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ON EVALUATION OF HIGHER ORDER COHOMOLOGY OPERATIONS

LEIF KRISTENSEN and IB MADSEN

1. Introduction.

It is the purpose of the present paper to treat the problem of evaluation of secondary and tertiary operations in low dimensions, generalizing the statement

$$Sq^{n+1}(\hat{x}) = 0 \quad \text{if} \quad \deg \hat{x} \leqq n\,,$$
$$Sq^{n+1}(\hat{x}) = \hat{x}^2 \quad \text{if} \quad \deg \hat{x} = n+1\,.$$

In the case of secondary operations, this was done in [4]; however, the treatment given here is very simple and easy. The result in the case of tertiary operations is new. The proof in this case is somewhat involved. We hope to be able to simplify the proof and to generalize the result further. Some applications of the results will be given in a forthcoming paper. Applications of the result for secondary operations have been given by Mahowald [6]. A general definition of operations of the N-th kind has been given by Maunder in [7].

Let A denote the Steenrod algebra (mod 2). Let

$$A(m) = A/B(m)\,,$$

where $B(m)$ is the left ideal of elements of excess $\geqq m+1$. To each relation in the left A-module $A(m)$,

$$r:\ 0 = \sum \hat{\alpha}_\nu \hat{a}_\nu\,,$$

there is associated a secondary operation Qu^r, defined in degrees less than $m+1$. The operation Qu^r is determined up to a primary operation. Let us consider the relation

$$r:\ 0 = \hat{a}Sq^{n+1} + \sum Sq^{n+1+\deg(\hat{a}_0^p)}\,\hat{a}_0^p + \sum \hat{\alpha}_\nu \hat{a}_\nu$$

in $A(m)$, $m > n$, where excess $(\hat{\alpha}_\nu \hat{a}_\nu) > n+1$ and $\sum \hat{a}_0^p$ appears as middle term in the Cartan formula for $\hat{a}$:

$$\Delta(\hat{a}) = \sum \hat{a}' \otimes \hat{a}'' + \sum \hat{a}_0^p \otimes \hat{a}_0^p + \sum \hat{a}'' \otimes \hat{a}'\,.$$

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Then we have

THEOREM. *There exists an operation Qu^r associated with r taking the values*

$$Qu^r(\hat{x}) = \begin{cases} 0 & \text{if } \deg \hat{x} < n\,, \\ \sum \hat{a}''(\hat{x})\hat{a}'(\hat{x}) & \text{if } \deg \hat{x} = n\,. \end{cases}$$

Applied to the stable operations $\Phi_{i,j}$, $i \leqq j$ and $i \neq j-1$, introduced by Adams [1], it yields

$$\Phi_{i,j}(\hat{x}) = 0 \quad \text{if } \deg \hat{x} \leqq \begin{cases} 2^i - 1 & \text{if } i = j\,, \\ 2^j - 2^i - 1 & \text{if } i < j-1\,. \end{cases}$$

It is well known that a tertiary operation is associated with a relation between primary and secondary operations:

$$\sum \hat{a}_i Qu^{r_i} + Qu^{\varepsilon} = 0\,,$$

where $\hat{a}_i \in A$, r_i and ε are relations in A. The tertiary operation is defined in dimensions less than the excess of ε minus 1 (for definition see Section 2). It is defined at least on classes $\hat{x}$ annihilated by all primary operations $\hat{\alpha} \in A$ and by the secondary operations Qu^{r_i}.

Let us consider the relation

$$\mathscr{R}\colon\ \sum \hat{a}_i Qu^{r_i} + Sq^{n+1+\deg r_0} Qu^{r_0} + Qu^{\varepsilon}\,,$$

where ε has excess $\geqq n+2$ and r_i is the relation

$$\begin{aligned} r_i\colon\ 0 &= \hat{b}_i Sq^{n+1} + \sum Sq^{n+1+\deg \hat{b}^q_{i,0}} \hat{b}^q_{i,0} + (\text{terms of excess } \geqq n+2)\,, \\ r_0\colon\ 0 &= \sum \hat{a}^p_{i,0} \hat{b}^q_{i,0}\,. \end{aligned}$$

Here $\hat{a}^p_{i,0}$ (resp. $\hat{b}^q_{i,0}$) appears as middle term in the Cartan formula for $\hat{a}_i$ (resp. $\hat{b}_i$), that is,

$$\begin{aligned} \Delta(\hat{a}_i) &= \sum \hat{a}'_{i,s} \otimes \hat{a}''_{i,s} + \sum \hat{a}^p_{i,0} \otimes \hat{a}^p_{i,0} + \sum \hat{a}''_{i,s} \otimes \hat{a}'_{i,s}\,, \\ \Delta(\hat{b}_i) &= \sum \hat{b}'_{i,t} \otimes \hat{b}''_{i,t} + \sum \hat{b}^p_{i,0} \otimes \hat{b}^p_{i,0} + \sum \hat{b}''_{i,t} \otimes \hat{b}'_{i,t}\,, \end{aligned}$$

where $\deg \hat{a}'_{i,s} < \deg \hat{a}''_{i,s}$ and $\deg \hat{b}'_{i,t} < \deg \hat{b}''_{i,t}$. Then we have

THEOREM. *There exists a tertiary operation $Qu^{\mathscr{R}}$ associated with $\mathscr{R}$ taking the values*

$$Qu^{\mathscr{R}}(\hat{x}) = \begin{cases} 0 & \deg \hat{x} \leqq n-1\,, \\ Qu^r(\hat{x})\,\hat{x} & \deg \hat{x} = n\,, \end{cases}$$

where Qu^r is the secondary operation associated with the relation $r\colon 0 = \sum \hat{a}_i \hat{b}_i$.

A more detailed statement of this theorem is given in Section 4, Theorem 2.

In Section 2 we review the definitions of secondary and tertiary operations in terms of cochain operations. In this connection we define a certain "Massey product" in the Steenrod algebra.

Section 3 contains cochain formulas used in the proof of the theorems mentioned above. Detailed statements of the theorems and proofs are given in Section 4.

The method of proof is based on exact sequences of the form

(1) $$\mathcal{O}^m \xrightarrow{\nabla} Z\mathcal{O}^m \xrightarrow{\varepsilon} A\oplus\ldots\oplus A \to 0 \quad (m \text{ summands})$$

and

(2) $$\mathcal{O}_{(2)} \xrightarrow{\nabla} Z\mathcal{O}_{(2)} \xrightarrow{\varepsilon} A\otimes A \to 0\,.$$

For definitions and proofs see [4] or [3] in case (1), and [5] or [3] in case (2). (In [5], $\mathcal{O}_{(2)}$ is denoted by Q.)

