

*for any $f, g \in m(S)$ and each $x \in S$.*

5. *$S$ is left amenable and* each *extreme point of the set of left invariant means on $m(S)$ is multiplicative.*
6. *$S$ is left amenable and $K$ is uniformly dense in $H$.*

For subalgebras $A \subset m(S)$ one has

THEOREM B. *Let $A$ be a uniformly closed left invariant subalgebra of $m(S)$ with $1 \in A$. The following conditions* 1.–5. *on $A$ are equivalent*:

1. *$A$ is* ELA.
2. $\inf\{\|1-h\|;\ h \in H_A\} = 1$.
3. $\sup\{h(s);\ s \in S\} \geqq 0$ *for each $h \in H_A$.*
4. *$H_A$ is not uniformly dense in $A$.*
5. *There is a mean $\varphi$ on $A$ such that $\varphi(gg_x) = \varphi(g^2)$ for each $g \in A$ and $x \in S$* or such that $\varphi(gg_x) = [\varphi(g)]^2$ for each $g \in A$, $x \in S$.

Conditions 1.–4. of theorem B have known analogues for left amenable semigroups (compare for example with theorems 17.4 and 17.15 in Hewitt–Ross [12][2]).

Conditions 1, 3, 4, 5, 6 of theorem A do not have analogues for left amenable semigroups.

Applying theorem B and theorems 17.4 and 17.15 in Hewitt–Ross [12] to the space of (uniformly) continuous bounded functions on the abelian topological group $G$, $(U = UC(G))$ $C(G) = C$, one has the following

THEOREM. *Let $G$ have a nontrivial continuous homomorphic image $G'$ which is a subgroup of a locally compact abelian topological group. Then*:

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[1] Mitchell's proof does, however, not carry over to the extreme right stationary case.

[2] Theorem 17.4 in [12] is due to Dixmier [5] while theorem 17.15 in [12] in slightly different form is due to M. Day [24, pp. 281–282 and p. 286]. Theorems related to theorem 17.15 in [12] appear also in E. Følner [7] and R. Raimi [20].

(1) *$UC(G)$ and $C(G)$ are not* ELA *even though they admit an invariant mean.*

(2) *$H_U\,[H_C]$ is uniformly dense in $UC(G)$ $[C(G)]$ while $K_U\,[K_C]$ is not uniformly dense in $UC(G)$ $[C(G)]$.*

(3) $\sup\{h(x);\, x \in G\} \geqq 0$ *for each $h \in K_C$ while* $\sup\{h(x);\, x \in G\} < 0$ *for some $h \in H_U$.*

(4) $\inf\{\|1-h\|;\, h \in H_U\} = 0$ *while* $\inf\{\|1-h\|;\, h \in K_C\} = 1$.

Some refinements of this theorem are also obtained.

In the last part of this work we give a fairly general construction of a big class of ELA semigroups. This construction is inspired by the main example in [10]. It comes out from this construction that *any left cancellation semigroup* is a subsemigroup of some left cancellation ELA semigroup. The fact that the classe of ELA semigroups is so rich seems to us surprising in view of the fact that the only *right cancellation* ELA *semigroup* is the trivial $S=\{e\}$ with $e^2=e$. We also remind the reader of this section about some open questions on amenable semigroups. In what follows we specialise the above construction to build a certain left cancellation ELA semigroup. We study it by determining all its ELA subsemigroups and all its left amenable subsemigroups. We show that many of its subsemigroups are not left amenable.

## 1. Extremely right stationary semigroups.

SOME NOTATIONS. By $m_c(S)$ we denote the space of bounded complex valued functions on $S$ with

$$\|f\| = \sup\{|f(s)|;\, s \in S\}\,.$$

The $w^*$-topology on $m_c(S)$ is the weakest topology which makes all linear functionals of $l_1(S)$ continuous. This topology is not dependent on whether $l_1(S)$ are the complex or real functions on $S$ (with countable support and with $\{\sum |g(s)|;\, s \in S\} < \infty$. If $B \subset m_c(S)$ then $w^* - \mathrm{cl}\, B$ is the closure of $B$ in the $w^*$-topology. For $f \in m_c(S)$,

$$O(f) = w^* - \mathrm{cl}\{r_a f;\, a \in S\}$$

and $K(f)$ is the set of reals $c$ such that $c1$, the constant $c$-function on $S$, is in $O(f)$. The set $K(f)$ may be empty.

LEMMA 1. (a) *Let $B \subset m_c(S)$ be a norm bounded set. Then the $w^*$-topology and the pointwise convergence topology coincide on $B$. (See Mitchell* [17, p. 249, lemma 3]).

(b) *If $g \in O(f)$ where $f \in m_c(S)$, then $\|f\| \geqq \|g\|$. Hence $O(f)$ is $w^*$-compact.*

(c) *If $f \in m_c(S)$ and $g \in O(f)$, then $O(g) \subset O(f)$.*

PROOF. (a) Let $B_r=\{f\in m_c(S); \|f\|\leqq r\}$. Then $B_r$ is $w^*$-compact (see [15, p. 22]). But the pointwise convergence topology on $B_r$ is Hausdorff and is weaker than the $w^*$-topology. Hence these two topologies coincide on $B_r$. Now $B\subset B_r$ for some $r>0$, which proves (a). This proof is simpler than that of [17].

(b) If $a,x\in S$, then $|r_af(x)|=|f(xa)|\leqq\|f\|$. If now $r_{a_\alpha}f\to g$ in $w^*$, then $f(xa_\alpha)\to g(x)$ for each $x$ in $S$ by (a). Hence $\|g\|\leqq\|f\|$.

(c) $r_a$: $m_c(S)\to m_c(S)$ is $w^*$ continuous (see Mitchell [17, pp. 246–247] proof of lemma 1(c)). Hence

$$r_a[Of]\subset w^*-\mathrm{cl}\{r_{ax}f;\, x\in S\}\subset O(f)\,.$$

REMARK. (a) is equivalent to: If $f_\alpha$ is a norm bounded net in $m_c(S)$ and $f\in m_c(S)$, then $f_\alpha\to f$ in $w^*$ if and only if $f_\alpha(x)\to f(x)$ for each $x$ in $S$. Furthermore, if $f_\alpha$ is a norm bounded net, there is a subnet which converges pointwise (that is $w^*$) to some $f$ in $m_c(S)$.

