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COCHAIN FUNCTORS FOR GENERAL COHOMOLOGY THEORIES II

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1. Introduction.

This is a continuation of the paper [11] (referred to as Part I).

The central part is the construction of higher cochain functors $C_{(r)}$ on the category of CSS-complexes yielding general cohomology theories $H_{(r)}$ of finite type. This is done in Section 5. Roughly, a $C_{(r)}$ is constructed from $C_{(r-1)}$ and $C_{(1)}$ as a mapping cone for a natural transformation $C_{(r-1)} \to C_{(1)}$ ($C_{(1)}$ is the ordinary cochain functor). In general, $C_{(r)}(X)$ has the algebraic structure of a graded differential loop (a loop is a set with a non-commutative and non-associative addition with zero and inverses).

In Section 2 we describe a class of cochain functors called c.g. functors comprising the $C_{(r)}$'s, and prove that a functor from the class gives rise to a cohomology theory. In Section 3 the existence of an exact sequence

$$\mathcal{O} \to Z\mathcal{O} \to \mathcal{a} \to 0$$

is proved for certain c.g. functors. This sequence enables us to prove that $C_{(r)}$ is a c.g. functor provided $C_{(r-1)}$ is, and to define cochain formulas for higher order cohomology operations.

For a more detailed description of the content, the reader is referred to the introduction of Part I.

2. Axioms for cochain functors.

We first put up the four axioms for cochain functors and then derive various consequences of various combinations of them. All four axioms together imply the result we want: The cochain functor gives rise to a cohomology theory (on the category of CSS-complexes).

Let us recall that a set with a binary composition (written $+$) is a *loop* if

(i) there is a two-sided neutral element (written 0).

(ii) for any a and b in the set, there are unique solutions to the equations $x+a=b$ and $a+y=b$. We denote the solutions $b \div a$ and $b \dot{-} a$, respectively, so that the elements $b \div a$ and $b \dot{-} a$ satisfy the equations

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$$(b \div a) + a = b ,$$
$$a + (b \dot{-} a) = b .$$

A homomorphism between loops is a mapping preserving $+$ and 0. It will then automatically preserve $\dot{-}$ and $\div$ (considered as binary compositions). A *chain complex* in the category $\mathscr{L}$ of loops is a Z-graded loop with a homomorphism δ of degree $+1$ such that $\delta \circ \delta$ is the zero homomorphism. We let $\mathscr{C}\mathscr{L}$ denote the category of Z-graded chain complexes in the category of loops.

We shall consider contravariant functors

$$C\colon \ \mathrm{CSS} \to \mathscr{C}\mathscr{L} . \tag{1}$$

We denote by C^n the n'th component of C. It is a functor CSS $\to \mathscr{L}$. For $f\colon X \to Y$ a CSS-map, we shall of course denote $C(f)$ by $f^\#$. Also, denote by Z: CSS $\to \mathscr{C}\mathscr{L}$ the functor defined by

$$Z(X) = \mathrm{Ker}\,\delta\colon \ C(X) \to C(X) .$$

(The value of this cocycle functor is of course a chain complex with zero differential.)

Let us recall the following definitions (see e.g. [7]). A *difference cokernel* for morphisms $f,f'\colon A \to B$ in a category is a morphism $e\colon B \to C$ such that any morphism $d\colon B \to D$ with $d \circ f = d \circ f'$ has a unique factorization over e. A *free sum* for the objects A_λ, $\lambda \in \Lambda$, is an object C with morphisms $i_\lambda\colon A_\lambda \to C$ such that given a set of morphisms $d_\lambda\colon A_\lambda \to D$, $\lambda \in \Lambda$, there is a unique morphism $g\colon C \to D$ with $g \circ i_\lambda = d_\lambda$ for all $\lambda \in \Lambda$. We use the French notation $\coprod A_\lambda$ for such a free sum. *Difference kernel* and *direct product* $\prod A_\lambda$ are defined dually. Difference cokernels and free sums are special cases of "right roots" or "direct limits". Difference kernels and direct products are special cases of "left roots" or "inverse limits". A contravariant functor F is termed *right-left continuous* if it transforms difference cokernel diagrams

$$A \underset{f'}{\overset{f}{\rightrightarrows}} B \overset{e}{\to} C$$

into difference kernel diagrams, and free sum diagrams into direct product diagrams. There are analogous definitions for left-right continuous (contravariant) functors, and for left continuous, respectively right continuous (covariant) functors. Recall that a left adjoint functor is right continuous and a right adjoint functor is left continuous.

Note that the category CSS has both difference cokernels and free sums, as well as difference kernels and direct products (for, this is true for the category Ens, and it is easy to see that one may construct the limits in question dimensionwise in the category Ens). Also $\mathscr{C}\mathscr{L}$ has difference cokernels and direct products, and the obvious forgetful functors

$$(2) \qquad \square_n \colon \mathscr{C}\mathscr{L} \xrightarrow{n\text{-component}} \mathscr{L} \to \text{Ens}$$

preserve these. (This tells us how to construct the inverse limits in $\mathscr{C}\mathscr{L}$.) For short $\square_n C$ is denoted by C^n.

We shall need the following concepts concerning algebraic expressions in the theory of loops.

Definition 2.1. Let $F(x_1,\ldots,x_n)$ be an n-ary operation built up by means of $+$, $\dot{-}$, $\dot{-}$ and 0. We say that F is *abelian zero*, denoted

$$F(x_1,\ldots,x_n) \equiv 0 \mod \text{Ab},$$

if $F(x_1,\ldots,x_n)=0$ is a theorem for abelian groups when $\dot{-}$ and $\dot{-}$ are interpreted as $-$.

Recall that $\Delta[q]$ denotes the standard q-simplex and that $\Lambda[q]$ denotes the subcomplex of $\Delta[q]$ consisting (geometrically) of all $(q-1)$-faces except one. Again, we refer to [6] for the exact definitions. We used $\Delta[q]$ and $\Lambda[q]$ in Part I, Section 2.

Given a functor C as in (1), a cochain operation θ on C in n variables (an n-ary operation) is the obvious thing: a family of functor transformations

$$C^m \times \ldots \times C^m \to C^{m+q},$$

m running through the integers, and with n factors on the left. We do not require additivity, but we want θ to be 0 if all the arguments are 0. The F considered in Definition 2.1 may be thought of as a cochain operation in n variables.

Finally, let $P=P_1 \cup \ldots \cup P_k$ be a covering of $\{1,\ldots,n\}$ (i.e. P_j is a subset of $\{1,\ldots,n\}$ and $\bigcup P_j=\{1,\ldots,n\}$).

Definition 2.2. An n-ary operation $F(x_1,\ldots,x_n)$ is called P-normed if for all $1 \leqq j \leqq k$ we have

$$F(x_1,\ldots,x_n) = 0 \quad \text{if} \quad x_i = 0 \text{ for } i \notin P_j .$$

If P is the "trivial" covering $\{1\} \cup \ldots \cup \{n\}$, then we will use the word normed instead of P-normed.

Definition 2.3. A contravariant functor C: CSS $\to \mathscr{C}\mathscr{L}$ will be called a *cohomology generating functor* (or c.g. functor) if it satisfies the following four axioms:

Axiom 1. *C is right-left continuous.*

Axiom 2. *Let F be an n-ary operation and let P be any covering of* $\{1,\ldots,n\}$. *Suppose that F is abelian zero (Definition* 2.1) *and P-normed (Definition* 2.2). *Then there exists an n-ary P-normed cochain operation D on C with*

$$D(\delta x_1,\ldots,\delta x_n)+\delta D(x_1,\ldots,x_n) = F(x_1,\ldots,x_n)\,.$$

Axiom 3. *If A is a subcomplex of X, then the induced map*

$$i^{\#}\colon C(X)\to C(A)$$

is onto.

Axiom 4. *For all* $q\geqq 0$, $C(\Delta[q],\Lambda[q])$ *is acyclic. (Here* $C(\Delta[q],\Lambda[q])$ *of course means the kernel of the restriction* $C(\Delta[q])\to C(\Lambda[q])$. *It is a* $\mathscr{C}\mathscr{L}$*-object, and to say that it is acyclic is simply to say that every cocycle is a coboundary.)*

An alternative formulation of Axiom 1 is given by the obvious

Proposition 2.4. *Axiom* 1 *holds for C if and only if each*

$$C^n\colon \mathrm{CSS}\to \mathrm{Ens}$$

is right-left continuous.

We deduce some consequences of the various axioms. The first of these does not use anything about CSS but its being a diagram category $\mathrm{Ens}^{\mathscr{D}}$, i.e. a category of functors $\mathscr{D}\to\mathrm{Ens}$. This description of CSS is given by Kan [9]. The category $\mathscr{D}$ has as objects the non-negative integers $p,q,r,\ldots$, and as morphisms the wellknown face and degeneracy operators. We use the letters d, d' etc. to denote arbitrary morphisms in $\mathscr{D}$. If $X\colon\mathscr{D}\to\mathrm{Ens}$ is a CSS-complex, then $X(p)$ is the set of its p-simplices. If $d\colon p\to q$ is a morphism in $\mathscr{D}$, $X(d)$ is the application of the operator d to the p-simplices; of course, we shall write d instead of $X(d)$. The proposition to be proved now will be well known to category theorists.