2. Definitions.

Let F denote the free associative Z_2-algebra with unit generated by sq^i, $i=1,2,\ldots$. Let R be the subalgebra generated by the Adem relations. Then $F/R=A$, such that we have an exact sequence

(1) $$0 \to R \to F \to A \to 0\,.$$

Elements of F will be denoted by small letters $\alpha,\beta,a,b,\ldots$. The corresponding elements in A will be denoted by $\hat{\alpha},\hat{\beta},\hat{a},\hat{b},\ldots$. Recall that the excess of a monomial $sq^{i_1}sq^{i_2}\ldots sq^{i_r}$ in F is

$$\max_j (i_j - i_{j+1} - \ldots - i_r)\,,$$

and that the excess of a sum $\sum m_i$ of monomials is $\min_i \operatorname{exc}(m_i)$.

Also, to each element α in F we associate a primary cochain operation $\alpha \in Z\mathcal{O}^1$, such that if α has excess n, then $\alpha(u)=0$ if u is a cochain of degree less than $n-1$ or a cocycle of degree $n-1$ (see p. 58 in [4]). We shall often identify F with its image in $Z\mathcal{O}^1$.

Let (b_{ij}) be a $n\times n$-matrix in elements of F and let a_i, c_i, e_i and f_i, $i=1,2,\ldots,n$, be elements from F such that

(2) $$r_i = \sum b_{ij}c_j + f_i \quad \text{and} \quad s_j = \sum a_i b_{ij} + e_j$$

map to zero in A, that is, r_i and s_j are relations in A. We obtain an identity in F:

(3) $$\mathcal{R}\colon\ \sum a_i r_i + \sum s_j c_j + \varepsilon = 0\,,$$

where

$$\varepsilon = \sum a_i f_i + \sum e_j c_j$$

is a relation. We shall refer to (3) as a relation among relations with an unfactorized relation ε.

Let us define N to be the largest number such that the inequalities

$$N \leqq \operatorname{exc} f_i - 2, \qquad N \leqq \operatorname{exc} e_j - \operatorname{exc} c_j - 2$$

are fulfilled.

If we consider the relations r_i, s_j and ε as cochain operations, then because of the exactness of the sequence

$$\mathcal{O} \xrightarrow{\nabla} Z\mathcal{O} \xrightarrow{\epsilon} A \to 0\,, \tag{4}$$

where $\nabla\theta = \delta\theta + \theta\delta$, there exist operations R_i, S_j and E such that

$$\nabla R_i = r_i, \quad \nabla S_j = s_j, \quad \nabla E = \varepsilon\,.$$

Moreover, E can and shall be chosen such that $E(u) = 0$ if u is a cochain of degree less than N or an N-dimensional cocycle [4, Theorem 3.7]. This is of importance in the definition of tertiary cohomology operations associated with $\mathscr{R}$ (see below).

We now proceed to the definitions of secondary and tertiary cohomology operations. First we describe what we shall understand by a secondary operation associated with the relation

$$r_i = \sum b_{ij} c_j + f_i\,.$$

Let X be a CSS-complex and $(x, \{w_j\})$ a system of cochains on X with $\deg x \leqq \operatorname{exc} f_i - 1$, satisfying

$$\delta x = 0, \quad \delta w_j = c_j(x)\,. \tag{5}$$

Define a cocycle $qu^{r_i}(x, \{w_j\})$ on X by

$$qu^{r_i}(x, \{w_j\}) = R_i(x) + \sum b_{ij}(w_j)\,.$$

It is clear that $qu^{r_i}(x, \{w_j\})$ depends on the choice of R_i. However, since two choices differ at most by a primary operation, it follows that $qu^{r_i}(x, \{w_j\})$ is determined by r_i modulo a cocycle $\alpha(x)$, $\alpha \in Z\mathcal{O}$.

Define natural additive relations Qu^{r_i} from $H^*(X)$ to $H^*(X)$ as follows:

The definition domain $\operatorname{Def}(Qu^{r_i})$ is the set of classes in $H^*(X)$ of degree less than $\operatorname{exc} f_i$ and annihilated by $\hat{c}_j$, $j = 1, \ldots, n$.

For $\hat{x}$ in $\operatorname{Def}(Qu^{r_i})$ put $Qu^{r_i}(\hat{x})$ equal to the factor set of cohomology classes of $qu^{r_i}(x, \{w_j\})$ where $(x, \{w_j\})$ runs over all systems of cochains satisfying (5). The indeterminacy subgroup $\operatorname{Ind}(Qu^{r_i})$ is the relevant component of the graded group $\sum \hat{b}_{ij} H^*(X)$. By an earlier remark it is clear that Qu^{r_i}, although not pointed out in the notation, is only determined up to a primary cohomology operation.

An easy computation (carried out in [4]) shows that Qu^{r_i} satisfies the following additivity relations

$$(6) \qquad Qu^{r_i}(\hat{x}+\hat{y}) = Qu^{r_i}(\hat{x}) + Qu^{r_i}(\hat{y}) \quad \text{if } \deg\hat{x}=\deg\hat{y}<\operatorname{exc} f_i-1\,,$$
$$Qu^{r_i}(\hat{x}+\hat{y}) = Qu^{r_i}(\hat{x}) + Qu^{r_i}(\hat{y})+d(f_i;\, x,y)^\wedge \quad \text{if } \deg\hat{x}=\deg\hat{y}=\operatorname{exc} f_i-1\,.$$

Here $d(f_i;\, x,y)$ is a cochain operation measuring the deviation from additivity of f_i, that is,

$$(7) \qquad \delta d(f_i;\, x,y) + d(f_i;\, \delta x,\delta y) = f_i(x+y) + f_i(x) + f_i(y)\,,$$
$$d(f_i;\, x,0) = 0 \quad \text{and} \quad d(f_i;\, 0,y) = 0\,.$$

The existence of such an operation is a trivial consequence of the exact sequence

$$\mathcal{O}^2 \xrightarrow{\nabla} Z\mathcal{O}^2 \xrightarrow{\varepsilon} A \oplus A \to 0$$

mentioned in Section 1. However, in order to evaluate $d(f_i;\, x,y)$ one has to give a more constructive definition. This is done in (6) and (7) of Section 3.

REMARK. The identities (6) shall be understood with care, i.e. to each choice of the operations Qu^{r_i} and $d(f_i;\, x,y)$ the identities hold modulo the total indeterminacy involved.

If $f_i = sq^n$, then one can choose the operation $d(f_i;\, x,y)$ such that

for $\deg x = \deg y < \operatorname{exc} f_i - 1$:

$$d(f_i;\, x,y) = 0;$$

for $\deg x = \deg y = \operatorname{exc} f_i - 1$:

$$d(f_i;\, x,y) = xy\,.$$

This was proved already in [4]; (6) of Section 3 repeats this proof.