LEMMA 2. *Let $f_0\in m(S)$, $\alpha_0\in K(f)$, and $F\in m_c(S)$ be any function which can be represented as*

$$F=\sum_1^n f_j(g_j-l_{a_j}g_j)+g(f_0-\alpha_0 1)\,,$$

*where $f_j$, $g_j$ and $g$ belong to $m_c(S)$ while $a_j$ are elements of $S$. Then, if $S$ is extremely right stationary,*

$$1\in O[1-F]\,.$$

PROOF. By decomposing the $g_j$'s into real and imaginary parts we can assume that $F$ is represented as

$$F=\sum_1^n f_j(g_j-l_{a_j}g_j)+g(f_0-\alpha_0 1)\,,$$

where $g_j\in m(S)$ are real valued and $f_j\in m_c(S)$. We show at first that some function

$$1-\sum_1^n f_j^{(1)}(g_j^{(1)}-l_{a_j}g_j^{(1)})\in O[1-F],\qquad \text{where}\quad f_j^{(1)}\in m_c(S),\ g_j^{(1)}\in m(S)\,.$$

There is a net $\{a_\alpha\}\subset S$ such that $(r_{a_\alpha}f_0)(x)\to\alpha_0$ for each $x\in S$. By possibly taking a subnet we can assume that $r_{a_\alpha}f_1\to f_1^{(1)}$ in $w^*$ (that is pointwise) and (by a further subnet), that $r_{a_\alpha}g_1\to g_1^{(1)}$ (pointwise) and so on, for each $f_j, g_j$, $1\leqq j\leqq n$. Hence we can assume that $r_{a_\alpha}(f_0-\alpha_0 1)\to 0$, (pointwise), (see lemma 1(b)) and $r_{a_\alpha}f_j\to f_j^{(1)}$ (pointwise), $r_{a_\alpha}g_j\to g_j^{(1)}$ (pointwise) for some $f_j^{(1)}$ in $m_c(S)$ and $g_j^{(1)}$ in $m(S)$, for $1\leqq j\leqq n$. Therefore

$$\lim_\alpha (r_{a_\alpha}(1-F))(x) = 1 - \sum_1^n f_j^{(1)}(g_j^{(1)} - l_{a_j} g_j^{(1)})(x)$$

for each $x$ in $S$. Since on $O[1-F]$ the pointwise topology coincides with the $w^*$-topology, we have that

$$1 - \sum_1^n f_j^{(1)}(g_j^{(1)} - l_{a_j} g_j^{(1)}) \in O[1-F] .$$

We show now that

$$1 - \sum_1^{n-1} f_j^{(2)}(g_j^{(2)} - l_{a_j} g_j^{(2)}) \in O[1-F]$$

for some $f_j^{(2)} \in m_c(S)$ and $g_j^{(2)} \in m(S)$, $1 \leqq j \leqq n-1$. In fact there is a net $r_{b_\beta}$ such that $\lim_\beta r_{b_\beta} g_n^{(1)}(x) = c$ for each $x$ in $S$. This is true since $g_n$ is real valued and $S$ is extremely right stationary. Hence

$$\lim_\beta r_{b_\beta}(g_n^{(1)} - l_{a_n} g_n^{(1)})(x) = 0$$

for each $x$ in $S$. By possibly taking a subnet we can assume that

$$\lim_\beta r_{b_\beta} g_j^{(1)}(x) = g_j^{(2)}(x) \quad \text{and} \quad \lim_\beta r_{b_\beta} f_j^{(1)}(x) = f_j^{(2)}(x)$$

for each $x$ in $S$ and each $1 \leqq j \leqq n-1$. Hence

$$\lim_\beta r_{b_\beta} \left[1 - \sum_1^n f_j^{(1)}(g_j^1 - l_{a_j} g_j^1)\right](x) = \left[1 - \sum_1^{n-1} f_j^{(2)}(g_j^{(2)} - l_{a_j} g_j^{(2)})\right](x) .$$

Therefore

$$1 - \sum_1^{n-1} f_j^{(2)}(g_j^{(2)} - l_{a_j} g_j^{(2)} \in O\left[1 - \sum_1^n f_j^{(1)}(g_j^{(1)} - l_{a_j} g_j^{(1)})\right] \subset O[1-F]$$

by (c) of lemma 1. We can hence assume that a function of type $1 - f(g - l_{a_1} g)$, where $g \in m(S)$ and $f \in m_c(S)$, belongs to $O[1-F]$. Using the same argument as above there is a net $r_{c_\nu}$ such that $\lim (r_{c_\nu} g)(x) = d$ for each $x$ in $S$ for some real $d$. Hence

$$\lim r_{b_\beta}[1 - f(g - l_{a_1} g)](x) = 1$$

for each $x$ in $S$ which shows that $1 \in O[1-F]$ by (c) of lemma 1.

Theorem 1. *Let $S$ be a semigroup. Then $S$ is extremely right stationary if and only if $S$ is extremely left amenable. In this case, if $f_0 \in m(S)$ then $\varphi(f_0) = \alpha$ for some multiplicative left invariant mean $\varphi$ if and only if $\alpha \in K(f_0)$.*

PROOF. Let $S$ be extremely right stationary and $\alpha_0 \in K(f_0)$. If

$$(*) \qquad F = \sum_1^n f_j(g_j - l_{a_j} g_j) + g(f_0 - \alpha_0),$$

where $g_j \in m(S)$, $g, f_j \in m_c(S)$ and $a_j \in S$, then $1 \in O[1 - F]$ by the previous lemma. Hence $\|1 - F\| \geqq \|1\| = 1$. This shows that the set of all functions which have a representation (*) (which is an ideal of $m_c(S)$ denoted by $J$) is not norm dense in $m_c(S)$. Hence there is a maximal ideal $M$ containing $J$ ([15, p. 58, 20A]). $M$ is necessarily closed, the quotient Banach algebra $m_c(S)/M$ is the complex field and the natural homomorphism $m_c(S) \to m_c(S)/M$ is a nontrivial multiplicative linear functional $\varphi$ such that $\varphi(M) = 0$ ([15, p. 60, 20D]). Hence $\varphi(J) = 0$. If now $B \subset S$ then $1_B{}^2 = 1_B$. Hence $\varphi(1_B) = [\varphi(1_B)]^2$. Thus $\varphi(1_B) = 0$ or 1. Hence if $f \geqq 0$ is a step function then $\varphi(f) \geqq 0$. If now $f \geqq 0$ and $f \in m(S)$ then $f$ can be approximated uniformly by non-negative step functions. Thus $\varphi(f) \geqq 0$. Furthermore $\varphi(1) = 0$ or 1. If $\varphi(1) = 0$ then $\varphi(f) = 0$ for each $f \in m(S)$ with $f \geqq 0$ and by decomposing we would have $\varphi = 0$, which cannot be. Thus $\varphi(1) = 1$ and $\varphi(f)$ is real for $f \in m(S)$. Thus $\varphi$ is a mean on $m(S)$ such that $\varphi(J) = 0$. Taking in (*) $f_j = 0$ and $g = 1$ one has $\varphi(f_0) = \alpha_0$. Since $f - f_a \in J$, $\varphi(f) = \varphi(f_a)$, hence $\varphi$ is a multiplicative left invariant mean on $m(S)$ such that $\varphi(f_0) = \alpha_0$ (and $S$ is ELA).

Conversely, assume that $\varphi$ is a multiplicative left invariant mean on $m(S)$. Then we can regard $\varphi$ as belonging to the Stone–Čech compactification of $S$. Hence there is a net of point measures $1_{s_\alpha} \in l_1(S)$ such that

$$1_{s_\alpha}(f) = f(s_\alpha) \to \varphi(f)$$

for each $f$ in $m(S)$. Hence

$$1_{s_\alpha}(f_x) = f(xs_\alpha) = (r_{s_\alpha} f)(x) \to \varphi(f_x) = \varphi(f)$$

for each $x \in S$. Therefore $S$ is extremely right stationary and moreover $\varphi(f) \in K(f)$ for each $f \in m(S)$, which finishes this proof.