Proposition 2.5. *A contravariant functor* $C\colon \mathrm{CSS}\to\mathrm{Ens}$ *is corepresentable if and only if it is right-left continuous.*

By the two propositions 2.4 and 2.5 we immediately get

Corollary 2.6. *If C satisfies Axiom* 1, *then each* C^n *is corepresentable.*

Proof of Proposition 2.5. Recall the pair of adjoint functors from Part I, Section 2,

$$\mathrm{CSS}\ \underset{\square}{\overset{\Delta}{\rightleftarrows}}\ \mathrm{Grad\ Ens}\,.$$

Let ε denote the end-adjunction $\varepsilon\colon \Delta\Box \to \mathrm{Id}_{\mathrm{CSS}}$, and η the front-adjunction $\eta\colon \mathrm{Id}_{\mathbf{Grad\,Ens}} \to \Box\Delta$. Recall that $\Delta[p]$ is Δ acting on a graded set with one point $*$ in dimension p and $\emptyset$ in other dimensions; $\eta(*) \in \Delta[p]$ is the basic simplex of the standard simplex and is denoted $t_{[p]}$. If $d\colon p \to q$, denote by $\bar{d}\colon \Delta[q] \to \Delta[p]$ the CSS-mapping given by $\bar{d}(t_{[q]}) = dt_{[p]}$.

The construction of a candidate L for a corepresenting object for C goes as follows:

$$\begin{aligned} L(p) &= C(\Delta[p])\,, \\ L(d) &= C(\bar{d}) \quad \text{for } d\colon p \to q\,. \end{aligned}$$

(Note that in case C is the usual n-cochain functor, this is precisely the construction of $\mathscr{L}_n$ (acyclic Eilenberg–MacLane complex) given in e.g. [6].) Then L corepresents C on all standard simplices

$$C(\Delta[p]) = L(p) \cong \mathrm{Hom}_{\mathrm{CSS}}(\Delta[p], L)\,.$$

What we have to do to get this isomorphism in general is to write an arbitrary CSS-complex X as a direct limit of standard simplices and use the limit preserving properties of C. Intuitively this can be done: take a disjoint union (=free sum) of standard simplices, one for each simplex in X, and glue them together correctly. This is a direct limit construction. Formally, this argument looks as follows.

LEMMA 2.7. *The following diagram is a difference cokernel diagram, functorial in X, in the category* CSS:

$$\Delta\Box\Delta\Box X \underset{\varepsilon_{\Delta\Box X}}{\overset{\Delta\Box\varepsilon_X}{\rightrightarrows}} \Delta\Box X \xrightarrow{\varepsilon_X} X\,.$$

PROOF. The following equation always holds for front- and end-adjunctions:

$$\Box\varepsilon_Y \circ \eta_{\Box Y} = id_{\Box Y}\,. \tag{3}$$

Suppose we have a CSS-map $h\colon \Delta\Box X \to Z$ with $h \circ \Delta\Box\varepsilon_X = h \circ \varepsilon_{\Delta\Box X}$. Apply the functor $\Box$. Then the map

$$\Box h \circ \eta_{\Box X}\colon \ \Box X \to \Box Z \tag{4}$$

gives a factorization of $\Box h$ over $\Box\varepsilon_X$, for

$$\begin{aligned} \Box h \circ \eta_{\Box X} \circ \Box\varepsilon_X &= \Box h \circ \Box\Delta\Box\varepsilon_X \circ \eta_{\Box\Delta\Box X} \\ &= \Box h \circ \Box\varepsilon_{\Delta\Box X} \circ \eta_{\Box\Delta\Box X} = \Box h; \end{aligned}$$

the first equation is naturality of η with respect to $\Box\varepsilon_X$, the next is the assumption on h, and the last is just (3). Also, (4) is the only possible factorization of $\Box h$ over $\Box\varepsilon_X$, as is seen by multiplying a possible factorization on the right by $\eta_{\Box X}$. By (3), $\Box\varepsilon_Y$ is an epimorphism, and therefore one can conclude that (4) is of the form $\Box k$, $k\colon X \to Z$. Since $\Box$ is faithful, k is a unique factorization of h over ε_X. This proves the lemma.

The proof of proposition 2.5 is completed as follows. Since Δ commutes with free sums, we may write

$$\Delta\Box Y \quad \text{as} \quad \coprod_{s\in Y} \Delta[g(s)]\,,$$

where $g(s)$ is the dimension of the simplex s. From the lemma we therefore get the difference cokernel diagram, natural in X (for suitable p's and q's in the summands)

$$\coprod_{\Box\Delta\Box X} \Delta[p] \rightrightarrows \coprod_{\Box X} \Delta[q] \to X\,. \tag{5}$$

Let P_i, $i=1,2$, denote the functors C and $\mathrm{Hom}_{\mathrm{CSS}}(\cdot,L)$ respectively. Both are contravariant and right-left continuous. Therefore, applying them to (5), we get two difference-kernel diagrams in $\mathscr{L}$.

$$\prod P_i(\Delta[p]) \leftleftarrows \prod P_i(\Delta[q]) \leftarrow P_i(X)\,. \tag{6}$$

Now,

$$P_1(\Delta[p]) = C(\Delta[p]) = L(p) \cong \mathrm{Hom}_{\mathrm{CSS}}(\Delta[p],L) = P_2(\Delta[p])\,,$$

so that the two diagrams (6) are isomorphic, natural in X. In particular, $P_1(X)\cong P_2(X)$. Proposition 2.5 is proved.

In general, one cannot define the homology of an object in $\mathscr{C}\mathscr{L}$. If, however, Axiom 2 holds for $C\colon \mathrm{CSS} \to \mathscr{C}\mathscr{L}$, we can define a cohomology functor H corresponding to C, $H\colon \mathrm{CSS} \to \mathrm{Grad\ Z\ Ab}$. For any complex X, we define on $Z(X)$ (i.e. on the kernel of $\delta\colon C(X) \to C(X)$) a relation $\sim$,

$$x\sim y \iff \exists\, c \text{ with } x=y+\delta c\,. \tag{7}$$

Axiom 2 will guarantee that this is a congruence relation. Reflexivity is obvious. Symmetry: apply Axiom 2 and get a normed operation D with

$$D(\delta x_1,\delta x_2,\delta x_3)+\delta D(x_1,x_2,x_3) = (x_1 \dot{-} (x_2+x_3)) \dot{-} ((x_1 \dot{-} x_2)+x_3)\,.$$

If $x=y+\delta c$ and y is a cocycle, then

$$y = x + \delta(c+D(y,y,\delta c))\,.$$

Transitivity: apply Axiom 2 to the expression

$$\big(((x_1+x_2)+x_3) \dot{-} x_4\big) \dot{-} ((x_1 \dot{-} x_4) + (x_2+x_3))$$

and find the corresponding normed D. Then, if

$$x = y+\delta b, \qquad y = z+\delta c$$

and z is a cocycle, we have

$$x = z + \delta\big((c+b) + D(z,\delta c,\delta b,z)\big) .$$

Finally, $\sim$ is compatible with $+$; apply Axiom 2 to the expression

$$\big(((x_1+x_2)+(x_3+x_4)) \dot{-} (x_1+x_3)\big) \dot{-} (x_2+x_4) ,$$

and find the corresponding normed D. Then, if x and y are cocycles,

$$(x+\delta a) + (y+\delta b) = (x+y) + \delta[(a+b) + D(x,\delta a; y,\delta b)] .$$

So we define $H(X)$ to be the graded loop obtained by factoring out by $\sim$ in $Z(X)$. Obviously, H is a functor. The loop structure on $H(X)$ is actually the structure of an abelian group, as is easily seen using again Axiom 2. For a pair (X,A) we define $C(X,A)$ to be the kernel of the restriction $i^\#: C(X) \to C(A)$. Not only is $C(X,A) \in \mathscr{C}\mathscr{L}$, but the naturality and normalization properties in the D's, used in the proofs so far, show that (7) also defines a congruence relation on $Z(X,A)$. We thus have

PROPOSITION 2.8. *Let* C: CSS $\to \mathscr{C}\mathscr{L}$ *satisfy Axiom* 2. *Then* (7) *defines a congruence relation on* $Z(X)$, *respectively on* $Z(X,A)$. *The quotient is in* Grad Z Ab *and defines a functor* $H=H(C)$

$$H\colon \text{CSS} \to \text{Grad Z Ab} ,$$

respectively a functor, also denoted H, *from the category of pairs of* CSS-*complexes to* Grad Z Ab.

Also, for $z \in Z^n(X)$ denote by $\hat{z}$ its equivalence class in $H^n(X)$.

PROPOSITION 2.9. *Let* C: CSS $\to \mathscr{C}\mathscr{L}$ *satisfy Axioms* 2 *and* 3. *Then the functors* H^n *defined in the preceding proposition gives for any pair* $A \subseteq X$ *of* CSS-*complexes a long exact sequence*

$$\ldots \to H^n(X,A) \xrightarrow{j^*} H^n(X) \xrightarrow{i^*} H^n(A) \xrightarrow{\delta^*} H^{n+1}(X,A) \to \ldots .$$

PROOF. By Axiom 3 we have a short exact sequence of $\mathscr{C}\mathscr{L}$'s

$$0 \to C(X,A) \to C(X) \to C(A) \to 0 .$$

Now the proof is, with a little care, as for abelian groups. Axiom 2 is not used except to assure that H is defined.

Next, we combine Axioms 1 and 2. If $\emptyset \neq A \subseteq X$, the CSS-complex X/A can be described as a difference cokernel

$$A \underset{p}{\overset{i}{\rightrightarrows}} X \xrightarrow{q} X/A\,, \tag{8}$$

where p maps A constantly to a point $*$ in X.