Next, let us turn to the case of tertiary operations. Define a cochain operation M by

$$(8) \qquad M(x) = \sum a_i R_i(x) + \sum S_j c_j(x) + E(x) + \sum d(a_i;\, \delta R_i x, R_i \delta x) +$$
$$+ \sum d\big(a_i;\, b_{i1}c_1(x),\ldots,b_{in}c_n(x), f_i(x)\big)\,.$$

An easy check shows that $\nabla M = 0$; that is, M is a primary cochain operation.

The element of Steenrod's algebra associated with M is called the Massey product of the relation among relations $\mathscr{R}$. The cochain operations R_i and S_j are determined only modulo primary cochain operations (i.e. elements in $Z\mathcal{O}$). Furthermore, it was assumed that E was chosen such that $E(x)=0$ whenever x is a N-dimensional cocycle or a cochain of dimension less than N. Therefore, the Massey product of $\mathcal{R}$ has the indeterminacy

$$\sum \hat{a}_i A + \sum A \hat{c}_j + B(N) ,$$

where $B(N)$ is the left ideal in A of elements of excess larger than N.

Now, assume that $\hat{M} = \sum \hat{\mu}_k \hat{m}_k$ is a factorization of $\hat{M}$ in A. Then $M + \sum \mu_k m_k$ is in $Z\mathcal{O}$ and maps to zero in A (cf. the exact sequence (4)). Choose by exactness a cochain operation χ with

$$\nabla \chi = M + \sum \mu_k m_k . \tag{9}$$

Let us consider systems of cochains $(x, \{w_j\}, \{u_i\}, \{v_k\})$, where

$$\delta x = 0, \quad \delta w_j = c_j(x), \quad \delta u_i = qu^{r_i}(x, \{w_j\}), \quad \delta v_k = m_k(x) . \tag{10}$$

For each such system satisfying in addition $\deg x \leqq N$, the cochain $qu^{\mathcal{R}}(x, \{w_j\}, \{u_i\}, \{v_k\})$ given by

$$\begin{aligned} qu^{\mathcal{R}}(x, \{w_j\}, \{u_i\}, \{v_k\}) = \chi(x) + \sum a_i(u_i) + \sum S_j(w_j) + \sum \mu_k(v_k) + \\ + \sum d(a_i; b_{i1}(w_1), \ldots, b_{in}(w_n), qu^{r_i}(x, \{w_j\})) \end{aligned} \tag{11}$$

is actually a cocycle. One may therefore define a tertiary cohomology operation corresponding to $\mathcal{R}$ in the following way:

The domain of definition $\mathrm{Def}(Qu^{\mathcal{R}})$ consists of all elements in $H^*(X)$ of degree less than or equal to N annihilated by the operations $\hat{c}_j$, Qu^{r_i} and $\hat{m}_k$.

If $\hat{x} \in \mathrm{Def}(Qu^{\mathcal{R}})$, then put $Qu^{\mathcal{R}}(\hat{x})$ equal to the factor set of cohomology classes of

$$qu^{\mathcal{R}}(x, \{w_j\}, \{u_i\}, \{v_k\}) ,$$

where $(x, \{w_j\}, \{u_i\}, \{v_k\})$ runs over all systems of cochains satisfying (10).

The indeterminacy subgroup $\mathrm{Ind}(Qu^{\mathcal{R}})$ is the appropriate component of the graded group

$$\sum \hat{a}_i H^*(X) + \sum Qu^{s_j} H^*(X) + \sum \hat{\mu}_k H^*(X) .$$

Although it has not been made clear in the notation, the definition of the cochain

$$qu^{\mathcal{R}}(x, \{w_j\}, \{u_i\}, \{v_k\})$$

involves choices of the cochain operations M and χ. Hence the natural additive relation $Qu^{\mathcal{R}}$ is not uniquely determined by $\mathcal{R}$. Two choices of $Qu^{\mathcal{R}}$ differ at most by a secondary operation.

As in the secondary case, one can prove by a lengthy but straightforward computation that

$$(12)\quad \begin{aligned} Qu^{\mathcal{R}}(\hat{x}+\hat{y}) &= Qu^{\mathcal{R}}(\hat{x}) + Qu^{\mathcal{R}}(\hat{y}) && \text{if } \deg\hat{x}=\deg\hat{y}<N\,, \\ Qu^{\mathcal{R}}(\hat{x}+\hat{y}) &= Qu^{\mathcal{R}}(\hat{x}) + Qu^{\mathcal{R}}(\hat{y}) + d(E;x,y)^{\wedge} && \text{if } \deg\hat{x}=\deg\hat{y}=N\,, \end{aligned}$$

where $d(E;x,y)$ is a cochain operation of excess N, for which

$$(13)\quad \delta d(E;x,y) + d(E;\delta x,\delta y) = E(x+y) + E(x) + E(y) + d(\varepsilon;x,y)\,,$$

and $d(E;x,y)=0$ if $x=0$ or $y=0$.

Again, (12) shall be understood in the following way: To each choice of the operations $Qu^{\mathcal{R}}$ and $d(E;x,y)$, the identities are fulfilled modulo the total indeterminacy involved.

If $\varepsilon=\nabla E$ has the form

$$\varepsilon = esq^{N+2} + \sum sq^{N+2+\deg e_0^p} e_0^p + (\text{terms of higher excess})\,,$$

then there is a choice of $d(E;x,y)$ such that

for $\deg x = \deg y < N$:

$$d(E;x,y) = 0;$$

for $\deg x = \deg y = N$ and $\delta x = \delta y = 0$:

$$d(E;x,y) = \sum e_i''(x)\cdot e_i'(y) \quad (\tfrac{1}{2}\text{ Cartan formula})\,,$$

where

$$\Delta(\hat{e}) = \sum \hat{e}_i'\otimes\hat{e}_i'' + \sum\hat{e}_0^p\otimes\hat{e}_0^p + \sum\hat{e}_i''\otimes\hat{e}_i' \quad (\deg\hat{e}_i'<\deg\hat{e}_i'')$$

is the Cartan formula for $\hat{e}$. This is proved in Section 3.

Note that $Qu^{\mathcal{R}}$ is a stable operation if $\varepsilon=0$.