REMARK. Using the same argument as in lemma 2 and theorem 1 above one could give a much simpler and entirely different proof to theorem 1 of Mitchell [17, p. 250]. The explicit construction of the sublinear functional $p_R$ of [17] prior to which lemmas 2, 3, 4, 5 had to be proved ([17, pp. 247–250]) would thus be saved. Using this idea we would, however, lose the information about the sublinear functional $p_R$ which is interesting for its own sake.

**2. Extremely amenable subalgebras of $m(S)$ and applications.**

LEMMA 3. *Let $X$ be a set, $\{t_\alpha;\ \alpha \in I\}$, a semigroup of mappings from $X$ to $X$, and $T_\alpha$: $m(X) \to m(X)$ be the linear operators defined by*

$$(T_\alpha f)(x) = f(t_\alpha x) \quad \text{for} \quad x \in X .$$

*Let $A$ be a uniformly closed subalgebra of $m(X)$ with $1 \in A$, and such that $T_\alpha A \subset A$ for each $\alpha \in I$. Let $H_A$ be the ideal of $A$ of all $h$ having a representation*

$$h = \sum_1^n f_j(g_j - T_{\alpha_j} g_j)$$

*for some $f_j, g_j \in A$, $\alpha_j \in I$, $n = 1, 2, \ldots$.*

*Then the following statements are equivalent:*

(a) *There is a multiplicative mean $\varphi$ on $A$ such that $\varphi(T_\alpha f) = \varphi(f)$ for each $f \in A$, $\alpha \in I$.*

(b) $\sup\{h(x);\ x \in X\} \geqq 0$ *for each $h$ in $H_A$.*

(c) $\inf\{\|1 - h\|;\ h \in H_A\} = 1$.

(d) *$H_A$ is not uniformly dense in $A$.*

(e) *There is a mean $\varphi$ on $A$ such that $\varphi[g(T_\alpha g)] = \varphi(g^2)$ for each $g \in A$, $\alpha \in I$,* or $\varphi[g(T_\alpha g)] = [\varphi(g)]^2$ for all $g \in A$, $\alpha \in I$.

PROOF. Let $\varphi$ be a multiplicative mean on $A$. If $h \in H_A$ and $d = \sup\{h(x);\ x \in X\}$ then $d1 - h \geqq 0$ and $d1 - h \in A$. Thus $0 \leqq \varphi(d1 - h) = d$. Hence (a) implies (b).

(b) $\Rightarrow$ (c): Let $h \in H_A$. Then $-h \in H_A$ and $\sup\{-h(x);\ x \in X\} \geqq 0$. Hence $\inf\{h(x);\ x \in X\} \leqq 0$. If $\varepsilon > 0$ there is some $x_0 \in X$ such that $h(x_0) < \varepsilon$. Hence $1 - h(x_0) > 1 - \varepsilon$ which implies that $\|1 - h\| \geqq 1 - \varepsilon$. Therefore $\|1 - h\| \geqq 1$ for each $h \in H_A$ and since $0 \in H_A$, (c) holds true. That (c) implies (d) is clear.

We show now that (d) implies (a): $A + iA$ is a subalgebra of $m_c(X)$ which is norm closed. One has in fact

$$\max\{\|\mathrm{Re} f\|, \|\mathrm{Im} f\|\} \leqq \|f\| \leqq \|\mathrm{Re} f\| + \|\mathrm{Im} f\|$$

for any $f \in m_c(X)$. Also $H_A + iH_A$ is an ideal of $A + iA$ which is not norm dense. For, if $f_0 \in A$ and $\|f_0 - (h + ih')\| < \varepsilon$ with $h$, $h'$ in $H_A$, then $\|f_0 - h\| < \varepsilon$, which would imply that $H_A$ is dense in $A$. Hence there is a maximal ideal $M$ of $A + iA$, which is necessarily closed, such that $H_A + iH_A \subset M$ (see Loomis [15, p. 58, 20A]). Therefore $(A + iA)/M$ is the complex field and the natural homomorphism $A + iA \to (A + iA)/M$ is a nontrivial multiplicative linear functional $\varphi$ on $A + iA$ such that $\varphi(M) = 0$ ([15, p. 60, 20D and p. 68, 22F]). This shows that $\varphi(H_A) = 0$ and in particular that $\varphi(f - T_\alpha f) = 0$ for each $f$ in $A$ and $\alpha$ in $I$. We show now that $\varphi$

restricted to $A$ is real. If $g \in A$ is such that $\varphi(g) = \alpha + i\beta$, $\beta \neq 0$, then $f = (g - \alpha 1)\beta^{-1}$ is in $A$ and $\varphi(f) = i$. Since $\varphi$ is a nontrivial homomorphism and $1 \in A$, $\varphi(1) = 1$. The series

$$\sum_0^\infty \frac{1}{n!}(-if)^n(x)$$

converges uniformly in $x \in X$ to $e^{if}$. Hence $e^{-if} \in A + iA$ and $\varphi(e^{-if}) = e^{-i\varphi(f)} = e > 1$. But $\|e^{-if}\| \leqq 1$, which contradicts the fact that $\|\varphi\| \leqq 1$. (This is a well known standard argument.) This shows that $\varphi$ restricted to $A$ is real. If now $f \in A$ and $0 \leqq f \leqq 1$ and $\varphi(f) < 0$, then $\varphi(1-f) > 1$. This cannot be since $\|\varphi\| \leqq 1$. Hence $\varphi(f) \geqq 0$, and $\varphi$ is a multiplicative mean on $A$ such that $\varphi(T_\alpha f) = \varphi(f)$ for each $\alpha \in I$. We might add here that $A \subset m(X)$ has to be a lattice as is well known (for a simple proof see R. G. Douglas [6]). If $\varphi$ is the mean in (a) then it clearly satisfies (e). Conversely, introduce $T_\alpha f = f_\alpha$ for $f \in A$, $\alpha \in I$ and let at first $\varphi$ be a mean such that

$$\varphi(gg_\alpha) = \varphi(g^2) \quad \text{if} \quad g \in A,\ \alpha \in I\,.$$

Then

$$\varphi[(f+g)(f_\alpha + g_\alpha)] = \varphi[(f+g)^2]\,,$$

hence

$$\varphi(fg_\alpha) + (gf_\alpha) = 2\varphi(fg)$$

for $f, g \in A$, $\alpha \in I$. Taking $f = 1$ one gets $\varphi(g_\alpha) = \varphi(g)$ for all $g \in A$, $\alpha \in I$. Now

$$\varphi[(g - g_\alpha)^2] = \varphi(g^2) + \varphi[(g_\alpha)^2] - 2\varphi(gg_\alpha) = 0$$

since $(g_\alpha)^2 = (g^2)_\alpha$. Thus

$$|\varphi(f(g - g_\alpha))|^2 \leqq \varphi(f^2)\varphi[(g - g_\alpha)^2] = 0$$

$\big(|\varphi(f_1 f_2)|^2 \leqq \varphi(f_1^2)\varphi(f_2^2)$ as in the Cauchy-Schwarz inequality$\big)$. Hence $\varphi(H_A) = 0$, $\varphi(1) = 1$. Thus $H_A$ is not dense in $A$ and (d) holds.