PROPOSITION 2.10. *Let C: CSS $\to \mathscr{CL}$ satisfy Axioms* 1 *and* 2. *Then for a pair (X,A), the following "excision" holds: q in* (8) *induces an isomorphism*

$$H^n(X/A, *) \cong H^n(X,A)\,.$$

PROOF. Applying C to (8), we get by right-left continuity of C a difference kernel diagram in $\mathscr{CL}$

$$C(A) \underset{p^\#}{\overset{i^\#}{\leftleftarrows}} C(X) \leftarrow C(X/A)\,.$$

From this one gets by chasing elements another difference kernel diagram

$$C(A,*) \underset{p^\#}{\overset{i^\#}{\leftleftarrows}} C(X,*) \xleftarrow{q^\#} C(X/A,*)\,, \tag{9}$$

but here $p^\#$ is the zero map, so that $q^\#$ is the kernel for $i^\#$. However, one easily sees that $C(X,A)$ also might be described as the kernel of $i^\#$ in (9), so that $q^\#$ in (9) is an isomorphism onto $C(X,A)$. The proposition then follows.

If C^n is right-left continuous, we denote the corepresenting objects (existing by Corollary 2.6) by L_n. It is well known and easily seen that transformations τ between the corepresented functors give rise to homomorphisms $\tilde{\tau}$ between the corepresenting objects (and conversely); we have e.g.

$$\begin{aligned} \tilde{+}, \tilde{\dot{-}}, \tilde{\div}:\ & L_n \times L_n \to L_n\,, \\ \tilde{\delta}:\ & L_n \to L_{n+1}\,. \end{aligned} \tag{10}$$

Each L_n is thus a loop in the category CSS. Denote the kernel of $\tilde{\delta}$ in (10) by K_n; it is a sub-CSS-loop of L_n, and it corepresents the functor Z^n. From $\delta\delta = 0$ follows $\tilde{\delta}\tilde{\delta} = 0$. So $\tilde{\delta}$: $L_n \to L_{n+1}$ factors over i

$$\tilde{\delta} = i \circ \tilde{\delta}\,, \tag{11}$$

where i is the inclusion $K_{n+1} \subseteq L_{n+1}$. Axioms 3 and 4 will give propositions on the "geometric structure" on L_n and K_n. We recall the geometric concepts from Part I, Section 2.

PROPOSITION 2.11. *Let C: CSS $\to \mathscr{CL}$ satisfy Axioms* 1 *and* 3. *Then for all n, the corepresenting object L_n for C^n is contractible.*

PROOF. Axiom 3 gives that L_n is injective: If $A \subseteq X$, a CSS-map from A to L_n can be extended over X. Now the proof is as for Proposition 2.8 in Part I.

PROPOSITION 2.12. *Let* C: CSS $\to \mathscr{C}\mathscr{L}$ *satisfy Axioms* 1, 3 *and* 4. *Then for all* n, $\tilde{\delta}: L_n \to K_{n+1}$ *is a Kan-fibering, and* K_n *and* L_n *have the Kan-property.*

PROOF. Copy the proof of Proposition 2.7 in Part I, replacing everywhere $C^n(\cdot\,; U)$ by C^n. Of course, one must be careful in the diagram chase, since now the objects in the diagrams are loops only. All the exactness properties in the diagrams are now given by definition or by Axiom 3 or 4.

Bringing in all four axioms for C, we may finally get at the main theorem, justifying the name c.g. functor ("cohomology generating") for such a C.

THEOREM 2.13. *Let* C: CSS $\to \mathscr{C}\mathscr{L}$ *be a c.g. functor* (*Definition* 2.2). *Let* K_n *denote the corepresenting object for* Z^n, *and* φ *the corresponding natural transformation* $\mathrm{Hom}(\cdot, K_n) \to Z^n$. *Then we have the following diagram, natural in* $X \in$ CSS,

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{CSS}}(X,K_n) & \xrightarrow[\varphi]{\cong} & Z^n(X) \\ \big\downarrow & & \big\downarrow \widehat{\ } \\ [X,K_n] & \xrightarrow[\chi]{} & H^n(X)\,, \end{array}$$

where the vertical maps are taking equivalence classes under homotopy and cohomology, respectively. Furthermore, χ *is an isomorphism, and* H *is a cohomology theory.*

PROOF. If the diagram commutes, χ must be defined by

$$\chi([f]) = f^*(\hat{z}_n)\,, \tag{12}$$

where z_n is the basic cocycle. We must prove that (12) well-defines χ. Suppose $f \simeq g$ by a homotopy h. Then $f \mathbin{\tilde{-}} g \simeq 0$ by a homotopy

$$h' = h \mathbin{\tilde{-}} (g \circ \mathrm{proj}_X), \qquad \mathrm{proj}_X\colon X \times \Delta[1] \to X\,.$$

Then, since $\tilde{\delta}$ has the homotopy lifting property (i.e., the Kan-property), $f \mathbin{\tilde{-}} g$ lifts over $\delta: L_{n-1} \to K_n$, that is,

$$f \mathbin{\tilde{-}} g = \tilde{\delta} \circ k\,,$$

with $k: X \to L_{n-1}$. Then

$$f^{\#}z_n \dot{-} g^{\#}z_n = (f \tilde{-} g)^{\#}z_n = (\tilde{\delta} \circ k)^{\#}z_n = k^{\#}\tilde{\delta}^{\#}(z_n) = k^{\#}\delta e_{n-1} \sim 0 ,$$

and thus $f^{\#}z_n \sim g^{\#}z_n$. Hence χ is well defined by (12). To see that χ is a monomorphism, put a loop structure on $[X, K_n]$ by means of $\tilde{+}$ on K_n. By the very definition of $\tilde{+}$ on K_n, χ becomes a homomorphism, so we need only check that it has zero kernel. Let $f\colon X \to K_n$ have $f^{\#}z_n = \delta b$. Then $f = \tilde{\delta} \circ \tilde{b}$, that is, f factors through L_{n-1} which is contractible (Proposition 2.11); thus $f \simeq 0$. Clearly, χ is onto. The fact that H^* is a cohomology theory is proved already: long exact sequence and excision in Propositions 2.7 and 2.8. Theorem 2.13 establishes the homotopy axiom. The theorem is proved.

3. The exact sequence again.

The existence of an exact sequence of (one-variable) operations can be proved under rather general circumstances. Let C and C' be c.g. functors (Definition 2.2), and H, H' the corresponding cohomology theories. Let L_n, K_n, respectively L_n', K_n' denote the corresponding Eilenberg–MacLane complexes. A family λ of functor-transformations (no additivity required, but 0 must be preserved)

$$\lambda_n\colon\ H^n \to H'^{n+q}, \qquad -\infty < n < \infty ,$$

is called *stable* if for $A \subseteq X$ and all n

$$\lambda_{n+1}\delta^*(\hat{x}) = (-1)^q \delta^* \lambda_n(\hat{x}), \qquad \hat{x} \in H^n(A) .$$

Let $\mathfrak{A}(H, H')$ denote the graded abelian group of stable cohomology operations between H and H'. Also, let $\mathcal{O}(C, C')$ denote the set of everywhere defined functor-transformations $\theta = \{\theta_n\}$ (no additivity is required, but 0 must be preserved),

$$\theta_n\colon\ C^n \to C'^{n+q}, \qquad -\infty < n < \infty .$$

Obviously $\mathcal{O}(C, C')$ inherits a loop structure from C'. We can put a differential ∇ of degree $+1$ on the graded loop $\mathcal{O}(C, C')$,

$$\nabla\theta(x) = \begin{cases} \delta\theta(x) \dot{-} \theta\delta(x) & \text{for } \deg\theta \text{ even} \\ \theta\delta(x) + \delta\theta(x) & \text{for } \deg\theta \text{ odd} . \end{cases} \tag{1}$$

Then $\nabla\nabla = 0$; ∇ will be a homomorphism if $C'(X)$ is a chain complex of abelian groups for all X. Let $Z\mathcal{O}(C, C')$ denote the kernel of ∇. Define

$$\varepsilon\colon\ Z\mathcal{O}(C, C') \to \mathfrak{A}(H, H')$$

in the standard manner as follows. If $\theta \in Z\mathcal{O}(C, C')$, let θ act on the basic cocycles $z_n \in C^n(K_n)$; this defines a family of H'-classes on the K's and thus a cohomology operation by the corepresentation theorem. It is easily seen to be stable. Also $\varepsilon \circ \nabla = 0$. We shall need two properties which a pair of c.g. functors C and C' may have.

(A) For each q the cohomology suspension

$$\sigma\colon\ H'^{n+q+1}(K(n+1),*) \to H'^{n+q}(K(n),*) \tag{2}$$

is an isomorphism if n is sufficiently large .

Let pr_i be the projection from $K(n) \times \ldots \times K(n)$ to the i'th factor.

(B) For each q the homomorphism

$$\coprod pr_i^*\colon\ \coprod H'^{n+q}(K(n),*) \to H'^{n+q}(\prod K(n),*)$$

is an isomorphism if n is sufficiently large .

Remark 3.0. If $H^n(\text{point}) = 0$ for n large, say $n \geqq N$, then $K(N+i)$ is i-connected. Further if H' is a usual cohomology theory, one may show (e.g. by a spectral sequence argument) that (A) and (B) are fulfilled. It will follow from the material of Section 5 that (A) and (B) are fulfilled even if H' is not a usual cohomology theory, but only a cohomology theory of finite type (that is, $H'^n(\text{point}) = 0$ for all but a finite number of n).

Theorem 3.1. *Assume that C and C' are such that* (A) *holds. Then the sequence*

$$\mathcal{O}(C,C') \xrightarrow{\nabla} Z\mathcal{O}(C,C') \xrightarrow{\varepsilon} \mathcal{A}(H,H') \to 0$$

is exact.