We close up this section by constructing universal examples for the stable tertiary operations. Consider the Postnikov system

$$\begin{array}{ccccc} F_2(n) \to & K_{(3)}(n) & & \\ & \downarrow{\scriptstyle \pi_3} & & \\ F_1(n) \to & K_{(2)}(n) & \xrightarrow{\ \Pi Qu^{r_i}\times\Pi \hat{m}_k\ } & B_{(2)}(n) \\ & \downarrow{\scriptstyle \pi_2} & & \\ & K(Z_2,n) & \xrightarrow{\ \Pi \hat{c}_j\ } & B_{(1)}(n)\,, \end{array}$$

where

$$\begin{aligned} F_{(2)}(n) &= \prod K(Z_2, n+\deg Qu^{r_i}-1) \times \prod K(Z_2, n+\deg \hat{m}_k - 1)\,, \\ F_{(1)}(n) &= \prod K(Z_2, n+\deg \hat{c}_j - 1)\,, \\ B_{(2)}(n) &= \prod (Z_2, n+\deg Qu^{r_j}) \times \prod K(Z_2, n+\deg \hat{m}_k)\,, \\ B_{(1)}(n) &= \prod K(Z_2, n+\deg \hat{c}_j)\,. \end{aligned}$$

As indicated on the diagram, the fibration

$$K_{(2)}(n) \xrightarrow{\pi_2} K(Z_2, n)$$

has k-invariant $\prod \hat{c}_j$, and the fibration

$$K_{(3)}(n) \xrightarrow{\pi_3} K_2(n)$$

has k-invariant $\prod Qu^{r_i} \times \prod \hat{m}_k$.

Let $\hat{z}^e$ denote the fundamental class in $H^e(K(Z_2, e); Z_2)$. Then the class

$$\sum \hat{a}_i(\hat{z}^{n+\deg Qu^{r_i}-1}) + \sum \hat{\mu}_k(\hat{z}^{n+\deg \hat{m}_k - 1})$$

in the fiber $F_{(2)}(n)$ of the fibration

$$F_{(2)}(n) \to K_{(3)}(n) \xrightarrow{\pi_3} K_{(2)}(n)$$

is transgressiv and transgresses into

$$\sum \hat{a}_i Qu^{r_i} \pi_2{}^*(\hat{z}^n) + \sum \hat{\mu}_k \hat{m}_k \pi_2{}^*(\hat{z}^n) = 0\,.$$

If $\hat{u} \in H(K_{(3)}(n); Z_2)$ is a class which restricts to the above mentioned class in the fiber, then $(K_{(3)}(n), \hat{u})$ is clearly a universal example for $Qu^{\mathscr{R}}$.

3. Some formulas.

In this section we shall specialize the relations and the relation among relations in consideration in order to simplify the formulas.

Let $\hat{a} \in A$ and let

$$\varDelta(\hat{a}) = \sum \hat{a}'_i \otimes \hat{a}''_i + \sum \hat{a}^p_0 \otimes \hat{a}^p_0 + \sum \hat{a}''_i \otimes \hat{a}'_i \qquad (\deg \hat{a}'_i < \deg \hat{a}''_i)$$

be the Cartan formula for $\hat{a}$. Then $\sum \hat{a}^p_0$ has the property $\hat{a}(x^2) = \sum \hat{a}^p_0(x)^2$ for all cohomology classes $\hat{x}$. Hence there is a relation in A:

$$s\colon\ 0 = \hat{a}\, Sq^{n+1} + \sum Sq^{n+1+\deg \hat{a}^p_0} \hat{a}^p_0 + \hat{e}\,, \tag{1}$$

where $\hat{e}$ has excess larger than $n+1$. Let

$$s = a sq^{n+1} + \sum sq^{n+1+\deg \hat{a}^p_0} a^p_0 + e$$

denote the corresponding cochain operation (in $Z\mathscr{O}$). (Note, that this, in particular, means that $e(x) = 0$ whenever x is an n-dimensional cocycle or a cochain of dimension less than n.) As mentioned in Section 1 there is an exact sequence

$$\mathcal{O}_{(2)} \xrightarrow{\nabla} Z\mathcal{O}_{(2)} \xrightarrow{\varepsilon} A \otimes A \to 0\,, \tag{2}$$

∇ being the differential

$$(\nabla\theta)(x,y) = \delta\theta(x,y) + \theta(\delta x,y) + \theta(x,\delta y)\,.$$

From (2) we conclude that there exists a cochain operation $T_a \in \mathcal{O}_{(2)}$ with the property

$$\begin{aligned} &\delta T_a(x,y) + T_a(\delta x,y) + T_a(x,\delta y) \\ &\quad = a(xy) + \sum a_i'(x)a_i''(y) + \sum a_0^p(x)a_0^p(y) + \sum a_i''(x)a_i'(y) + \\ &\quad + d(a;\delta xy, x\delta y) + \deg x\, d(a; x\delta y, x\delta y)\,. \end{aligned} \tag{3}$$

Using T_a we can now (at least in low dimensions) give an explicit expression for a cochain operation $\theta(x)$ with

$$\delta\theta(x) + \theta(\delta x) = s(x)\,.$$

First, let $\theta'(x)$ be the following partially defined cochain operation:

for $\deg x \leqq n-2$:

$$\theta'(x) = 0\,,$$

for $\deg x = n-1$:

$$\theta'(x) = \sum a_i''(x)a_i'(\delta x)\,, \tag{4}$$

for $\deg x = n$:

$$\theta'(x) = \sum a_i''(x)a_i'(x) + \sum a_i'(x) \cup_1 a_i''(\delta x) + T_a(x,\delta x)\,,$$

for $\deg x = n+1$ and $\delta x = 0$:

$$\theta'(x) = \sum a_i'(x) \cup_1 a_i''(x) + T_a(x,x)\,.$$

An easy computation shows that

$$\delta\theta'(x) + \theta'(\delta x) = a\,sq^{n+1}(x) + \sum sq^{n+1+\deg a_0^p} a_0^p(x) + e(x)\,,$$

whenever the left-hand side is defined.

We now apply Lemma 3.4 of [4]. This gives us the existence of an overall defined cochain operation θ (that is, $\theta \in \mathcal{O}$), taking the following values in low dimensions:

for $\deg x \leqq n-2$:

$$\theta(x) = 0\,,$$

for $\deg x = n-1$:

$$\theta(x) = \sum a_i''(x)a_i'(\delta x)\,, \tag{5}$$

for $\deg x = n$ and $\delta x = 0$:

$$\theta(x) = \sum a_i''(x)a_i'(x)\,.$$

We recall from Section 2 ((6) and (12)) that (in appropriate dimensions) $d(e; x,y)$ measure the deviation from additivity of the secondary operation Qu^s, and that $d(\theta; x,y)$ measure the deviation from additivity of certain tertiary operations. It is therefore of importance to evaluate these cochain operations (in low dimensions).