In the second case let $\varphi$ be a mean such that $\varphi(gg_\alpha) = [\varphi(g)]^2$. Then

$$\varphi[(f+g)(f_\alpha + g_\alpha)] = [\varphi(f+g)]^2\,.$$

Hence

$$(*) \qquad \varphi(fg_\alpha) + \varphi(f_\alpha g) = 2\varphi(f)\varphi(g)\,.$$

Taking $f = 1$ one gets $\varphi(g_\alpha) = \varphi(g)$ for $g \in A$, $\alpha \in I$. Taking $f = g_\alpha$ in (*) one has

$$\varphi(g_\alpha^2) + \varphi((g_\alpha)_\alpha g) = 2[\varphi(g)]^2\,.$$

But $(g_\alpha)_\alpha = g_\beta$ for some $\beta \in I$. Thus

$$\varphi(g^2) = \varphi(g_\alpha^2) = [\varphi(g)]^2,$$

and replacing $g$ by $f+g$ and expanding one gets that $\varphi(fg) = \varphi(f)\varphi(g)$.

Thus $\varphi$ is even a multiplicative mean on $A$ which satisfies $\varphi(T_\alpha g) = \varphi(g)$ for $\alpha \in I$ and $g \in A$.

Specialising to semigroups one has

THEOREM 2. *Let $S$ be a semigroup, $A \subset m(S)$ be a norm closed subalgebra with $1 \in A$ and $f_s \in A$ for each $f \in A$ and $s \in S$. Let $H_A$ be the ideal of all $h$ in $A$ which have a representation*

$$h = \sum_1^n f_k(g_k - l_{a_k} g_k)$$

*for $f_k, g_k$ in $A$ and $a_k$ in $S$, $n = 1, 2, \ldots$.*

*Then the following statements are equivalent:*

(a) *There is a multiplicative left invariant mean on $A$.*

(b) $\inf\{\|1 - h\|;\ h \in H_A\} = 1$.

(c) $\sup\{h(s);\ s \in S\} \geqq 0$ *for each $h$ in $H_A$.*

(d) *The uniform closure of $H_A$ is not equal to $A$.*

(e) *There is a mean $\varphi$ on $A$ such that $\varphi(gg_s) = \varphi(g^2)$ for each $s$ in $S$* and $g \in A$ or such that $\varphi(gg_s) = [\varphi(g)]^2$ for all $s \in S$ and $g \in A$.

REMARKS. 1. Conditions (a) to (d) of this theorem are analogues for extreme left amenable subalgebras $A \subset m(S)$ of the conditions for left amenable subspaces $A \subset m(S)$ given in theorem 17.4 (due to Dixmier) and 17.15 (due to M. Day) in [12,p. 231 and p. 235].

We do though not assume that $S$ is a left cancellation semigroup in order to get that (d) implies (a). This condition is imposed on $S$ in theorem 17.5 of [12] in order to show that the analogue of (d) implies that $A$ admits a left invariant mean.

2. It has been shown by S. P. Lloyd in [14] that if $S$ is left amenable, then a mean $\varphi$ is an extreme point of the set of left invariant means on $m(S)$ if and only if for each $f,g$ in $m(S)$ the function $F(s) = \varphi(fg_s)$ is left almost convergent to $\varphi(f)\varphi(g)$, that is,

$$\mu(F) = \varphi(f)\varphi(g)$$

for each left invariant mean $\mu$ on $m(S)$. The constant function $c \cdot 1$ is obviously left almost convergent to $c$. Hence $S$ is ELA if and only if there is some mean $\varphi$ for which the function $F(s) = \varphi(fg_s)$ is *identical* with the constant function $\varphi(f)\varphi(g)$, for each, $f, g \in m(S)$.

3. Let $A_\alpha$ be left invariant subalgebras of $m(S)$ containing 1, and such that for each $\alpha, \beta$ there is a $\gamma$ such that $A_\alpha \cup A_\beta \subset A_\gamma$. Let $A = \bigcup A_\alpha$. If each $A_\alpha$ admits a multiplicative left invariant mean so does $A$. In fact, let

$$h = \sum_1^n g_i(f_i - l_{a_i} f_i)$$

with $f_i, g_i \in A$ and $a_i \in S$. Then $f_i, g_i \in A_\alpha$ for some $\alpha$. Hence

$$\sup\{h(x);\, x \in S\} \geqq 0 .$$

**3. Application to topological groups.**

Let $G$ be a topological group and $LUC(G) \subset m(G)$ be the space of left uniformly continuous functions on $G$ (that is, $f \in LUC(G)$ iff for any $\varepsilon > 0$ there is a neighborhood $V$ of the origin $e$ of $G$ such that $|f(vx) - f(x)| < \varepsilon$ for any $x \in G$ and $v \in V$).

LEMMA 4. *Let $G, G'$ be topological groups and $\varrho: G \to G'$ a continuous homomorphism onto. If $LUC(G)$ admits a multiplicative left invariant mean, so does $LUC(G')$.*

PROOF. Let $\varrho(g) = g'$ for $g$ in $G$ and let $\varphi$ be a multiplicative left invariant mean on $LUC(G)$. Define $\varphi'$ on $LUC(G')$ by $\varphi'(f') = \varphi(f'(\varrho))$ where $f'(\varrho)(g) = f'(g')$ for $f'$ in $LUC(G)$.

Since $\varrho$ is continuous, $f'(\varrho) \in LUC(G)$ for any $f' \in LUC(G)$. Hence $\varphi'$ is a well defined mean which is multiplicative, as easily checked. Now

$$f'_{a'}(\varrho)(g) = f'(\varrho a\, \varrho g) = [f'(\varrho)]_a(g) \,.$$

Hence $\varphi'$ is left invariant.

LEMMA 5. *Let $G$ be a totally bounded group for which $LUC(G)$ admits a multiplicative left invariant mean $\varphi$. Then $G = \{e\}$.*

PROOF. If $a \in G$ and $a \neq e$ let $U$ be a symmetric neighborhood of $e$ such that $a^{-1} \notin U^2$. Let $f \in LUC(G)$ be such that $f(e) = 1, f(x) = 0$ if $x \notin U$, and $0 \leqq f \leqq 1$ (see A. Weil [23, p. 13]). If $V = \{x; f(x) > \frac{1}{2}\}$ then $G = \bigcup_{i=1}^n a_i^{-1} V$ for some $\{a_1, \ldots, a_n\} \subset G$. Hence $\sum_1^n f_{a_i}(x) \geqq \frac{1}{2}$ for each $x$ in $G$. Thus $n\varphi(f) \geqq \frac{1}{2}$ which shows that $\varphi(f) > 0$.

If now $f(x) > 0$ for some $x \in G$, then $x \in U$ and so $x \notin a^{-1}U$. Hence $f_a(x) = 0$. Therefore $ff_a = 0$ which implies that

$$0 = \varphi(ff_a) = [\varphi(f)]^2 > 0$$

and cannot be.

LEMMA 6. *Let $G_0$ be any subgroup of a locally compact abelian group $G$ with identity $e$. If $G_0 \neq \{e\}$ then $LUC(G_0)$ does not admit a multiplicative left invariant mean.*

PROOF. Let $a \in G_0 \subset G$ be such that $a \neq e$. Let $\chi$ be a continuous character of $G$ such that $\chi(a) \neq 1$ (see Hewitt–Ross [10, p. 345]). Then the continuous homomorphic image of $G_0$ by $\chi$ is a nontrivial totally bounded subgroup of the circle group. Apply now the previous two lemmas.