Proof. The proof that ε is onto can be copied almost word by word from the proof of Lemma 3.5 in Part I, interpreting $(-1)^q y$ as y if q is even, and as $(0 \dot- y)$ if q is odd. In the last case, the equation (6) in that proof is replaced by

$$b_{M+1} = 0 \dot- \big((0 \dot- b'_{M+1}) + z\big) .$$

We proceed to prove exactness in the middle. Let $r \in Z\mathcal{O}(C,C')$ with $\varepsilon(r) = 0$; let, for example, r be of odd degree q. We shall construct C'-cochains $\theta(e_n)$ on L_n satisfying

$$\delta\theta(e_n) \dot- \tilde{\delta}^{\#}\theta(e_{n+1}) = r(e_n) \quad \forall n .$$

Choose N so big that (2) is an isomorphism for $n \geqq N$. Choose a C'-cochain $\theta(z_{N+1})$ on K_{N+1}, so that $\delta\theta(z_{N+1}) = r(z_{N+1})$. (This can be done, since $\varepsilon(r) = 0$.) Now it is easy to construct C'-cochains $\theta(e_p)$ on L_p, $p \leqq N$, and $\theta(z_p)$ on K_p, $p \leqq N+1$, satisfying the equations

$$(3)_n \qquad \delta\theta(z_{n+1}) = r(z_{n+1}), \quad n \leqq N\,,$$

$$(4)_n \qquad \delta\theta(e_n) \dot{-} \tilde{\delta}^{\#}\theta(z_{n+1}) = r(e_n), \quad n \leqq N\,,$$

$$(5)_n \qquad i^{\#}\theta(e_n) = \theta(z_n), \quad n \leqq N\,.$$

We shall construct $\theta(z_{N+2})$, $\theta(e_{N+1})$ so that the equations $(3)_n$, $(4)_n$, and $(5)_n$ hold for $n \leqq N+1$. Choose $\theta'(z_{N+2})$ and $\theta'(e_{N+1})$ so that $(3)_{N+1}$ and $(4)_{N+1}$ hold. We do not know that $(5)_{N+1}$ holds. But $\theta(z_{N+1}) \dot{-} i^{\#}\theta'(e_{N+1})$ is a cocycle on K_{N+1}. Now, use that the suspension is an isomorphism to get a cocycle $z \in Z'(K_{N+2})$ and cochains $w \in C'(L_{N+1})$, $w' \in C'(L_{N+1})$ with

$$\tilde{\delta}^{\#}z = w\,,$$
$$i^{\#}(w+\delta w') = \theta(z_{N+1}) \dot{-} i^{\#}\theta'(e_{N+1})\,.$$

Let $D(x,y,z)$ be a normed operation with

$$(6) \qquad \nabla D(x,y,z) = \big((z+y)+(x\dot{-}z)\big) \dot{-} (x+y)\,.$$

By Axiom 2 such a D exists. Now put

$$\theta(z_{N+2}) = \theta'(z_{N+2}) + z\,,$$
$$\theta(e_{N+1}) = \big(\theta'(e_{N+1})+(w+\delta w')\big) + D\big(\delta\theta'(e_{N+1}), \delta w, \delta^{\#}\theta'(z_{N+2})\big)\,.$$

Then $(3)_{N+1}$ still holds, and $(5)_{N+1}$ holds since $i^{\#}$ on the D-term is zero. And $(4)_{N+1}$ holds since in our case $\delta D = \nabla D$.

Now, construct $\theta_{N+2}(e_{N+2})$, $\theta_{N+3}(e_{N+3})$ etc. in the same way. The family thus constructed defines a cochain operation θ, and by $(4)_n$, $\nabla\theta = r$. This proves the theorem for $\deg r$ odd. For $\deg r$ even, the proof is in principle the same, but of course the D to be used is not that in (6).

We shall derive some consequences of this theorem. First, let $C^{\times m}$ be the c.g. functor defined by:

$$(7) \qquad \begin{aligned} (C^{\times m})^p(X) &= C^p(X) \times \ldots \times C^p(X) \ (m \text{ factors}) \\ \delta(x_1,\ldots,x_m) &= (\delta x_1,\ldots,\delta x_m) \\ (x_1,\ldots,x) + (y_1,\ldots,y_m) &= (x_1+y_1,\ldots,x_m+y_m)\,. \end{aligned}$$

It is obvious that the corepresenting objects of $(C^{\times m})^p$ (respectively $(Z^{\times m})^p$) are $L(p) \times \ldots \times L(p)$ (respectively $K(p) \times \ldots \times K(p)$). Moreover, if (C,C') has the properties (A) and (B), then so does $(C^{\times m},C')$. The cohomology theory $H^{\times m}$ associated to $C^{\times m}$ is nothing but $H \oplus \ldots \oplus H$ (m summands).

LEMMA 3.2. *If the pair* (C,C') *has the property* (B), *then the homomorphism*

$$(8) \qquad \mathfrak{a}(H,H') \oplus \ldots \oplus \mathfrak{a}(H,H') \to \mathfrak{a}(H^{\times m},H')\,,$$

taking $(\hat{\lambda}_1,\ldots,\hat{\lambda}_m)$ *into the operation* $\hat{\lambda}_1(\hat{x}_1)+\ldots+\hat{\lambda}_m(\hat{x}_m)$, *is an isomorphism.*

We leave the proof to the reader.

Let us now assume that both (A) and (B) hold for (C,C'). Then by Lemma 3.2 every stable cohomology operation is additive. Combining Theorem 3.1 and Lemma 3.2, we get an exact sequence

$$\mathcal{O}(C^{\times m},C') \xrightarrow{\nabla} Z\mathcal{O}(C^{\times m},C') \xrightarrow{\varepsilon} \mathfrak{a}(H,H')\oplus\ldots\oplus\mathfrak{a}(H,H') \to 0\,. \tag{9}$$

Note, that $\mathcal{O}(C^{\times m},C')$ is the set of n-ary operations from C to C' (see Section 2). Using (9), we may measure the deviation from additivity of elements in $Z\mathcal{O}(C,C')$. Let λ be a "primary" cochain operation from C to C', that is, $\lambda\in Z\mathcal{O}(C,C')$. Then the operation $(\lambda(x+y)\dot{-}\lambda(x))\dot{-}\lambda(y)$ in $Z\mathcal{O}(C^{\times 2},C')$ maps to zero by ε. Hence there exists an operation $d(\lambda;\cdot,\cdot)$ in $\mathcal{O}(C^{\times 2},C')$ so that

$$\nabla d(\lambda;x,y) = (\lambda(x+y)\dot{-}\lambda(x))\dot{-}\lambda(y)\,. \tag{10}$$

Moreover, one may choose $d(\lambda;x,y)$ such that in addition it has the property

$$d(\lambda;x,y) = 0 \quad \text{if} \quad x=0 \text{ or } y=0\,. \tag{11}$$

This is trivial if C' is an abelian-valued cochain complex, since for each $d(\lambda;\cdot,\cdot)$ satisfying (10) the operation

$$d(\lambda;x,y)-d(\lambda;x,0)-d(\lambda;0,y)$$

has the desired property. In the general case one has, of course, to use Axiom 2. The argument goes as follows. Let $d(\lambda;x,y)$ be a cochain operation satisfying (10) and let us e.g. suppose that $\deg\lambda$ is odd.

The operation

$$F(x_1,x_2,x_3,x_4) = \Big(x_2+\big(((x_3+x_1)\dot{-}(x_4+x_2))\dot{-}(x_3\dot{-}x_4)\big)\Big)\dot{-}x_1$$

is abelian zero and P-normed, where $P=\{1\}\cup\{2\}\cup\{3,4\}$. Hence, there exists a P-normed cochain operation $E(x_1,x_2,x_3,x_4)$, such that $\nabla E=F$. We substitute in E

$$\begin{aligned} x_1 &= \delta d(\lambda;x,y)\dot{-}\delta d(\lambda;x,0)\,,\\ x_2 &= d(\lambda;\delta x,\delta y)\dot{-}d(\lambda;\delta x,0)\,,\\ x_3 &= \delta d(\lambda;x,0)\,,\\ x_4 &= d(\lambda;\delta x,0)\,, \end{aligned}$$

and get a cochain operation $e(x,y)$. Now, observe that the operation $d_1(x,y)$,

$$d_1(x,y) = \big(d(\lambda; x,y) \dot{-} d(\lambda; x,0)\big) + e(x,y)\,,$$

has the properties

$$\nabla d_1(x,y) = \nabla d(\lambda; x,y)\,,$$
$$d_1(x,0) = 0\,.$$

In a similar way one may define a cochain operation $d_2(x,y)$ such that

$$\nabla d_2(x,y) = \nabla d_1(x,y)\,,$$
$$d_2(x,y) = 0 \quad \text{if } x=0 \text{ or } y=0\,.$$

This completes the argument.

4. Some constructions.

Let C_π be the usual cochain functor with coefficients in the graded abelian group π (see I) and let C' be a c.g. functor. We suppose throughout this section that the pair (C',C_π) has the properties (A) and (B) of Section 3. As in Part I, in the formulas we shall denote the degree of x by $g(x)$.

Let $k\colon C' \to C_\pi$ be a "primary" cochain operation, i.e. $k \in Z\mathcal{O}(C',C_\pi)$. Put

$$C^n[k](X) = C'^n(X) \times C_\pi^{\,n+g(k)-1}(X) \quad \text{(as sets)}$$

and define coboundary and addition in $C[k](X)$ by

$$\text{(1)}\qquad \begin{aligned} \delta(x,w) &= \big(\delta x,\, k(x)-(-1)^{g(k)}\delta w\big) \\ (x,w)+(x',w') &= \big(x+x',\, w+w'+(-1)^{g(k)}d(k;x,x')\big)\,, \end{aligned}$$

where $d(k;\,\cdot,\,\cdot)$ is a fixed cochain operation measuring the additivity defect of k (cf. (10) and (11) of Section 3). It is easy to see, that (1) gives $C[k](X)$ the structure of a graded loop, and that δ is a homomorphism of degree $+1$.

