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ON THE STABILITY OF LINEAR DIFFERENTIAL EQUATIONS IN SPACES WITH AN INDEFINITE METRIC

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Introduction.

The purpose of this paper is to investigate the stability of the differential equation

$$u'(t) = A(t)u(t) .$$

Here, for every real, non-negative t, the function $u(t)$ belongs to a separable Hilbert space H, and the periodical operator $A(t)$ has domain on and range in H. Besides some regularity conditions which guarantee the existence of a unique solution, the operator $A(t)$ satisfies the symmetry condition

$$DA(t) + A^*(t)D = 0 ,$$

where D is a constant operator in $[H]$.

We will define two kinds of stability: the equation is stable if for every initial value its solution is bounded in norm, uniformly in t; the equation is weakly G-stable, if $|(Du(t),v)|$ is bounded uniformly in t for v in the set $G \subset H$.

The paper is divided into four sections. In the first one, we quote some basic facts about the differential equation; in the second one we introduce a sesquilinear form $[x,y]=(Dx,y)$ on the space, following [6], [8] and [9]. We obtain some analogues of Bessel's inequality and Parseval's relation.

In the third section we make a construction extending a method originally designed by G. Borg [3]. This construction is fundamental for the last section in which our main theorem (Theorem 4.1.) about G-weak stability is proved. The result extends an investigation by Boman [2]. Let $A(t)=A+B(t)$, where A is a constant operator of a special kind and $B(t)$ is a compact operator belonging to the interior of the unit sphere in the sense of Proposition 4.1. Then there exists a non-empty set $G \subset H$ so that the equation is weakly G-stable. The result is best possible, i.e. a perturbation $B(t)$ on or outside the unit sphere may give instability.

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1. Basic facts about the differential equation.

Let H denote a separable Hilbert space and R^+ the set of positive reals, including zero. Consider the equation

$$u'(t) = A(t)u(t)\,, \tag{1.1}$$

where $A(t)$ is a periodic function from R^+ to $[H]$, the Banach space of continuous endomorphisms of H, such that $A(t)$ is Bochner integrable with respect to the measure $\mu(t)=t$ in every finite subinterval, and, for almost every $t \in R^+$, $A(t)$ is a compact operator. Assume for the sake of simplicity that $A(t)=A(t+1)$. Here $u'(t)$ denotes the strong derivative of a function $u(t)$ from R^+ to H.

For every initial value $u_0 \in H$ the equation (1.1) has a unique continuous solution $u(t)$ such that $u(0)=u_0$. We define the solution operator $U(t)$ by the relation

$$u(t) = U(t)u(0)$$

and observe that the inverse operator $U^{-1}(t)$ defined on the range of $U(t)$ exists for all $t \in R^+$ (see Massera–Schäffer [10]). It follows by the uniqueness of the solution that

$$U(n+t) = (U(1))^n U(t), \qquad n=1,2,\ldots\,.$$

Set for convenience $U(1)=U$.

We further assume that there exists a continuously invertible operator $D \in [H]$, not depending on t, satisfying

$$DA(t) = -A^*(t)D\,. \tag{1.2}$$

Here $A^*(t)$ denotes the adjoint of $A(t)$. We observe that $A(t)$ is a closed operator and that D hence can be chosen self-adjoint. This can be done by exchanging D for $\frac{1}{2}(D+D^*)$ if D is not already self-adjoint provided that $D+D^* \neq 0$. Otherwise D is exchanged for iD. Hence, from now on we assume D self-adjoint. It is well known that the solution operator U corresponding to the equation (1.1) with the symmetry condition (1.2) has the property

$$DU = (U^*)^{-1}D\,. \tag{1.3}$$

2. Indefinite metric.

A complex linear space X is said to be a space with G-metric, if there exists a sesquilinear form $G(x,y)$ defined on X [6]. We can now define various topologies on X. We will consider the case where $X=H$ with its ordinary topology. The sesquilinear form, denoted by $[x,y]$ is given by

$$(2.1) \qquad [x,y] = (Dx,y)$$

where $D \in [H]$ is the operator, mentioned above. The form $[x,y]$ is thus hermitian. The elements $x,y \in H$ are called *D-orthogonal* if $[x,y]=0$. A set of vectors $x_i \in H$ is called *D-orthonormal* if $[x_i,x_j]= \pm\delta_{ij}$, where

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i=j, \end{cases} \qquad i,j=1,2,\ldots .$$

An operator $V \in [H]$, satisfying the relation $[Vx,y]=[x,Vy]$ for all $x,y \in H$ is called *D-self-adjoint* or D-s.a. Analogously, an operator $U \in [H]$, satisfying the relation $[Ux,Uy]=[x,y]$ for all $x,y \in H$ is called *D-isometric*. It is easy to see that these conditions are equivalent to $DV=V^*D$ and $U^*DU=D$ respectively. Let $\mathrm{sp}(A)$ denote the spectrum of an operator A. It is a standard observation that if U is D-isometric then $\mathrm{sp}(U)=(\overline{\mathrm{sp}(U)})^{-1}$ and if V is D-self-adjoint then $\mathrm{sp}(V)=\overline{\mathrm{sp}(V)}$.

D-positiveness.

We will now study the implications of the concept D-positiveness.

DEFINITION 2.1. An operator V, defined on H, is said to be *D-positive* if $[Vx,x] \geqq 0$ for all $x \in H$.

PROPOSITION 2.1. *Let V be an invertible, D-s.a. and D-positive operator. Then $[Vx,x]=0$ implies $x=0$.*

PROOF. First we observe that DV is positive and symmetric. Then DV has a symmetric square root R. (See e.g. Riesz–Nagy [11, p. 250].) Now

$$0 = [Vx,x] = (R^2x,x) = \|Rx\|^2 .$$

Then

$$DVx = R^2x = R(Rx) = 0$$

and consequently $x=0$.