THEOREM 3. *Let $G$ be a topological group which has a non-trivial continuous homomorphic image $G'$ such that*

(a) *$G'$ is a subgroup of a locally compact abelian group, or*

(b) *$G'$ is a totally bounded group, or*

(c) *$G'$ is a discrete group.*

*Then $LUC(G)$ and a fortiori $C(G)$ do not admit a multiplicative left invariant mean.*

PROOF. (a) Invoke lemmas 4, 6.

(b) Invoke lemmas 4, 5.

(c) Invoke lemma 4 and theorem 2 of Mitchell [16] (for a different proof see [8]).

REMARK. Furthermore, if $G$ is any nontrivial additive subgroup of a locally convex linear topological space or any subgroup of a direct product of groups $G_\alpha$ which satisfy (a) or (b) or (c) of the previous theorem, then $LUC(G)$ (and hence $C(G)$) does not admit a multiplicative left invariant mean. (Locally convex linear topological spaces have sufficiently many characters while a nontrivial subgroup of such a direct product has a nontrivial continuous homomorphic image $G'$ which satisfies (a) or (b) or (c) of the previous theorem.)

It would be interesting to characterise those topological groups for which $LUC(G)$ or $C(G)$ admits a multiplicative left invariant mean.

Let $G$ be a topological group $K_U$ $[K_C]$ be the linear subspace of functions in $LUC(G)$ $[C(G)]$ which can be represented as

$$h = \sum_{1}^{n} (f_k - l_{a_k} f_k)$$

for some $f_k \in LUC(G)$ $[f_k \in C(G)]$, $a_k \in G$, $n = 1, 2, \ldots$. Let $H_U$ $[H_C]$ be the ideal of $LUC(G)$ $[C(G)]$ of all functions $h$ which can be represented as

$$h = \sum_{1}^{n} f_k(g_k - l_{a_k} g_k)$$

for some $f_k, g_k \in LUC(G)$ $[f_k, g_k \in C(G)]$ and $a_k \in G$, $n = 1, 2, \ldots$.

THEOREM 4. *Let $G$ be a topological group which is amenable as a discrete group and has a nontrivial continuous homomorphic $G'$ which satisfies* (a) *or* (b) *or* (c) *of the previous theorem. Then:*

(a) *$K_U$ $[K_C]$ is not norm dense in $LUC(G)$ $[C(G)]$ but $H_U$ $[H_C]$ is norm dense in $LUC(G)$ $[C(G)]$.*

(b) $\sup\{h(x); x \in G\} \geqq 0$ *for any* $h \in K_C$ *but* $\sup\{h(x); x \in G\} < 0$ *for some* $h \in H_U$.

(c) $\inf\{\|1-h\|;\ h \in K_C\}=1$ *but* $\inf\{\|1-h\|;\ h \in H_U\}=0$.

(d) $LUC(G)$ $[C(G)]$ *admits a left invariant mean but not a multiplicative left invariant mean.* (See [12, theorem 17.4 and 17.15].)

REMARK. In particular $G \neq \{e\}$ may be any locally finite or solvable group (or a weak direct product of such groups) which satisfies (a) or (b) or (c) of theorem 3. Such groups are amenable as discrete, see von Neumann [19] and Day [2, p. 516–517].

EXAMPLE. Let $G$ be the set of positive reals with usual multiplication. Let $A \subset C(G)$ be the set of all functions which are uniformly continuous with respect to the metric $|x-y|=d(x,y)$. (These functions are not necessarily in $LUC(G)$.)

If $a \in G$ and $f \in A$ then $f_a(x)=f(ax)$ and $f_a \in A$ as can directly be checked. For $f \in A$ let

$$\varphi(f) = \lim_{x\to 0} f(ax) = \lim_{x\to 0} f(x) = \varphi(f) .$$

Hence the "huge" subalgebra $A \subset C(G)$ admits a multiplicative invariant mean while $LUC(G)$ does not. Hence $H_A$ is not dense in $A$ while $H_{LUC(G)}$ is dense in $LUC(G)$.

## 4. Further information on ELA semigroups.

THEOREM 5. *Let $S$ be a semigroup. Then the following statements are equivalent:*

(a) *$S$ is* ELA.

(b) *For each finite set $\{g_1,\dots,g_n\} \subset m(S)$ and $\{a_1,\dots,a_n\} \subset S$ the functions $g_1 - l_{a_1}g_1,\dots,g_n - l_{a_n}g_n$ have a common zero in $S$, that is for some $x_0 \in S$,*

$$g_k(x_0) - g_k(a_k x_0) = 0$$

*for each $k$, $1 \leqq k \leqq n$.*

(c) *Each $h \in H$ has a zero in $S$.*

PROOF. (a) $\Rightarrow$ (b): By [10, (5) of theorem 3] the finite subset $\{a_1,\dots,a_n\} \subset S$ has a common right zero, that is $a_k x_0 = 0$ for each $1 \leqq k \leqq n$. That (b) $\Rightarrow$ (c) is clear. Now (c) $\Rightarrow$ (a) since if $h \in H$, then $h(x)=0$ for some $x \in S$. Thus $\sup\{h(s);\ s \in S\} \geqq 0$ and so $A = m(S)$ admits a multiplicative left invariant mean.

REMARKS. 1. Condition (c) does not hold for left invariant norm closed subalgebras $A \subset m(S)$ which contain the constants and admit a multiplicative left invariant mean. In fact, let $S$ be the additive positive integers and $A=c$ be the set of $f\in m(S)$ for which $\lim_{n\to\infty}f(n)=\varphi(f)$ exists. $\varphi$ is a multiplicative left invariant mean on $A$. Consider $f(n)=1/n$. Then $f(n)-f(n+1)$ does not have a zero in $S$.

2. There is no analogue to conditions (b) and (c) for amenable semigroups or even for the additive integers $Z$. In this case $H$ is replaced by the linear space of all $h$ such that

$$h(n) = \sum_{k=1}^{m} (f_k(n)-f_k(j_k+n))$$

for some $\{j_1,\ldots,j_k\}\subset Z$ and $\{f_1,\ldots,f_k\}\subset m(Z)$. If $h(n)=\exp(-n^2)-\exp(-(n+1)^2)$, then $h(n)=0$ implies that $n+1=\pm n$, that is, $n=\frac{1}{2}\notin Z$.

3. If $\bar{H}$ is the norm closure of $H$ in $m(S)$ then there may be $f$ in $\bar{H}$ such that $f(s)\neq 0$ for each $s$ in $S$. In fact let $S=\{1,2,3,\ldots\}$ with the multiplication $i\vee j=\max(i,j)$. Let $g_n(s)=1$ if $s=n$ and 0 otherwise. Let

$$f = \sum_{1}^{\infty} 2^{-n}(g_n-l_ng_n)^2 .$$

Then $f(n_0)=0$ if and only if $g_n(n_0)=g_n(n\vee n_0)$ for each $n=1,2,\ldots$.. If $n>n_0$, then $n\vee n_0=n$. Hence $0=g_n(n_0)=g_n(n)=1$ which cannot be. This example also shows: the fact that $S$ is ELA does not imply that $\{g_n-l_{a_n}g_n\}_1^\infty$ have a common zero in $S$, when $\{g_n\}\subset m(S)$ and $\{a_n\}\subset S$.