The two difference $\dot{-}$ and $\div$ are given by

$$\text{(2)}\qquad \begin{aligned} (x,w) \dot{-} (x',w') &= \big(x \dot{-} x',\, w-w'-(-1)^{g(k)}d(k;x',x\dot{-}x')\big)\,, \\ (x,w) \div (x',w') &= \big(x \div x',\, w-w'-(-1)^{g(k)}d(k;x\div x',x')\big)\,. \end{aligned}$$

We shall call $C[k]$ the cone construction.

PROPOSITION 4.1. *The cone construction* $C[k]$ *is a* c.g.*-functor.*

PROOF. It is almost trivial that $C[k]$ satisfies the Axioms 1, 3 and 4, and the proofs will be omitted.

In order to verify Axiom 2, let $F(\xi_1,\ldots,\xi_m)$ be an m-ary operation which is abelian zero (Definition 2.1) and P-normed (Definition 2.2), P

being any covering of $\{1,\ldots,m\}$. The cochain functor C_π is abelian-valued. Hence,

$$F((x_1,w_1),\ldots,(x_m,w_m)) = (F(x_1,\ldots,x_m),d_F(x_1,\ldots,x_m)),$$

where $d_F(x_1,\ldots,x_m)$ is some linear combination of terms of the form $d(k;\cdot,\cdot)$.

Since δ commutes with $+$, $\dot{-}$, $\div$ and 0, and since F has even degree, we have $\nabla F(\xi_1,\ldots,\xi_m)=0$. Thus

$$\nabla d_F(x_1,\ldots,x_m) = (-1)^{g(k)}kF(x_1,\ldots,x_m).$$

Let $D_F(x_1,\ldots,x_m)$ be a P-normed operation with

$$\nabla D_F(x_1,\ldots,x_m) = F(x_1,\ldots,x_m).$$

Then

$$kD_F + (-1)^{g(k)}d(k; D_F\delta, \delta D_F) - d_F$$

is "primary" and maps to zero in $\mathcal{A}((H')^{\times m},H_\pi)$. We apply the exact sequence

$$\mathcal{O}((C')^{\times m},C_\pi) \xrightarrow{\nabla} Z\mathcal{O}((C')^{\times m},C_\pi) \xrightarrow{e} \mathcal{A}((H')^{\times m},H_\pi) \to 0,$$

and get a P-normed operation E in $\mathcal{O}((C')^{\times m},C_\pi)$ having

$$\nabla E = kD_F + (-1)^{g(k)}d(k; D_F\delta, \delta D_F) - d_F.$$

A straightforward computation shows that the operation $\bar{D}_F$ in $\mathcal{O}((C[k])^{\times m},C_\pi)$, given by

$$\bar{D}_F((x_1,w_1),\ldots,(x_m,w_m)) = (D_F(x_1,\ldots,x_m),(-1)^{g(k)}E(x_1,\ldots,x_m)),$$

has the desired properties. This completes the proof.

REMARK 4.2. The $\bar{D}_F$ constructed has the further property that $j\bar{D}_F = D_F$, where $j\colon C[k] \to C'$ is the projection.

Let $L'(n)$, respectively $L(\pi,n)$, be the corepresenting objects for C'^n, respectively C^n_π, and let $K'(n)$, respectively $K(\pi,n)$, be the corepresenting object for Z'^n, respectively Z^n_π. ($L(\pi,n)$ is an abelian CSS group, since C^n_π is an abelian group.) The corepresenting object for $C^n[k]$ is

$$L(n) = L'(n) \times L(\pi, n+g(k)-1),$$

and the map $\tilde{\delta}$ from $L(n)$ to $L(n+1)$ is given by

$$\tilde{\delta} = \tilde{\delta}' \times (k \,\tilde{-}\, (-1)^{g(k)}\tilde{\delta}_\pi) \tag{3}$$

(cf. (10) of Section 2).

In Section 2 we proved that the CSS-map $\tilde{\delta}_\pi$ is a Kan fibration

(4) $$K(\pi, n+g(k)-1) \to L(\pi, n+g(k)-1) \to K(\pi, n+g(k)) .$$

Since k is a "primary" cochain operation, the CSS-map $\tilde{k}$ defines a map from $K'(n)$ to $K(\pi, n+g(k))$. Let

(5) $$K(\pi, n+g(k)-1) \to K(n) \to K'(n)$$

be the fibration induced from (4) by means of $\tilde{k}$. The formula (3) tells us that the total space $K(n)$ of (5) is the representing object of $Z^n[k]$. The cohomology functor $H[k]$ associated with $C[k]$ is by Theorem 2.10 naturally equivalent to the functor $[\cdot, K(n)]$. Since $K(n)$ (up to homotopy type) depends only on the homotopy class of $\tilde{k}$, it follows that the cohomology functor $H[k]$ depends only on the cohomology operation $\hat{k}: H' \to H_\pi$, and not on the choice of a representing cochain operation. We proceed to the description of some exact sequences associated with the cone-construction. First, let us consider the obvious short exact sequence of c.g. functors

(6) $$0 \to C_\pi \xrightarrow{\alpha} C[k] \xrightarrow{j} C' \to 0 ,$$

where $\deg \alpha = -g(k)+1$; $\deg j = 0$. Associated with (6) there is a long exact sequence on cohomology level (the cone-sequence)

(7) $$\ldots \to H_\pi{}^{n+g(k)} \xrightarrow{\hat{\alpha}} H^{n+1}[k] \xrightarrow{\hat{j}} H'^{n+1} \xrightarrow{\hat{k}} H_\pi^{n+g(k)+1} \to \ldots .$$

Clearly, (7) is natural with respect to induced maps and coboundary δ^*.

From (7) we see that if $H'^n(\text{point}) = 0$ for n sufficient large, the same is true for $H[k]$.

Moreover, using (7) one may prove that if C is a c.g. functor so that properties (A) and (B) of Section 3 hold for the pair (C, C'), then the same is true for $(C, C[k])$. Let C'' be another c.g. functor, and let H'' be the associated cohomology functor. The functor $\mathfrak{a}(H'', \cdot)$ applied to the sequence (7) yields a new sequence

(8)
$$\ldots \to \mathfrak{a}(H'', H_\pi) \xrightarrow{\hat{\alpha}_*} \mathfrak{a}(H'', H[k]) \xrightarrow{\hat{j}_*} \mathfrak{a}(H'', H') \xrightarrow{\hat{k}_*} \mathfrak{a}(H'', H_\pi) \to \ldots ,$$

and we have

THEOREM 4.3. *The sequence* (8) *is exact provided* (A) *and* (B) *holds for the pairs* (C'', C'), (C'', C_π) *and* $(C'', C[k])$.

PROOF. We shall not go into details with the proof but just mention that it is based on the following diagram with exact columns

(9)
$$\begin{array}{ccccccccc}
 & 0 & & 0 & & 0 & & 0 & \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\ldots \to & \mathcal{A}(H'',H_\pi) & \xrightarrow{\hat{\alpha}_*} & \mathcal{A}(H'',H[k]) & \xrightarrow{\hat{j}_*} & \mathcal{A}(H'',H') & \xrightarrow{\hat{k}_*} & \mathcal{A}(H'',H_\pi) & \to \ldots \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\ldots \to & Z\mathcal{O}(C'',C_\pi) & \xrightarrow{\alpha_*} & Z\mathcal{O}(C'',C[k]) & \xrightarrow{j_*} & Z\mathcal{O}(C'',C') & \xrightarrow{k_*} & Z\mathcal{O}(C'',C_\pi) & \to \ldots \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
 & \mathcal{O}(C'',C_\pi) & & \mathcal{O}(C'',C[k]) & & \mathcal{O}(C'',C') & & \mathcal{O}(C'',C_\pi) &
\end{array}$$

where

(10)
$$\begin{aligned}
\alpha_*(\theta) &= \alpha \circ \theta , \\
j_*(\theta) &= j \circ \theta , \\
k_*(\theta) &= \begin{cases} k \circ \theta & \text{if } \deg\theta \text{ is even ,} \\ k \circ \theta + (-1)^{g(k)} d(k;\, \theta\delta, \delta\theta) & \text{if } \deg\theta \text{ is odd .} \end{cases}
\end{aligned}$$

Here $d(k;\,\cdot,\,\cdot)$ is a cochain operation such that

(11)
$$\nabla d(k;x,y) = k(x+y) - (k(x)+k(y))$$

(see Section 3 (10) and (11)).

Similarly, one may apply the functor $\mathcal{A}(\cdot\,; H'')$ to (7) to get a sequence

(12)
$$\ldots \leftarrow \mathcal{A}(H_\pi,H'') \xleftarrow{\hat{\alpha}^*} \mathcal{A}(H[k],H'') \xleftarrow{\hat{j}^*} \mathcal{A}(H',H'') \xleftarrow{\hat{k}^*} \ldots ,$$

and we have

Theorem 4.3 a. *The sequence* (12) *is exact provided the properties* (A) *and* (B) *hold for the pairs of* c.g. *functors involved.*

Proof. We restrict ourselves to prove exactness of the sequence

$$\mathcal{A}(H,H'') \xleftarrow{\hat{\alpha}^*} \mathcal{A}(H[\hat{k}],H'') \xleftarrow{\hat{j}^*} \mathcal{A}(H',H'') ,$$

leaving the two other cases to the reader.