It is, of course, clear that $d(\psi; x,y)$, $\psi \in \mathcal{O}$, is not uniquely characterized by the property

$$\nabla d(\psi; x,y) + d(\nabla\psi; x,y) = \psi(x+y) + \psi(x) + \psi(y) .$$

Anyway, we may define $d(\lambda; x,y)$ for $\lambda \in F$ in the following way:

$$(6) \qquad d(sq^{n+1}; x,y) = x \cup_1 y + \delta x \cup_{i+1} y \quad (i = \deg x - n)$$

$$(7) \qquad \begin{aligned} d(\alpha a; x,y) &= \alpha d(a; x,y) + d(\alpha; a(x), a(y)) + \\ &\quad + d(\alpha; \nabla d(a; x,y), a(x)+a(y)) + \\ &\quad + d(\alpha; \delta d(a; x,y), d(a; \delta x, \delta y)) , \\ d(\alpha + a; x,y) &= d(\alpha; x,y) + d(a; x,y) . \end{aligned}$$

Furthermore, let us define

$$d(a; x_1,\ldots,x_n) = \sum_{i=1}^{n-1} d(a; x_i, x_{i+1} + \ldots + x_n);$$

then

$$\delta d(a; x_1,\ldots,x_n) + d(a; \delta x_1,\ldots,\delta x_n) = a(\sum x_i) + \sum a(x_i) .$$

It is a consequence of (6) and (7) that $d(e; x,y)$ vanishes on cochains of dimension less than n and on coycles of dimension n, whenever e has excess larger than $n+1$. Note also that $d(a; x,y) = 0$ if $x = 0$ or $y = 0$.

By means of (6) and (7) we can give the following table for $d(s; x,y)$, where $s = a\, sq^{n+1} + \sum sq^{n+1+\deg a_0^p} a_0^p + e$:

for $\deg x = \deg y < n-1$:

$$d(s; x,y) = 0 ,$$

(8) for $\deg x = \deg y = n-1$:

$$\begin{aligned} d(s; x,y) &= a(\delta x y) + \sum a_0^p(\delta x) a_0^p(y) + d(a; \delta x \delta y, \delta x \delta y) + \\ &\quad + \sum \delta d(a_0^p; \delta x, \delta y)(a_0^p(x) + a_0^p(y)) , \end{aligned}$$

for $\deg x = \deg y = n$ and $\delta x = \delta y = 0$:

$$d(s; x,y) = a(xy) + \sum a_0^p(x) a_0^p(y) + \sum \delta d(a_0^p; x,y)(a_0^p(x) + a_0^p(y)).$$

Next, let $\bar{d}(\theta; x,y)$ be the following partially defined cochain operation:

for $\deg x = \deg y \leqq n-3$:

$$\bar{d}(\theta; x,y) = 0,$$

for $\deg x = \deg y = n-2$:

$$\bar{d}(\theta; x,y) = \sum a_i''(x) a_i'(\delta y),$$

for $\deg x = \deg y = n-1$:

$$\begin{aligned}\bar{d}(\theta; x,y) = {} & \sum a_i''(x) a_i'(y) + \sum a_i''(y) \cup_1 a_i'(\delta x) + T_a(\delta x, y) + \\ & + \sum d(a_i''; x,y)(a_i'(\delta x) + a_i'(\delta y)) + \sum (a_i''(x) + a_i''(y)) d(a_i'; \delta x, \delta y) + \\ & + \sum d(a_0^p; \delta x, \delta y)(a_0^p(x) + a_0^p(y)) + \deg x\, d(a; x\delta y, x\delta y) + \\ & + \sum d(a_i''; x,y)\, \delta d(a_i'; \delta x, \delta y),\end{aligned} \tag{9}$$

for $\deg x = \deg y = n$ and $\delta x = \delta y = 0$:

$$\begin{aligned}\bar{d}(\theta; x,y) = {} & T_a(x,y) + \sum a_i''(y) \cup_1 a_i'(x) + \\ & + \sum d(a_i''; x,y)(a_i'(x) + a_i'(y)) + \sum (a_i''(x) + a_i''(y)) d(a_i'; x,y) + \\ & + \sum d(a_0^p; x,y)(a_0^p(x) + a_0^p(y)) + \sum d(a_i''; x,y)\, \delta d(a_i'; x,y).\end{aligned}$$

A check gives at once that

$$\delta \bar{d}(\theta; x,y) + \bar{d}(\theta; \delta x, \delta y) = \theta(x) + \theta(y) + \theta(x+y) + d(s; x,y), \tag{10}$$

whenever the left-hand side is defined. Here $d(s; x,y)$ is given in (8) and $\theta(x)$ is given in (5). As before, applying the extension theorem we get an every-where defined cochain operation satisfying (10). It takes the following values in low dimensions:

(11)

for $\deg x = \deg y \leqq n-3$:

$$d(\theta; x,y) = 0,$$

for $\deg x = \deg y = n-2$:

$$d(\theta; x,y) = \sum a_i''(x) a_i'(\delta y),$$

for $\deg x = \deg y = n-1$ and $\delta x = \delta y = 0$:

$$d(\theta; x,y) = \sum a_i''(x) a_i'(y).$$

We shall now turn to the problem of evaluation (in low dimensions) of tertiary cohomology operations. The way we attack this problem is parallel to the case of secondary operations but fairly more complicated and not quite satisfactory.

Let r: $0 = \sum \hat{a}_i \hat{b}_i$ be a relation in A, and let

$$
(12)\qquad \begin{aligned} \Delta(\hat{a}_i) &= \sum \hat{a}'_{i,s}\otimes\hat{a}''_{i,s} + \sum \hat{a}^p_{i,0}\otimes\hat{a}^p_{i,0} + \sum \hat{a}''_{i,s}\otimes\hat{a}'_{i,s}\,, \\ \Delta(\hat{b}_i) &= \sum \hat{b}'_{i,t}\otimes\hat{b}''_{i,t} + \sum \hat{b}^q_{i,0}\otimes\hat{b}^q_{i,0} + \sum \hat{b}''_{i,t}\otimes\hat{b}'_{i,t}\,, \end{aligned}
$$

where $\deg\hat{a}'_{i,s} < \deg\hat{a}''_{i,s}$ and $\deg\hat{b}'_{i,t} < \deg\hat{b}''_{i,t}$. Then we have relations

$$
(13)\qquad \begin{aligned} r_i:\ \ 0 &= \hat{b}_i Sq^{n+1} + \sum Sq^{n+1+\deg\hat{b}^q_{i,0}}\hat{b}^q_{i,0} + (\text{terms of excess } \geqq n+2)\,, \\ r'_{i,q}:\ \ 0 &= \hat{a}_i Sq^{n+1+\deg\hat{b}^q_{i,0}} + \sum Sq^{n+1+\frac{1}{2}\deg(\hat{a}_i\hat{b}_i)}\hat{a}^p_{i,0} + \\ &\qquad + (\text{terms of excess } \geqq n+2)\,, \\ r_0:\ \ 0 &= \sum \hat{a}^p_{i,0}\hat{b}^q_{i,0}\,, \\ r:\ \ 0 &= \sum \hat{a}_i\hat{b}_i\,. \end{aligned}
$$

The cochain operations corresponding to the relations r, r_0, r_i and $r'_{i,q}$ will be denoted by the same symbols. The relations (13) give rise to a relation among relations with an unfactorized relation ε of excess larger than $n+1$:

$$
(14)\qquad \mathscr{R}:\ \ 0 = \sum a_i r_i + sq^{n+1+\deg r_0} r_0 + r\,sq^{n+1} + \sum r'_{i,q} b^q_{i,0} + \varepsilon\,.
$$

(The equation is an equation in the free associative algebra with unit generated by $1, sq^1, sq^2, \ldots$.)