THEOREM 6. *Let $S$ be a semigroup. The following conditions are equivalent:*

(a) *$S$ is left amenable and $\mu(fg_x)=\mu(fg)$ for each left invariant mean $\mu$ on $m(S)$ and each $f,g\in m(S)$ and $x\in S$.*

(b) *$S$ is left amenable and each extreme point of the set of left invariant means is multiplicative.*

(c) *$S$ is left amenable and $K$ is uniformly dense in $H$.*

(d) *$S$ is* ELA.

PROOF. (d) $\Rightarrow$ (a): Let $R\subset S$ be a right ideal and $f\in m(S)$. If $a\in R$, then

$$\mu(1_Rf) = \mu(l_a(1_Rf)) = \mu[(l_a1_R)(l_af)] = \mu(f_a) = \mu(f)$$

for any left invariant mean $\mu$, since $1_R(ax)=1$ for each $x\in S$.

Let $a\in S$ be fixed and $R=\{s\in S;\ as=s\}$. Then $R$ is not empty, (see [10, theorem 3]) and is clearly a right ideal. Furthermore $1_R(s)g(as)=1_R(s)g(s)$ for each $s\in S$ and $g\in m(S)$. Hence $1_Rg_a=1_Rg$ and

$$\mu(fg_a) = \mu(f1_Rg_a) = \mu(f1_Rg) = \mu(fg)$$

for each $a\in S$, $f,g\in m(S)$ and each left invariant mean $\mu$ on $m(S)$. If (a) holds then $H$ is not dense in $m(S)$, since $\mu(H)=0$ and $\mu(1)=1$ for each left invariant mean $\mu$ on $m(S)$, and by theorem 2 one gets that (d) holds. Hence (a) $\Leftrightarrow$ (d).

(d) $\Rightarrow$ (b): Let $\mu$ be an extreme point of the set of left invariant means. Then by a result of S. P. Lloyd (see [12]), for any $f,g \in m(S)$, the function $F(x)=\mu(fg_x)$ is left almost convergent to $\mu(f)\,\mu(g)$, while from the first part $F(x)=\mu(fg)$ for each $x \in S$. Hence

$$\mu(fg) = \mu(F) = \mu(f)\mu(g) .$$

Thus $\mu$ is multiplicative.—A different proof which does not use Lloyd's result (whose proof is rather difficult) runs as follows: Let $\mu$ be an extreme left invariant mean and $f \in m(S)$, $0 \leqq f \leqq 1$, be fixed. Let $\nu \in m(S)^*$ be defined by

$$\nu(g) = \mu(fg) - \mu(f)\,\mu(g) .$$

One has $\nu(1)=0$ and $\nu(g_s)=\nu(g)$ for $s \in S$ and $g \in m(S)$ (since ($a$) holds). If $g \geqq 0$ then

$$(\mu+\nu)(g) = \mu(g)\,[1-\mu(f)] + \mu(fg) \leqq 0$$

and

$$(\mu-\nu)(g) = \mu[(1-f)g] + \mu(f)\,\mu(g) \geqq 0 ,$$

since $0 \leqq f \leqq 1$. Since $\mu$ is extreme, $\nu=0$ and so $\mu(fg)=\mu(f)\,\mu(g)$ for all $f,g \in m(S)$ with $0 \leqq f \leqq 1$. If $f \in m(S)$ is arbitrary, then $f=\alpha f_1 - \beta f_2$ with $\alpha, \beta \geqq 0$, $0 \leqq f_i \leqq 1$. This readily implies that $\mu$ is multiplicative.

(b) $\Rightarrow$ (d): The set of left invariant means is $w^*$ compact and convex and as well known has to have an extreme point, by the Krein–Milman theorem ([12, p. 460]).

(d) $\Rightarrow$ (c): $K \subset H$ is clear. If $K$ is not norm dense in $H$ then there is some $h_0 \in H$ and a bounded linear functional $\varphi$ on $m(S)$ such that $\varphi(K)=0$ and $\varphi(h_0)=1$. Thus $\varphi(f)=\varphi(f_a)$ for $a \in S, f \in m(S)$, and $\varphi$ is left invariant. Hence $\varphi=\alpha\varphi_1 - \beta\varphi_2$ where $\varphi_1, \varphi_2$ are left invariant means and $\alpha, \beta \geqq 0$ (see [9, p. 55 footnote]). By (a) $\Leftrightarrow$ (d) above $\varphi_1(H)=0$ and $\varphi_2(H)=0$ and so $\varphi(h_0)=0$ which cannot be.

(c) $\Rightarrow$ (d): If $\mu$ is a left invariant mean on $m(S)$ then $\mu(K)=0$ and $\mu(1)=1$. Hence $K$ (and therefore $H$) is not norm dense in $m(S)$. Apply now theorem 2.

REMARKS. 1. One is tempted to conjecture now that if $S$ is a left amenable semigroup and $\mu$ a left invariant mean on $m(S)$ then the function $F(x)=\mu(fg_x)$, if not equal to the constant $\mu(fg)$ for each $x \in S$, is at least left almost convergent to $\mu(fg)$ for $f,g \in m(S)$. That this is not true is shown as follows: Let $S$ be a left amenable semigroup which is not ELA (for example the additive integers) and let $\mu$ be an extreme point of the set of left invariant means. Let $f,g \in m(S)$ and $F(x)=\mu(fg_x)$.

Then by the result of S. P. Lloyd in [12], $\mu(F)=\mu(f)\mu(g)$. If the conjecture would hold then $\mu(F)=\mu(fg)$ which would imply that $S$ is ELA.

2. The mere fact that $K$ is dense in $H$ does not yet imply that $S$ is even left amenable. In fact if $G$ is a free group on two generators then $G$ is not left amenable (von Neumann [19]). Hence $K$ and a fortiori $H$ are dense in $m(G)$ (see [12, p. 235]). We see though from above that the fact that $H$ is *not* dense in $m(S)$ forces $K$ to be dense in $H$.

**Main example.**

Let $I$ be a linearly ordered set (by "$>$" say) with no last element. For each $i \in I$ let $S_i$ be a semigroup with identity $e_i$ and such that

$$e_i \notin [S_i-\{e_i\}][S_i-\{e_i\}] \quad \text{and} \quad S_i \neq \{e_i\},$$

where $S_i-\{e_i\} = \{s \in S_i;\ s \neq e_i\}$. (For example the additive non-negative integers or a free semigroup on any number of generators. In fact, if $T$ is any semigroup, one can adjoin to $T$ a new element $\{e\}$ and define in $T'=T \cup \{e\}$ the multiplication $et=te=t$ for any $t \in T$, $e^2=e$ and $t_1t_2$ to be as in $T$ if $t_1,t_2$ both belong to $T$. Then $T'$ is a semigroup which contains $T$ as a two sided ideal. $T'$ has left (right) cancellation if $T$ has (respectively). Clearly $e \notin TT$.)