Let $\hat{\lambda} \in \mathcal{A}(H[\hat{k}],H'')$ be an element which maps to zero by $\hat{\alpha}^*$. We may suppose (without real loss of generality) that $g(\hat{\lambda})$ and $g(\hat{k})$ are even. Let $\lambda \in Z\mathcal{O}(C[k],C'')$ represent $\hat{\lambda}$. Then $\lambda_1 \in Z\mathcal{O}(C_\pi,C'')$ given by

$$\lambda_1(w) = \lambda(0,w) + d(\lambda;\, (0,\delta w), (0,-\delta w))$$

represents $\hat{\alpha}^*(\hat{\lambda})$ (see (10) and (11) Section 3 for definition of $d(\lambda;\,\cdot,\,\cdot)$). By Theorem 3.1 there is a $\mu \in \mathcal{O}(C_\pi,C'')$ such that

$$\delta\mu(w) \dot{-} \mu\delta(w) = \lambda_1(w) .$$

Now, consider the operation $A\colon (C'')^{\times 5} \to C''$,

$$\begin{aligned} &A(x_1,x_2,x_3,x_4,x_5) \\ &\quad = (x_1 \dot{-} x_3) \dot{-} \Big(\Big(\big(x_1+(x_2+(x_4+x_5))\big) \dot{-} \big(x_3+(x_2+x_4)\big)\Big) \dot{-} x_5\Big) . \end{aligned}$$

A is abelian zero (Definition 2.1) and A is P-normed, P being the covering $\{1\}\cup\{2,4\}\cup\{3\}\cup\{5\}$. Choose by Axiom 2 a P-normed operation E in $\mathcal{O}((C'')^{\times 5},C'')$ with $\nabla E = A$.

Substituting in $E(x_1,x_2,x_3,x_4,x_5)$

$$\begin{aligned} x_1 &= \lambda(\delta x,0), \quad x_2 = \lambda(0,k(x))\,, \\ x_3 &= \mu\delta k(x), \quad x_4 = d\big(\lambda;\,(0,k(\delta x)),\,(0,-k(\delta x))\big)\,, \\ x_5 &= \delta d\big(\lambda;\,(\delta x,0),\,(0,k(x))\big) \end{aligned}$$

we get an operation $e\colon C' \to C''$ with the property $e(x)=0$ if $\delta x = 0$ and such that

$$\lambda_2(x) = \Big(\big(\lambda(x,0) \dot{-} \mu(k(x))\big) \dot{-} d\big(\lambda;\,(\delta x,0),\,(0,k(x))\big)\Big) + e(x)$$

is an operation in $Z\mathcal{O}(C',C'')$.

Finally we show that $j^*(\hat\lambda_2) = \hat\lambda$. Let (x,w) be a cocycle in $C[k]$ (i.e. $x=0$ and $\delta w = k(x)$). We have to show that $\lambda(x,0)\dot{-}\mu\big(k(x)\big)$, and $\lambda(x,w)$ are cohomologous. This follows from

$$\begin{aligned} \lambda(x,0)\dot{-}\mu(k(x)) &= \lambda(x,0) \,\dot{-}\, \Big(\delta\mu(w) \dot{\div} \big(\lambda(0,w) + d(\lambda;\,(0,\delta w),\,(0,-\delta w))\big)\Big) \\ &\sim \lambda(x,0) \,\dot{-}\, \Big(0 \dot{\div} \big(\lambda(0,w) + d(\lambda;\,(0,\delta w),\,(0,-\delta w))\big)\Big)\,, \\ \lambda(x,w) &= \lambda(x,0) + \big(\lambda(0,w) + \nabla d(\lambda;\,(x,0),\,(0,w))\big) \\ &\sim \lambda(x,0) + \big(\lambda(0,w) + d(\lambda;\,(0,\delta w),\,(0,-\delta w))\big)\,. \end{aligned}$$

together with a simple application of Axiom 2.

Now, suppose that (C',C_π) and (C',C_ϱ) have properties (A) and (B) of Section 3. Then, copying the definition of secondary operations for usual cohomology theory (see e.g. [12]), to each "relation"

$$\hat a \hat k = 0\,, \tag{13}$$

where $\hat a \in \mathcal{a}(H_\pi,H_\varrho)$, we shall assign an operation (in fact a set of operations)

$$Qu^{\hat a\hat k}\colon\ HC[k] \to HC_\varrho\,.$$

The construction goes as follows. Choose a representative $a \in Z\mathcal{O}(C_\pi,C_\varrho)$ for $\hat a$. Thus, by exactness of the sequence

$$\mathcal{O}(C',C_\varrho) \xrightarrow{\nabla} Z\mathcal{O}(C',C_\varrho) \xrightarrow{\varepsilon} \mathcal{a}(H',H_\varrho) \to 0\,,$$

there is an operation θ in $\mathcal{O}(C',C_\varrho)$ such that $\nabla\theta = ak$. Define $qu^{ak}\colon C[k] \to C_\varrho$ by

$$qu^{ak}(x,w) = \theta(x) + (-1)^{g'(\theta)}a(w) + d'\big(a;(-1)^{g(k)}\delta w, k(x)-(-1)^{g(k)}\delta w\big)\,. \tag{14}$$

Here $d'(a;x,y)$ is a (fixed) cochain operation such that

$$d'(a;x,y) = a(y)+(-1)^{g(k)}a((-1)^{g(k)}x)-a(x+y)\,,$$
$$d'(a;0,y) = 0\,.$$

Then $qu^{ak}\in Z\,\mathcal{O}(C[k],C_\varrho)$ and thus defines an operation $Qu^{\hat{a}\hat{k}}\in \mathfrak{a}(H[k],H_\varrho)$. Note that $Qu^{\hat{a}\hat{k}}$ is only determined by the relation $\hat{a}\hat{k}=0$ up to an element in $\mathfrak{a}(H',H_\varrho)$.

By Theorem 4.3 a we have an exact sequence

$$\ldots \leftarrow \mathfrak{a}(H_\pi,H_\varrho) \xleftarrow{\hat{\alpha}^*} \mathfrak{a}(H[k],H_\varrho) \xleftarrow{\hat{j}^*} \mathfrak{a}(H',H_\varrho) \xleftarrow{\hat{k}^*} \mathfrak{a}(H_\pi,H_\varrho) \leftarrow \ldots\,,$$

where $\hat{k}^*(\hat{\beta})=\hat{\beta}\hat{k}$. Let $\varphi\in\mathfrak{a}(H[k],H_\varrho)$ and put $\hat{\alpha}^*\varphi=\hat{a}$. The relation $\hat{k}^*\hat{\alpha}^*=0$ implies that $\hat{a}\hat{k}=0$. Therefore $Qu^{\hat{a}\hat{k}}$ is defined and one easily sees that

$$\hat{\alpha}^*Qu^{\hat{a}\hat{k}} = \hat{\alpha}^*\varphi = \hat{a}$$

for every choice of $Qu^{\hat{a}\hat{k}}$, that is, $Qu^{\hat{a}\hat{k}}-\varphi$ is in the image of $\hat{j}^*$. Thus there exists a choice of $Qu^{\hat{a}\hat{k}}$ which is equal to φ.

We have proved

PROPOSITION 4.4. *Every stable cohomology operation from $H[k]$ to H_ϱ is of the type $Qu^{\hat{a}\hat{k}}$.*

5. The higher cochain functors.

The higher cochain functors $C[k_1,\ldots,k_{r-1}]$ (shortly $C_{(r)}$), which are special c.g. functors of finite type, are defined by iterated use of the cone construction given in Section 4. The definition of $C_{(r)}$ is therefore inductive.

Let $\pi_0,\pi_1,\pi_2,\ldots$ be graded abelian groups of finite type, that is, π_i has only a finite number of components different from zero.

Put $C_{(1)}=C(\cdot\,;\pi_0)$ and define inductively $C[k_1,\ldots,k_{r-1}]$ to be the cone construction $C[k_1,\ldots,k_{r-2}][k_{r-1}]$, where k_{r-1} (the $(r-1)$th k-invariant) is in $Z\,\mathcal{O}(C[k_1,\ldots,k_{r-2}],C(\cdot\,;\pi_{r-1}))$. Since we are working with graded coefficients, we may without loss of generality assume that $g(k_i)=i+1$. Then from the very definition it follows that

$$C^n_{(r)}(X) = \prod_{i=0}^{r-1} C^{n+i}(X;\pi_i)(X) \quad \text{(as sets)}\,.$$

Coboundary and addition are given by

$$\text{(1)}\quad \begin{aligned} \bar{x} &= (x_1,k(x_1)-x_2,\ldots k_{r-1}(x_1,\ldots,x_{r-1})+(-1)^r x_r)\,,\\ \bar{x}+\bar{y} &= \big(\ldots,x_i+y_i+(-1)^i d((k_{i-1};(x_1,\ldots,x_{i-1}),(y_1,\ldots,y_{i-1})),\ldots\big)\,, \end{aligned}$$

where $\bar{x}=(x_1,\ldots,x_r)$ and $\bar{y}=(y_1,\ldots,y_r)$.

The string of c.g. functors

$$(2) \qquad \cdots \to C_{(r)} \xrightarrow{j} C_{(r-1)} \xrightarrow{j} C_{(r-2)} \xrightarrow{j} \cdots \xrightarrow{j} C_{(1)} ,$$

where j is the projection, has an inverse limit $C_{(\infty)}$ in the category of graded differential loops.