We recall that to $\mathscr{R}$ in (14) there is associated a Massey product M (see (8) in Section 2). Using the formulas (5), (6) and (7) we get the following expression for $M(x)$ in low dimensions:

for $\deg x \leqq n-2$:

$$
M(x) = 0\,,
$$

for $\deg x = n-1$:

$$
(15)\qquad \begin{aligned} M(x) &= \sum a_i\big(\sum b''_{i,t}(x)\, b'_{i,t}(\delta x)\big) + \sum a''_{i,s} b^q_{i,0}(x)\, a'_{i,s} b^q_{i,0}(\delta x) + \\ &\quad + \sum d\big(a_i;\, \sum b''_{i,t}(\delta x)\, b'_{i,t}(\delta x),\, \sum b''_{i,t}(\delta x)\, b'_{i,t}(\delta x)\big) + \\ &\quad + \sum a^p_{i,0} b^q_{i,0}(\delta x)\, a^u_{j,0} b^v_{j,0}(x)\,, \end{aligned}
$$

for $\deg x = n$ and $\delta x = 0$:

$$
\begin{aligned} M(x) &= \sum a_i\big(\sum b''_{i,t}(x)\, b'_{i,t}(x)\big) + \sum a''_{i,s} b^q_{i,0}(x)\, a'_{i,s} b^q_{i,0}(x) + \\ &\quad + \sum a^p_{i,0} b^q_{i,0}(x)\, a^u_{j,0} b^v_{j,0}(x)\,. \end{aligned}
$$

The last summation in both of the cases $\deg x = n-1$ and $\deg x = n$ is over all triples (i,p,q) and (j,u,v), where $(i,p,q) < (j,u,v)$ for some ordering of triples of indices (i,p,q). We make this a convention such that

$$
\sum a^p_{i,0} b^q_{i,0}(x)\, a^u_{j,0} b^v_{j,0}(y)
$$

in the rest of this paper will mean summation over all (i,p,q) and (j,u,v) with $(i,p,q) < (j,v,u)$.

As a consequence of (15) we see that the cohomology operation $\hat{M}$ associated to M has excess larger than $n-1$. Hence there is a formula

$$\hat{M} = \sum Sq^I Sq^{n+\deg J} Sq^J + \hat{e}\,, \tag{16}$$

where the terms $Sq^I Sq^{n+\deg J} Sq^J$ are admissible of excess n and $\hat{e}$ is admissible of excess larger than n. Note in particular that

$$sq^I sq^{n+\deg J} sq^J(x) = sq^{I_0} sq^J(x)\, sq^{I_0} sq^J(x) \tag{16a}$$

if x is an n-dimensional cocycle, and that

$$sq^I sq^{n+\deg J} sq^J(x) = sq^{I_0} sq^J(x)\, sq^{I_0} sq^J(\delta x) \tag{16b}$$

if x is a $(n-1)$-dimensional cochain.

The aim of the rest of this section is, in low dimensions, to give an explicit expression for a cochain operation χ, satisfying

$$\nabla\chi = M + \sum sq^I sq^{n+\deg J} sq^J + e\,. \tag{17}$$

Before we can state the result, we have to make some preparations. Let us define $\sum e_\nu'' f_\nu'' \otimes e_\nu' f_\nu'$ by

$$\begin{aligned} \sum e_\nu'' f_\nu'' \otimes e_\nu' f_\nu' = {} & \sum a_i b_i \otimes 1 + \sum a_{i,s}'' b_{i,t}'' \otimes a_{i,s}' b_{i,t}' + \\ & + \uparrow \sum a_{i,s}'' b_{i,t}' \otimes a_{i,s}' b_{i,s}'' + \uparrow \sum a_{i,s}' b_{i,t}'' \otimes a_{i,s}'' b_{i,t}' + \\ & + \sum a_{i,s}'' b_{i,0} \otimes a_{i,s}' b_{i,0} + \sum a_{i,0} b_{i,t}'' \otimes a_{i,0} b_{i,t}'\,. \end{aligned} \tag{18}$$

The equality is considered as an equality in $F\otimes F$ (see Section 2 (1) for definition of F). The Cartan formula (in F) of $\sum a_i b_i$ then takes the following form:

$$\begin{aligned} \Delta \sum a_i b_i = {} & 1\otimes \sum a_i b_i + \sum e_\nu' f_\nu' \otimes e_\nu'' f_\nu'' + \updownarrow \sum a_{i,s}' b_{i,t}'' \otimes a_{i,s}'' b_{i,t}' + \\ & + \updownarrow \sum a_{i,s}'' b_{i,t}' \otimes a_{i,s}' b_{i,t}'' + \sum a_{i,0}^p b_{i,0}^q \otimes a_{i,0}^p b_{i,0}^q + \\ & + \sum a_i b_i \otimes 1 + \sum e_\nu'' f_\nu'' \otimes e_\nu' f_\nu'\,. \end{aligned} \tag{19}$$

Here $\uparrow\sum(-\otimes-)$ $(\downarrow\sum-\otimes-,\ \updownarrow\sum-\otimes-)$ is a short notation for that part of the sum, where the degree of the left-hand term is larger (less than, equal to) the degree of the right-hand term.