Let $S$ consist of all functions $s$ defined on $I$ such that $s(i) \in S_i$ for each $i \in I$ and such that the set $\{i;\ s(i) \neq e_i\}$ has a last element say $i_s \in I$ (that is, $s(i_s) \neq e_{i_s}$ but $s(i)=e_i$ if $i > i_s$). We define $d(s)=i_s$ to be the degree of $s$. (The element $e$ such that $e(i)=e_i$ for each $i \in I$ does not belong to $S$.) We define in $S$ the following multiplication:

$$(st)(i) = \begin{cases} s(i) & \text{if} \quad i > d(t)\,, \\ s(i)t(i) & \text{as in } S_i \text{ if} \quad i=d(t)\,, \\ t(i) & \text{if} \quad i < d(t)\,. \end{cases}$$

In any case $s(i)t(i) \neq e_i$ if $i=d(t)$ since then $t_i \neq e_i$ and because of our condition imposed on $S_i$. Clearly, if $d(s) < d(t)$, then $st=t$. Furthermore

$$(*) \qquad d(st) = d(s) \vee d(t) = \max\{d(s),dt)\} \quad \text{for any } s,t \in S\,.$$

The associativity of multiplication is shown as that of the semigroup of example 2 in section 3 of [10]. Let now $s,t \in S$. Then there is some $i_0 > d(s) \vee d(t)$. ($I$ does not have a last element.) Let $c \in S_{i_0}$, $c \neq e_{i_0}$, and let $h \in S$ be defined by $h(i)=e_i$ if $i \neq i_0$ and $h(i_0)=c$. Then $d(h)=i_0$, thus $sh=th=h$. Hence any two elements of $S$ have a common right zero and so $S$ is ELA. If all $S_i$ have left cancellation then $S$ has. In fact let $st=su$. If $d(t) > d(u)$, then for $i=d(t)$,

$$s(i)t(i) = (st)(i) = (su)(i) = s(i) = s(i)e_i \,.$$

Hence $t(i)=e_i$ which cannot be. Hence $d(t) \leqq d(u)$ and by symmetry $d(t)=d(u)=i_0$ (say). If $i>i_0$ then $u(i)=e_i=t(i)$. If $i=i_0$ then

$$s(i_0)u(i_0) = (su)(i_0) = (st)(i_0) = s(i_0)t(i_0) \,.$$

Hence $u(i_0)=t(i_0)$. If $i<i_0$, then

$$t(i) = (st)(i) = (su)(i) = u(i)$$

which shows that $u=t$.

REMARKS. 1. We did not assume that the semigroups $S_i$ are left amenable. By the remark at the beginning of this example one has that *any* left cancellation semigroup can be a subsemigroup of a left cancellation ELA semigroup. This wealth of left cancellation ELA semigroups seems to us all the more surprising in view of the fact that there are *no right cancellation* ELA *semigroups* (except the trivial $S=\{e\}$ with $e^2=e$). This fact provides strength to the conjecture of John Sorenson that any left amenable right cancellation semigroup has to have left cancellation. Additional strength to Sorenson's conjecture is provided by lemma 4 of [9] to the effect that any left amenable right cancellation semigroup $S$ which is periodic (that is, if $c \in S$ then $c^{2n}=c^n$ for some $n$, [13, p. 113]) is a group (and is hence amenable). To the best of our knowledge Sorenson's conjecture is still open. We add to it the following weaker one: Let $S$ be a left amenable right cancellation semigroup which is in addition *extremely right amenable* (that is, if $a,b \in S$ then $ca=cb=c$ for some $c \in S$). Then $S$ has to have left cancellation and is hence the trivial semigroup containing identity only.

2. The situation is entirely different if one sided cancellation is replaced by two sided cancellation. It has been shown by A. H. Frey in his thesis [8] (which unfortunately has not been published):

THEOREM (A. H. Frey): *Let $S$ be a cancellation (that is two sided) semigroup containing no free subsemigroup on two generators. If $S$ is left amenable then so is every subsemigroup of $S$.*

The question of whether each subsemigroup of a cancellation left amenable semigroup is left amenable can be reduced to the, apparently easier, question of whether each subsemigroup of an amenable group is left amenable (it would then, by symmetry, be right amenable and hence amenable). Both these questions are raised in Frey's thesis [8, p. 90]. In fact if $S$ is a cancellation left amenable semigroup then any two right ideals of $S$ have nonvoid intersection. Using a theorem of Ore (Ljapin [13, p. 392]) one has that the semigroup $S$ can be embedded in a group $G_1$. If $G$ is the subgroup of $G_1$ generated by $S$, then $G$ is left amenable. In fact, one can find, by Day's generalization of the Markoff–Kakutani fixed point theorem (see Day [3]), a mean $\varphi$ on $m(G)$ such that $L_s\varphi=\varphi$ for each $s \in S$. Hence

$$L_{s^{-1}}\varphi = L_{s^{-1}}(L_s\varphi) = L_e\varphi = \varphi ,$$

where $e$ is the identity of $G$, for each $s\in S$. If now $g\in G$, then $g=s_1^{\varepsilon_1}\dots s_n^{\varepsilon_n}$, where $s_i \in S$ and $\varepsilon_i = \pm 1$. Hence $L_g\varphi=\varphi$ for any $g\in G$ and $G$ is left amenable. This argument seems to us more direct than that in (c) of theorem 2 in [17]. Thus the question of whether each subsemigroup of $S$ is left amenable is reduced to the question of whether each subsemigroup of the left amenable (hence amenable [12, p. 234]) group $G$ is left amenable. Now each subgroup of $G$ is left amenable ([12, p. 234]). Thus Frey's first question is reduced to whether the group $G$ can contain a free subsemigroup on two generators or not (again asked in [6]). As noted by Frey in [6], if $G$ contains such a subsemigroup, then the subgroup generated by it cannot be a free group (von Neumann [18] or [12, p. 236]). It is interesting to note that a group $G$ can be generated by a free semigroup on two generators and need not be a free group. Such an example is given by K. Appel and F. Djorup in [1]. An example of the same kind was also known to A. H. Frey (written communication).

We do not know the answer to Frey's question and we add to it the following easier one: Let $S$ be a left amenable subsemigroup of the amenable group $G$. Does this force $S$ to be right amenable? (Equivalently, is any cancellation left amenable semigroup also right amenable?) We refer the reader for further questions of the above type to M. Day [2, p. 520]. The analogues of the above questions for ELA semigroups become trivial, since there are no cancellation ELA semigroups (except the trivial one).

We continue now to study the semigroup $S$ constructed from the semigroups $S_i$ for $i$ in the linearly ordered set $I$.

We specialise $I$ to be the real line $I=R$ with its usual linear order and $S_x$ to be the free semigroup on two generators $\{a_x, b_x\}$ with adjoined identity $e_x$ for $-\infty<x<\infty$. Hence $S$ will consist of all functions $f$ defined on $R$ such that $f(x)\in S_x$ for each real $x$ and such that $\{x; f(x) \neq e_x\}$ has a last element $x_f \in R$, which is the degree $d(f)$ of $f$. We know already that $S$ is a left cancellation ELA semigroup. Our first purpose is to find all ELA subsemigroups of $S$. If $A\subset S$, denote $d(A)=\sup\{d(f); f\in A\}$, $-\infty<d(A)\leqq+\infty$.