It is clear that $C_{(\infty)}$ satisfies Axiom 1, Axiom 3 and Axiom 4 of Section 2. The remark 4.2 in Section 4 tells us that Axiom 2 is also fulfilled. Hence $C_{(\infty)}$ is a c.g. functor. As usual, let us denote by $L_{(r)}(n)$, respectively $K_{(r)}(n)$, $1 \leqq r \leqq \infty$, the corepresenting objects of $C^n_{(r)}$, respectively $Z^n_{(r)}$.

We have

$$L_{(\infty)}(n) = \prod_{i=0}^{\infty} L(\pi_i, n+i) .$$

The CSS-complexes $K_{(r)}(n)$ may be organized in a Moore–Postnikov tower (see [18]) with $K_{(\infty)}(n)$ as inverse limit:

$$(3) \qquad \begin{array}{rcl}
 & K_{(\infty)}(n) & \\
 & \downarrow & \\
 & \vdots & \\
 & \downarrow j & \\
K(\pi_{r-1}, n+r-1) \to & K_{(r)}(n) & \xrightarrow{\tilde{k}_r} K(\pi_r; n+r+1) \\
 & \downarrow j & \\
K(\pi_{r-2}, n+r-2) \to & K_{(r-1)}(n) & \xrightarrow{\tilde{k}_{r-1}} K(\pi_{r-1}; n+r) \\
 & \downarrow j & \\
 & \vdots & \\
 & \downarrow j & \\
K(\pi_1; n+1) \to & K_{(2)}(n) & \xrightarrow{\tilde{k}_2} K(\pi_2; n+3) \\
 & \downarrow j & \\
 & K_{(1)}(n) & \xrightarrow{\tilde{k}_1} K(\pi_1; n+2) .
\end{array}$$

Using a CSS-version of Browns representation theorem [4] and the Moore–Postnikov decomposition of a CSS-complex, one gets the following factorization theorem.

THEOREM 5.1. *Given a cohomology theory h with the property*:

$$(4) \qquad h^n(\text{point}) = 0 \textit{ for } n \textit{ sufficiently large} .$$

There exists a c.g. *functor $C_{(\infty)}$ such that $h = HC_{(\infty)}$.*

REMARK. Even if condition (4) does not hold, Theorem 5.1 may be of use since every cohomology theory can be "approximated" by theories satisfying (4). Leaving details to the reader we give the definition of such "approximating" theories $h_{(N)}$,

$$h^p_{(N)}(X) = \operatorname{Im}(h^p(X, X^{p-N-1}) \to h^p(X, X^{p-N-2})) .$$

COROLLARY 5.2. *Given a pair of* c.g. *functors* (C, C') *such that H satisfies* (4) *and H' is of finite type. Then conditions* (A) *and* (B) *of Section* 4 *are fulfilled.*

PROOF. By Theorem 5.1 we may assume that $C' = C_{(r)}$ for a suitable choice of k-invariants. If $r=1$, such that $C_{(r)}$ is a usual abelian chain complex, the corollary is just Remark 3.0. In the case $r>0$, apply the cone sequences (Section 4),

$$\ldots \to H_{(i)} \to H_{(i-1)} \to H_{(1)} \to H_{(i)} \to \ldots , \qquad i \leqq r \tag{7}$$

and induction over r. This completes the proof.

Since $C_{(\infty)} = \lim_{\leftarrow} C_{(r)}$, we get a natural transformation

$$HC_{(\infty)} \to \lim_{\leftarrow} HC_{(r)} . \tag{5}$$

Even though we cannot do homological algebra with the (loop valued!) c.g. functors, we can do enough hand work with Axiom 2 to get information about the kernel of (5); we can prove that the following sequence is exact for all n:

$$0 \to \lim_{\leftarrow}^{(1)} H_{(r)}^{n-1} \to H_{(\infty)}^{n} \to \lim_{\leftarrow} H_{(r)}^{n} \to 0 ,$$

where $\lim_{\leftarrow}^{(1)}$ is the first derived functor of $\lim_{\leftarrow}$ as defined in e.g. [17]. The proof is omitted.

We proceed to discuss cohomology operations of r-th kind. Let $\hat{k}_1, \hat{k}_2, \ldots, \hat{k}_{r-1}$ be k-invariants

$$\hat{k}_i\colon\ H[\hat{k}_1, \ldots, \hat{k}_{i-1}] \to H(\cdot\,; \pi_i) \tag{6}$$

with cochain representatives

$$\begin{aligned} k_i\colon&\ C[k_1, \ldots, k_{i-1}] \to C(\cdot\,; \pi_i), \\ k_1\colon&\ \qquad C(\cdot\,; \pi_0) \to C(\cdot\,; \pi_1). \end{aligned} \tag{7}$$

We state

DEFINITION 5.3. A stable cohomology operation of the r-th kind is an additive relation

$$\hat{k}_r (j^{r-1})^{-1}\colon\ H(\cdot\,; \pi_0) \to H(\cdot\,; \pi_r) ,$$

$\hat{k}_r$ being in $\mathfrak{A}(H_{(r)}, H(\cdot\,; \pi_r))$ and j^{r-1} being the projection $H_{(r)} \to H_{(1)}$ $(H_{(1)} = H(\cdot\,; \pi_0))$.

Note that the cone sequence gives information about indeterminacy and domain-of-definition of such r-th kind operations.

By Proposition 4.4 every $\hat{k}_r$ is of the form $Qu^{\hat{a}\hat{k}_{r-1}}$, $\hat{a} \in \mathfrak{A}(H(\cdot\,; \pi_{r-1}), H(\cdot\,; \pi_r))$ and $\hat{a}\hat{k}_{r-1} = 0$ (Theorem 4.3.).

We shall verify that the axioms given by Maunder [14] hold. This needs some preparations, namely the concept of derivation of the functors $C_{(r)}$ and $H_{(r)}$. The iterated projection $j^{r-1}\colon C_{(r)} \to C_{(1)}$ $(= C(\cdot\,; \pi_0))$

is a cochain operation, and one may therefore define a functor $C_{(r,1)}\colon \mathrm{CSS} \to \mathscr{C}\mathscr{L}$ (the first derivative of $C_{(r)}$) by

$$C_{(r,1)} = \operatorname{Ker} j^{r-1}\,. \tag{8}$$

If $k_r\colon C_{(r)} \to C(\cdot\,;\pi_r)$ is a primary cochain functor, then the restriction $k_{(r,1)}$ of k_r to $C_{(r,1)}$ (the derivative of k_r) is of course again a primary cochain operation. Thus we may describe $C_{(r,1)}$ as the c.g. functor

$$C_{(r,1)} = C[k_{(2,1)},\ldots,k_{(r-1,1)}]\,. \tag{9}$$

The procedure may be iterated to obtain c.g. functors

$$C_{(r,s)} = C[k_{(s+1,s)}),\ldots,k_{(r,s)}]$$

and operations

$$k_{(r,s)}\colon\ C_{(r,s)} \to C(\cdot\,;\pi_r)\,.$$

Alternatively, $C_{(r,s)} = \operatorname{Ker} j^{r-s}$.

Remark. The c.g. functor $C_{(t,t-1)}$ is nothing but $C(\cdot\,;\pi_{t-1})$, to be precise, there is a primary cochain isomorphism $\alpha\colon C(\cdot\,;\pi_{t-1}) \to C_{(t,t-1)}$ of degree $t-1$. In the sequel we shall not distinguish between $C(\cdot\,;\pi_{t-1})$ and $C_{(t,t-1)}$. Thus, in particular, the stable operation

$$\hat{k}_{(t,t-1)}\colon\ H_{(t,t-1)} \to H(\cdot\,;\pi_t)$$

is also considered as a stable operation from $H(\cdot\,;\pi_{t-1})$ to $H(\cdot\,;\pi_t)$.

The whole pyramid of cohomology operations

$$\begin{array}{llll} \hat{k}_r & \hat{k}_{r-1} & \ldots\hat{k}_2 & \hat{k}_1 \\ \hat{k}_{(r,1)} & \hat{k}_{(r-1,1)} & \ldots\hat{k}_{(2,1)} & \\ \vdots & & \cdot\ \cdot & \\ \hat{k}_{(r,r-2)} & \hat{k}_{(r-1,r-2)} & & \\ \hat{k}_{(r,r-1)} & & & \end{array} \tag{10}$$

has the following two properties:

$$\begin{aligned} \hat{k}_{(t+1,t)}\hat{k}_{(t,s)} &= 0, \quad 0 \leqq s < t \leqq r, \quad (\hat{k}_t = \hat{k}_{(t,0)})\,, \\ \hat{k}_{(t+1,s)} &= Qu^{\hat{k}_{(t+1,t)}\hat{k}_{(t,s)}}\,. \end{aligned} \tag{11}$$

To see this, e.g. for the case $t = r-1$, choose cochain representatives for the k's. Then

$$\begin{aligned} k_{(r,r-1)}k_{r-1}(x) &= k_r(0, k_{r-1}(x)) \\ &= \pm k_r\delta(0,x) = \pm\delta k_r(0,x) \sim 0\,, \end{aligned}$$

proving the first assertion. For the second assertion, see the proof of Proposition 4.4.

By the very definition of the $C_{(r,s)}$ there is a short exact sequence

$$(12)\qquad 0 \to C_{(r,s)} \xrightarrow{\alpha^s} C_{(r)} \xrightarrow{j^{r-s}} C_{(s)} \to 0$$

with an associated long exact sequence

$$(13)\qquad \ldots \to H_{(r,s)} \xrightarrow{\hat{\alpha}^s} H_{(r)} \xrightarrow{\hat{j}^{r-s}} H_{(s)} \to H_{(r,s)} \to \ldots .$$

(If $s=r-1$, then (11) reduces to the cone sequence (7) of Section 4.)