The admissible monomials constitute an additive basis for A. Hence there exist admissible monomials

$$m_k', \quad k\in \bigcup M_\nu'\,,$$
$$m_h'', \quad h\in \bigcup M_\nu''\,,$$

such that

$$\begin{aligned} s_\nu' &= e_\nu' f_\nu' + \sum m_k', & k\in M_\nu'\,, \\ s_\nu'' &= e_\nu'' f_\nu'' + \sum m_h'', & h\in M_\nu''\,, \end{aligned} \tag{20}$$

are relations. We have the following identities in $F\otimes F$

$$\begin{aligned}(21)\qquad &\sum e_\nu'' f_\nu''\otimes e_\nu' f_\nu' = \sum s_\nu''\otimes e_\nu' f_\nu' + \sum_\nu(\sum m_h'')\otimes s_\nu' ,\\ &\sum_\nu(\sum m_h'')\otimes(\sum m_k') = 0 .\end{aligned}$$

In (15) we gave a table for the Massey product $M(x)$. By a short computation one sees that $M(x)$ is cohomologous to

$$\updownarrow \sum a_{i,s}' b_{i,t}''(x)\, a_{i,s}'' b_{i,t}'(x) + \sum a_{i,0}^p b_{i,0}^q(x)\, a_{j,0}^u b_{j,0}^v(x) ,$$

when x is an n-dimensional cocycle. Thus by (16a) the following expression is cohomologous to zero:

$$\begin{aligned}(22)\qquad &\updownarrow \sum a_{i,s}' b_{i,t}''(x)\, a_{i,s}'' b_{i,t}'(x) + \sum a_{i,0}^p b_{i,0}^q(x)\, a_{j,0}^u b_{j,0}^v(x) +\\ &\qquad + \sum sq^{I_0} sq^J(x)\, sq^{I_0} sq^J(x) .\end{aligned}$$

Here I_0 denotes the "half of I". We introduce a short notation for (22):

$$\begin{aligned}\sum g_\lambda' h_\lambda'\otimes g_\lambda'' h_\lambda'' = &\updownarrow \sum a_{i,s}' b_{i,t}''\otimes a_{i,s}'' b_{i,t}' + \sum a_{i,0}^p b_{i,0}^q\otimes a_{j,0}^u b_{j,0}^v +\\ &+ \sum sq^{I_0} sq^J\otimes sq^{I_0} sq^J .\end{aligned}$$

Now, choose admissible monomials

$$\begin{aligned}&\overline{m}_\omega', \quad \omega\in \bigcup \overline{M}_\lambda',\\ &\overline{m}_\pi'', \quad \pi\in \bigcup \overline{M}_\lambda'',\end{aligned}$$

such that

$$t_\lambda' = g_\lambda' h_\lambda' + \sum \overline{m}_\omega', \quad \omega\in \overline{M}' ,$$

and

$$t_\lambda'' = g_\lambda'' h_\lambda'' + \sum \overline{m}_\pi'', \quad \pi\in \overline{M}'' ,$$

are relations. We have the following equation in $F\otimes F$:

$$(23)\quad \sum g_\lambda' h_\lambda'\otimes g_\lambda'' h_\lambda'' = \sum t_\lambda'\otimes g_\lambda'' h_\lambda'' + \sum_\lambda(\sum \overline{m}_\omega')\otimes t_\lambda'' + \sum_\lambda(\sum \overline{m}_\omega')\otimes(\sum \overline{m}_\pi'') .$$

Let $\hat{z}^n$ be the fundamental cohomology class in $H^n(K(Z_2,n))$. We have

$$\sum_\lambda(\sum \hat{\overline{m}}_\omega'(\hat{z}^n))(\sum \hat{\overline{m}}_\pi''(\hat{z}^n)) = 0$$

since $\sum g_\lambda' h_\lambda'(z^n)\, g_\lambda'' h_\lambda''(z^n)$ by (22) is null-cohomologous. Hence there exist admissible monomials n_μ' and n_μ'' such that (in $F\otimes F$)

$$(24)\qquad \sum_\lambda(\sum \overline{m}_\omega')\otimes(\sum \overline{m}_\pi'') = \sum n_\mu'\otimes n_\pi'' + \sum n_\mu''\otimes n_\mu' + \sum \varrho'\otimes\varrho'' ,$$

where each summand in the last term has the property that either ϱ' or ϱ'' has excess larger than n.

We are now ready to state the main result of this section.

First, let us choose cochain operations R, S_ν', S_ν'', T_λ' and T_λ'' in $\mathcal{O}$ such that

$$\nabla R = r, \quad \nabla S'_\nu = s'_\nu, \quad \nabla S''_\nu = s''_\nu, \quad \nabla T'_\lambda = t'_\lambda, \quad \nabla T''_\lambda = t''_\lambda .$$

Then define χ' as follows:

for $\deg x \leqq n-3$:

$$\chi'(x) = 0 ,$$

for $\deg x = n-2$:

$$\chi'(x) = \downarrow \sum a'_{i,s} b''_{i,t}(x)\, a''_{i,s} b'_{i,t}(\delta x) + \sum a^p_{i,0} b^q_{i,0}(x)\, a^u_{j,0} b^v_{j,0}(\delta x) + \\ + \sum n''_\mu(x) n'_\mu(\delta x) ,$$

for $\deg x = n-1$:

$$(25) \quad \begin{aligned} \chi'(x) = {} & \downarrow \sum a'_{i,s} b''_{i,t}(x)\, a''_{i,s} b'_{i,t}(x) + \sum a^p_{i,0} b^q_{i,0}(x)\, a^u_{j,0} b^v_{j,0}(x) + \\ & + \sum n''_\mu(x) n'_\mu(x) + R(x)\delta x + \sum S''_\nu(x) e'_\nu f'_\nu(\delta x) + \\ & + \sum_\nu \big(\sum m''_h(x)\big) S'_\nu(\delta x) + \sum T'_\lambda(x)\, g''_\lambda h''_\lambda(\delta x) + \sum_\lambda \big(\sum \overline{m}'_\omega(x)\big) T''_\lambda(\delta x) + \\ & + \downarrow \sum a'_{i,s} b''_{i,t}(\delta x) \cup_1 a''_{i,s} b'_{i,t}(x) + \sum n'_\mu(x) \cup_1 n''_\mu(\delta x) + \\ & + \sum T_{a_i}(b''_{i,t}(x), b'_{i,t}(\delta x)) + \sum d(a_i;\, \ldots, b''_{i,t}(x) b'_{i,t}(\delta x), \ldots) , \end{aligned}$$

for $\deg x = n$ and $\delta x = 0$:

$$\begin{aligned} \chi'(x) = {} & R(x)x + \sum S''_\nu(x)\, e'_\nu f'_\nu(x) + \sum_\nu \big(\sum m''_h(x)\big) S'_\nu(x) + \\ & + \sum T'_\lambda(x)\, g''_\lambda h''_\lambda(x) + \sum_\lambda \big(\sum \overline{m}'_\omega(x)\big) T''_\lambda(x) + \\ & + \downarrow \sum a'_{i,s} b''_{i,t}(x) \cup_1 a''_{i,s} b'_{i,t}(x) + \sum n'_\mu(x) \cup_1 n''_\mu(x) + \\ & + \sum T_{a_i}(b''_{i,t}(x), b'_{i,t}(x)) + \sum d(a_i; \ldots, b''_{i,t}(x) b'_{i,t}(x), \ldots) + \\ & + \sum d(a_i;\, \sum b''_{i,t}(x) b'_{i,t}(x),\, \sum b''_{i,t}(x) b'_{i,t}(x)) . \end{aligned}$$