PROPOSITION 1. *A subsemigroup* $T \subset S$ *is* ELA *if and only if* $d(T) > d(f)$ *for each* $f \in T$ *(i.e., this supremum is not attained).*

PROOF. If $d(T)$ is not attained and $f, g \in T$, then there is some $h \in T$ such that $d(h) > d(f) \vee d(g)$. Hence $fh = gh = h$ and $T$ is ELA. Conversely, if $d(g) = \sup\{d(f); f \in T\}$ for some $g \in T$, then we claim that $g$ and $g^2$ do not have a common right zero in $T$. In fact if $gh = g^2h$, then $h = gh$ and so

$$d(h) = d(g) \vee d(h) = d(g) .$$

If now $x_0 = d(g)$, then

$$e_{x_0} h(x_0) = h(x_0) = g(x_0) h(x_0) .$$

Hence $g(x_0) = e_{x_0}$ which contradicts the fact that $d(g) = x_0$. Thus $T$ is not ELA. In particular if $d(T) = +\infty$, $T$ is ELA.

We find out now which subsemigroups of $S$ are left amenable.

It is clear that those subsemigroups $T$ for which $d(T)$ is not attained are ELA and a fortiori left amenable. Hence we should decide which ones among the semigroups $T \subset S$ for which $d(T)$ is attained are left amenable.

PROPOSITION 2. *Let* $T$ *be a semigroup for which* $d(T) = x_0$ *is attained. Let*

$$T_0 = \{f \in T; d(f) = x_0\} \neq \emptyset$$

*and*

$$S^0 = \{f(x_0); f \in T_0\} \subset S_{x_0} .$$

*Then* $T$ *is left amenable if and only if* $S^0$ *is commutative.*

PROOF. $T_0$ is a two sided ideal of $T$ because of the equality $d(fg) = d(f) \vee d(g)$. Hence $T$ is left amenable if and only if $T_0$ is left amenable (see end of [10]). Define $\varphi \colon T_0 \to S^0$ by $\varphi(f) = f(x_0)$. Then, since $d(f) = d(g) = x_0$ for any $f, g \in T_0$, we have by the definition of multiplication in $S$ that

$$(fg)(x_0) = f(x_0) g(x_0) .$$

Hence $\varphi$ is a homomorphism onto $S^0$ and so $S^0$ is left amenable (see Day [2, p. 515 (c)]). Furthermore $S^0$ does not contain the identity $e_{a_0}$ because of the definition of $T_0$. Consider now $S_{x_0}$ as being embedded in the free group $G$ on the two generators $\{a_{x_0}, b_{x_0}\}$. Let $G_0$ be the subgroup of $G$ generated by $S^0$ ($\subset S_{x_0}$). By [14, p. 96], $G_0$ is a free group. But since $G_0$ is generated by the left amenable semigroup $S^0$, $G_0$ has to be left amenable (see the preceeding remark 2) and hence amenable [12, p. 234

(17.11)]. By Day [2, p. 516, $(G')$] $G_0$ is generated by one element. Hence $G_0$ and so $S^0$ are commutative.

Conversely, assume that $S^0$ is commutative. Let $f \in T_0$ be fixed (then $f(x_0) \in S^0$). If $h, g \in T_0$ then

$$(gf)(x) = \begin{cases} e_x & \text{if} \quad x > x_0\,, \\ g(x_0)f(x_0) & \text{if} \quad x = x_0\,, \\ f(x) & \text{if} \quad x < x_0\,. \end{cases}$$

Hence $(gf)(hf) = (hf)(gf)$ as directly checked. (It is only needed to check that the value of these functions coincide at $x = x_0$ since

$$(kf)(x) = e_x \quad \text{for } x > x_0$$

and

$$(kf)(x) = f(x) \quad \text{for } x < x_0$$

for any $k \in T_0$. For $s = s_0$ the commutativity of $S^0$ implies the rest.) Hence $T_0 f$ is commutative and so $T$ is left amenable.

Remarks. 1. Call a subsemigroup $T \subset S$ bounded if $d(T) < +\infty$. If $T$ is generated by the finite set $\{f_1, \ldots, f_n\} \subset S$ then

$$d(T) = \max\{d(f_i);\ 1 \leqq i \leqq n\}$$

is attained. In fact $d(f_i^k) = d(f_i)$ and so if $g \in T$, then $g$ is a product of powers of the $f_i$'s. Hence

$$d(g) \leqq \max\{d(f_i);\ 1 \leqq i \leqq n\} = d(f_{i_0})$$

say. Since $f_{i_0} \in T$, $d(T) = d(f_{i_0})$. In particular *any* finitely generated subsemigroup of $S$ is not ELA (as in example 2 of section 3 of [10]).

It can be directly checked that if $A$ is a subset of $S$ and $T$ the semigroup generated by $A$, then $d(A) = d(T)$.

2. Let $T$ be a bounded subsemigroup of $S$. Then there are $f, g \in S$ such that the subsemigroup $U$ generated by $\{T \cup \{f, g\}\}$ is not left amenable. In fact, let

$$x_0 = d(T) + 1$$

and $f, g$ be defined as: $f(x) = e_x$ if $x \neq x_0$ while $f(x_0) = a_{x_0}$, $g(x) = e_x$ if $x \neq x_0$ while $g(x_0) = b_{x_0}$. Then

$$d(f) = d(g) = x_0 = d(T \cup \{f, g\})\,.$$

Now $U_0 = \{h \in U;\ d(h) = x_0\}$ and $V^0 = \{h(x_0);\ h \in U_0\}$ is the semigroup generated by $\{a_{x_0}, b_{x_0}\}$ which is a free semigroup on two generators and hence by no means commutative. By proposition 2, $U$ is not left amenable.

We should notice here that the subsemigroup $X$ of $S$ generated by $\{T \cup \{f\}\}$ is left amenable. For this semigroup $X_0 = \{h \in X;\ d(h) = x_0\}$, and $V^0 = \{h(x_0);\ h \in X_0\}$ is the subsemigroup of $S_{x_0}$ generated by $\{a_{x_0}\}$ (which is commutative).

In view of the general construction of this section it seems that the structure of (left cancellation) ELA semigroups is extremely interesting and worth investigating. It would be especially interesting to know how far our construction is from representing the general ELA semigroup.

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## ERRATA AND ADDENTUM TO [10]

On p. 182: Last row $\psi$ should be replaced by $\varphi$.

On p. 190: In row 18: Delete the word »multiplicative«.—In row 21: Delete »Since $\varphi(A)$ is either 0 or 1, $\varphi(S_0)=1$«.—In row 22: Instead of »Since $\varphi(dS)=\varphi(S_0)=1$« should be »Since $\varphi(dS)=1$«.

On p. 184, Cor. 4: The assumption that each two right ideals of $S_0$ have non-void intersection can easily be removed. Since if $a,b \in S_0$, apply Cor. 4 as stated in [10] to the semigroups $\{a^n\}_1^\infty$, $\{b^n\}_1^\infty$. Let then $a_1, b_1 \in S$ satisfy $a(aa_1)=aa_1$. $b(bb_1)=bb_1$. Any $s \in b_1S \cap a_1S$ will satisfy $as=bs=s$ which is even more than asserted in this Cor. 4.

On p. 196 to Prop. 7: Remark: If $S$ is ELA (or LA) and $I \subset S$ is a *right* ideal then $I$ is ELA (LA) since $\varphi(I)=1>0$ for any left invariant mean $\varphi$ on $m(S)$. The converse is *not true* since if $S=\{e_1,e_2\}$ with $e_ie_j=e_i$, $i,j \leqq 2$, then the right ideal $I=\{e_1\}$ is ELA but $S$ is not even LA.

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