Now, assume that $\pi_0,\ldots,\pi_r$ are graded $\boldsymbol{Z}_p$ vector spaces (of finite type). Then the string

$$\hat{k}_{(r,r-1)}, \hat{k}_{(r-1,r-2)}, \ldots, \hat{k}_{(2,1)}, \hat{k}_1$$

yields a chain complex of $\mathcal{A}(H_{\boldsymbol{Z}_p}, H_{\boldsymbol{Z}_p})$-modules

$$(14)\qquad D_r \xrightarrow{d_r} D_{r-1} \to \ldots \to D_1 \xrightarrow{d_1} D_0 \to 0 ,$$

d_t being "matrix multiplication" by $\hat{k}_{(t,t-1)}$. The next proposition shows that our definition of operations of r-th kind agrees with that of Maunder's.

Proposition 5.4. *If $\hat{k}_{(t,s)}$ in* (10) *is replaced by $\hat{k}_{(t,s)}(\hat{j}^{t-s-1})^{-1}$, the resulting pyramid is a pyramid of cohomology operations associated with the chain complex* (14) *in the sense of Maunder* [14].

Proof. It is easily seen that Axioms 0–4 (of [14]) are satisfied. One only has to use the exact sequences

$$\ldots \to H_{(t)} \to H_{(t-1)} \xrightarrow{\hat{k}_{t-1}} H_{(1)} \to \ldots ,$$
$$\ldots \to H_{(t-1)} \to H_{(t)} \quad \to \quad H_{(1)} \to \ldots .$$

Proof of Axiom 5. Let Y be a subcomplex of X and let $\hat{x}$ be a class in $H_{(r-1)}(X)$ which restricts to a class $\hat{y}$ in $H_{(r-1)}(Y)$. Suppose that $\hat{k}_{r-1}(\hat{y})=0$, or equivalently, that

$$\hat{k}_{r-1}(\hat{x}) \in \operatorname{Im}(j^*\colon H(X,Y;\pi_{r-1}) \to H(X;\pi_{r-1})) .$$

It then follows from a suitable "cone-sequence" that $\hat{y}$ may be lifted to a class $(y,w)^\wedge$ in $H_{(r)}(Y)$. Let us denote by w' a cochain in $C(X;\pi_{r-1})$ which restricts to w. The cochain $k_r(0, k_{r-1}(x)-(-1)^r\delta w)$ in $C(X;\pi_r)$ is actually a cocycle and represents both $\delta^*\hat{k}_r(y,w)^\wedge$ and a class of $\hat{k}_{(r,r-1)}(j^*)^{-1}\hat{k}_{r-1}(\hat{x})$. We have proved that

$$\delta^*\hat{k}_r(y,w)^\wedge \subseteq \hat{k}_{(r,r-1)}(j^*)^{-1}\,\hat{k}_{r-1}(\hat{x}) .$$

This completes the proof of Axiom 5 and the proof of the proposition.

Next, we shall demonstrate how, by means of $H_{(r)}$, one may derive the spectral sequences of Atyiah–Hirzebruch [3] and Adams [1]. No new results are obtained.

The cone sequences

$$\ldots \to H^{n+r-1}(\cdot\,; \pi_r) \to H^n_{(r)} \to H^n_{(r-1)} \to H^{n+r}(-\,; \pi_r) \to \ldots$$

fit together in an exact couple

$$\begin{array}{ccccccccc}
 & \vdots & & & & \vdots & & & \\
 & \downarrow & & & & \downarrow & & & \\
\ldots \to & H^n_{(r+1)} & \to & H^{n+r+2}(\cdot\,; \pi_{r+1}) & \to & H^{n+1}_{(r+2)} & \to & H^{n+r+4}(\cdot\,; \pi_{r+2}) & \to \ldots \\
 & \downarrow & & & & \downarrow & & & \\
\ldots \to & H^n_{(r)} & \to & H^{n+r+1}(\cdot\,; \pi_r) & \to & H^{n+1}_{(r+1)} & \to & H^{n+r+3}(\cdot\,; \pi_{r+1}) & \to \ldots \\
 & \downarrow & & & & \downarrow & & & \\
\ldots \to & H^n_{(r-1)} & \to & H^{n+r}(\cdot\,; \pi_{r-1}) & \to & H^{n+1}_{(r)} & \to & H^{n+r+2}(\cdot\,; \pi_r) & \to \ldots \\
 & \downarrow & & & & \downarrow & & & \\
 & \vdots & & & & \vdots & & &
\end{array}$$

Assuming π_r to be homogeneous of degree zero, the exact couple reduces to the Atyiah–Hirzebruch spectral sequence

$$(16)\qquad \begin{aligned} E_2^{p,q} &= H^p(\cdot\,; H^q_{(\infty)}(\text{point})) = H^p(\cdot\,; \pi_{-q}), \\ E_2^{p,q} &\overset{p}{\Rightarrow} H^{p+q}_{(\infty)}(\cdot)\,. \end{aligned}$$

The differential d_r in this spectral sequence is the additive relation

$$(17)\qquad \begin{array}{ccccc}
 & & H^{p+q}_{(-q+r)} & \xrightarrow{k(-q+r)} & H^{p+r+1}(\cdot\,; \pi_{-q+r}) \\
 & & \big\downarrow \hat{j}^{r-1} & & \\
H^p(\cdot\,; \pi_{-q}) & \xrightarrow{\hat{\alpha}} & H^{p+q}_{(-q+1)} & &
\end{array}$$

There is a commutative diagram with exact columns

$$(18)\qquad \begin{array}{ccccc}
 & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow \\
 & & H_{(-q+r,\,-q)} & \to & H_{(-q+r)} \\
 & & \big\downarrow \hat{j}^{r-1} & & \big\downarrow \hat{j}^{r-1} \\
H(\cdot\,; \pi_{-q}) & = & H_{(-q+1,\,-q)} & \to & H_{(-q+1)} \\
 & & \big\downarrow & & \big\downarrow \\
 & & H_{(-q+r,\,-q+1)} & = & H_{(-q+r,\,-q+1)}\,, \\
 & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots
\end{array}$$

the left side vertical sequence being the long exact sequence associated with the short exact sequence of c.g. functors

$$0 \to C_{(-q+r,\,-q+1)} \to C_{(-q+r,\,-q)} \to C_{(-q+1,\,-q)} \to 0\,.$$

Thus we have

$$d_r = \hat{k}_{(-q+r,\,-q)}(\hat{j}^{r-1})^{-1}\,. \tag{19}$$

This result was obtained by Maunder in [15].

If we apply the "functor" $\mathfrak{a}(H_{(r)},\cdot)$ to the exact couple (15), we get a new exact couple (Theorem 4.2) and therefore a spectral sequence with

$$\begin{aligned} E_1 &= \mathfrak{a}(H_{(r)}, H(\cdot\,;\pi_s))\,,\\ E_\infty &= \mathfrak{a}(H_{(r)}, H_{(r)})\,. \end{aligned} \tag{20}$$

We shall not treat this spectral sequence in detail, but only mention that results on the differentials may easily be derived.

Finally we shall "functionalize" the r-th kind operations from Definition 5.3. Let $f\colon X \to Y$ be a CSS-map. Then the functionalization of $\hat{k}_r(\hat{j}^{r-1})^{-1}$ is defined to be the additive relation $\hat{\alpha}^{-r} f^* \hat{j}^{-r}$ or, in detail,

$$\begin{array}{ccccccc} H[k_1,\ldots,k_r](Y) & \xrightarrow{\hat{j}} \;\cdots\; \xrightarrow{\hat{j}} & H[k_1](Y) & \xrightarrow{\hat{j}} & H_{\pi_0}(Y) \\ \big\downarrow{\scriptstyle f^*} & & & & \big\downarrow \\ H[k_1,\ldots,k_r](X) & \xleftarrow{\hat{\alpha}} H[k_{(2,1)},\ldots,k_{(r,1)}] \xleftarrow{\hat{\alpha}} & \cdots & \xleftarrow{\hat{\alpha}} & H_{\pi_r}(X)\,. \end{array}$$

Information about indeterminacy and domain-of-definition is given by the sequences (5.13).

The following relations are easily checked on cochain level (notation as in (10))

$$\hat{k}_{(p,q)} \circ \hat{\alpha} = \hat{k}_{(p,q+1)} \tag{21}$$

and

$$\hat{j} \circ \hat{\alpha} = \hat{\alpha} \circ \hat{j}\,. \tag{22}$$

Using the definitions just given of r-th kind operation (Definition 5.3) and of functionalized operation together with (21) and (22), one can get a family of Peterson–Stein-like formulas using only the well-known rules for additive relations. We shall give only one such relation; let $s \leqq r-1$

$$\begin{aligned} f^* \hat{k}_r \hat{j}^{-(r-1)} &\subseteq \hat{k}_r \hat{j}^{-(r-1)+s} f^* \hat{j}^{-s} \\ &\supseteq \hat{k}_{(r,s)} \hat{\alpha}^{-s} \hat{j}^{-(r-1)+s} f^* \hat{j}^{-s} = \hat{k}_{(r,s)} \hat{j}^{-(r-1)+s} \hat{\alpha}^{-s} f^* \hat{j}^{-s}\,. \end{aligned} \tag{23}$$

The left hand side of this relation is an r-th kind operation followed by f^*; the right hand side is a functionalized s-th kind operation followed by an operation of kind $r-s$. The inclusion relations (23) then express that these two are congruent modulo the indeterminacy of the relation in the second line of (23).

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