An easy but rather lengthy computation shows that

$$(26) \quad \delta\chi'(x) + \chi'(\delta x) = M(x) + sq^I sq^{\deg J + n} sq^J(x) + e(x)$$

whenever this makes sense. By Lemma 3.4 of [4] there exists a cochain operation χ such that:

(27)

for $\deg x \leqq n-1$ and $\delta x = 0$:

$$\chi(x) = \chi'(x) ,$$

for $\deg x = n$ and $\delta x = 0$:

$$\chi(x) = \chi'(x) + \sum sq^{I_1}(x)\, sq^{I_2}(x) \cdot \ldots \cdot sq^{I_r}(x), \quad r \geqq 2 .$$

4. Theorems.

All references in this section are references to Section 3.

Let s be the relation (1) and $\mathscr{R}$ the relation among relations (14). Then we have

THEOREM 1. *There exists a secondary operation associated with the relation s, taking the following values in low dimensions:*

for $\deg\hat{x} \leqq n-1$:

$$Qu^s(\hat{x}) = 0 ,$$

for $\deg\hat{x} = n$:

$$Qu^s(\hat{x}) = \sum \hat{a}_i''(\hat{x})\,\hat{a}_i'(\hat{x}) \qquad (\tfrac{1}{2}\text{ Cartan formula}) .$$

PROOF. Apply (5).

THEOREM 2. *There is a tertiary operation associated with the relation among relations $\mathscr{R}$, taking the following values in low dimensions:*

a) *for* $\deg\hat{x} \leqq n-2$:

$$Qu^{\mathscr{R}}(\hat{x}) = 0 ,$$

b) *for* $\deg\hat{x} = n-1$:

$$Qu^{\mathscr{R}}(\hat{x}) = \downarrow\sum \hat{a}_{i,s}'\hat{b}_{i,t}''(\hat{x})\,\hat{a}_{i,s}''\hat{b}_{i,t}'(\hat{x}) + \sum \hat{a}_{i,0}^p\hat{b}_{i,0}^q(\hat{x})\,\hat{a}_{j,0}^u\hat{b}_{j,0}^v(\hat{x}) + \\ + \sum \hat{n}_\mu''(\hat{x})\,\hat{n}_\mu'(\hat{x}) ,$$

c) *for* $\deg\hat{x} = n$ *and* $\hat{x}$ *annihilated by all primary operations of degree less than* $\deg\mathscr{R} - n$:

$$Qu^{\mathscr{R}}(\hat{x}) = Qu^r(\hat{x})\,\hat{x} \qquad (\tfrac{1}{2}\text{ Cartan formula}) .$$

REMARK. The symbol $\downarrow\sum$ is explained in (19). The term

$$\sum a_{i,0}^p l_{i,0}^q \otimes a_{j,0}^u l_{j,0}^v$$

is defined in (15), and the term $\sum n_\mu' \otimes n_\mu''$ in (24). Note that these two terms are zero if $\deg(\hat{a}_i\hat{b}_i)$ is odd.

PROOF OF THEOREM 2. Part a) and part b) are immediate consequences of (25). In order to prove c) we introduce the notation $\delta^{-1}b$ for a cochain with $\delta(\delta^{-1}b) = b$. This is a convenient abuse of notation; handled with care it will not give rise to confusions. From (11) of Section 2 one sees that $Qu^{\mathscr{R}}(\hat{x})$ $(\deg\hat{x} = n)$ is represented by the cocycle:

$$qu^{\mathscr{R}}(x) = \chi(x) + \sum a_i\delta^{-1}qu^{r_i}(x) + \sum R_{i,q}'\delta^{-1}l_{i,0}^q(x) + \\ + \sum{}^* sq^I sq^{\deg J+n}\delta^{-1}sq^J(x) + sq^I\delta^{-1}sq^n(x) + \\ + d\left(sq^{n+1+\deg r_0};\ \ldots, a_{i,0}^p\delta^{-1}l_{i,0}^q(x), \ldots, qu^{r_0}(x, \{\delta^{-1}l_{i,0}^q(x)\})\right) ,$$

where $\nabla R_{i,q}' = r_{i,q}'$ and $\sum^* sq^I sq^{\deg J+n} sq^J$ is the part of the sum $\sum sq^I sq^{\deg J+n} sq^J$, where $sq^J \neq 1$. By (5) we have

$$qu^{\mathscr{R}}(x) \sim \chi(x) + \sum a_i(\sum \delta^{-1} b''_{i,t}(x)\, b'_{i,t}(x)) + \\ + \sum a''_{i,s} \delta^{-1} b^q_{i,0}(x)\, a'_{i,s} b^q_{i,0}(x) + \sum a^p_{i,0} b^q_{i,0}(x)\, a^u_{j,0} \delta^{-1} b^v_{j,0}(x) + \\ + \sum{}^* sq^I sq^{n+\deg J} \delta^{-1} sq^J(x) + sq^I \delta^{-1}(sq^n x) .$$

By (18), (19) and (25)

$$qu^{\mathscr{R}}(x) \sim R(x)x + \left(\sum a_i \delta^{-1} b_i(x)\right) x + \updownarrow \sum a'_{i,s} \delta^{-1} b''_{i,t}(x)\, a''_{i,s} b'_{i,t}(x) + \\ + \sum a^p_{i,0} b^q_{i,0}(x)\, a^u_{j,0} \delta^{-1} b^v_{j,0}(x) + \sum n'_\mu(x)\, \delta^{-1} n''_\mu(x) + \\ + \sum \delta^{-1} n''_\mu(x)\, n'_\mu(x) + \sum t'_\lambda(x)\, \delta^{-1} g''_\lambda h''_\lambda(x) + \sum_\lambda \left(\sum \delta^{-1} \overline{m}'_\omega(x)\right) t''_\lambda(x) + \\ + \sum{}^* sq^I sq^{n+\deg J} \delta^{-1} sq^J(x) + sq^I(\delta^{-1} sq^n x) .$$

Hence by (16a), (16b), (22), (23) and (24)

$$qu^{\mathscr{R}}(x) \sim R(x)x + \left(\sum a_i \delta^{-1} b_i(x)\right) x = qu^r(x, \{\delta^{-1} b_i(x)\})\, x .$$

This completes the proof.

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UNIVERSITY OF AARHUS, DENMARK