COROLLARY. *If x is an eigenvector of V and λ is the corresponding eigenvalue, then $[x,x] \neq 0$ and λ is real.*

With the assumptions of Proposition 2.1 we can hence introduce a new metric generated by the inner product

$$\{x,x\} = [Vx,x] .$$

We observe that V is a self-adjoint operator in this metric.

It is easy to see that eigenvectors of a D-positive and D-s.a. operator belonging to different eigenvalues are D-orthogonal. For the eigenvectors of V, D-orthogonality is equivalent to orthogonality with respect to $\{\cdot,\cdot\}$,

PROPOSITION 2.2. *Let V be a D-s.a. and D-positive operator, having a D-orthonormal set of eigenvectors $(\varphi_i)_{i=1}^{\infty}$ with the corresponding eigenvalues $(\lambda_i)_{i=1}^{\infty}$; then a generalized Bessel's inequality holds true:*

$$[Vx,x] \geqq \sum_{i=1}^{\infty} |\lambda_i| \, |[x,\varphi_i]|^2 . \tag{2.2}$$

PROOF. Consider

$$y_n = x - \sum_{i=1}^{n} [x,\varphi_i]\,[\varphi_i,\varphi_i]\varphi_i .$$

The result follows by direct calculation, since $[Vy_n,y_n] \geqq 0$ for all n.

We will now state an analogue of Parseval's relation.

PROPOSITION 2.3. *Let V be an invertible, D-s.a., and D-positive operator with a countable spectrum. Then there exists a D-orthonormal basis $(\varphi_i)_{i=1}^{\infty}$ of H consisting of eigenvectors of V, and*

$$[Vx,y] = \sum_{i=1}^{\infty} |\lambda_i| \, [x,\varphi_i]\,[\varphi_i,y] \quad \textit{for all } x,y \in H ,$$

where $(\lambda_i)_{i=1}^{\infty}$ are the corresponding eigenvalues.

PROOF. The existence of a basis $(\psi_i)_{i=1}^{\infty}$ of H, consisting of eigenvectors of V and orthogonal with respect to the inner product $\{\cdot,\cdot\}$ is clear by Dunford–Schwartz [4, p. 905]. Thus

$$x = \sum_{i=1}^{\infty} \{x,\psi_i\}\psi_i, \qquad y = \sum_{i=1}^{\infty} \{y,\psi_i\}\psi_i$$

and

$$[Vx,y] = \{x,y\} = \sum_{i=1}^{\infty} |\lambda_i|^2 \, [x,\psi_i]\,[\psi_i,y]$$

$$= \sum_{i=1}^{\infty} |\lambda_i| \, [x,\varphi_i]\,[\varphi_i,y] ,$$

where $\varphi_i = |[\psi_i,\psi_i]|^{-\frac{1}{2}}\psi_i$.

COROLLARY. *Let V be as above and denote its Cayley transform by U. Then*

$$|[U^m x, Vy]| \leqq [Vx,x]^{\frac{1}{2}}\,[Vy,y]^{\frac{1}{2}} , \qquad m = 1,2,3,\ldots .$$

PROOF. From

$$\{U^m x,y\} = \sum_{\nu=1}^{\infty} \mu_\nu{}^m \{x,\psi_\nu\}\{\psi_\nu,y\} ,$$

where

$$\mu_\nu = (i+\lambda_\nu)(i-\lambda_\nu)^{-1}, \qquad V\psi_\nu = \lambda_\nu \psi_\nu ,$$

it follows that

$$\begin{aligned}|[U^m x, Vy]| = |\{U^m x, y\}| &\leqq \sum_{\nu=1}^{\infty} |\{x,\psi_\nu\}\{\psi_\nu, y\}| \\ &\leqq \left(\sum_{\nu=1}^{\infty} |\{x,\psi_\nu\}|^2 \sum_{\nu=1}^{\infty} |\{\psi_\nu, y\}|^2\right)^{\frac{1}{2}} \\ &= ([Vx,x]\,[Vy,y])^{\frac{1}{2}} .\end{aligned}$$

3. Perturbation of the differential equation.

Basic facts.

We will now study the differential equation (1.1) by applying the Floquet procedure. Let μ be an eigenvalue of U, and x a corresponding eigenvector. Thus

$$Ux = \mu x \quad \text{and} \quad U(t+1) = U(t)\,Ux = \mu\, U(t)x .$$

Now define

$$w(t) = \exp(-t \log \mu)\cdot U(t)x ,$$

where $-\pi < \arg \mu \leqq \pi$. Then $w(t)$ has period 1. Since $w(t)\exp(t \log \mu)$ is a solution of (1.1) we obtain

$$w'(t) + \big(i\zeta I - A(t)\big) w(t) = 0, \tag{3.1}$$

where $\log \mu = i\zeta$, $-\pi < \operatorname{Re}\zeta \leqq \pi$.

Let us now make a perturbation of the equation (1.1):

$$u'(t) = \big(A(t)+B(t)\big) u(t) , \tag{3.2}$$

where $B(t)$ satisfies the same regularity conditions as $A(t)$. By the previous transformation we obtain

$$w'(t) + \big(i\zeta I - A(t)\big) w(t) = B(t) w(t) . \tag{3.3}$$

We now turn to some basic definitions.

Definition 3.1. We define $B_2(H)$ as the set of functions $x(\cdot): t \to x(t)$, $x(t) \in H$ for $t \in [0,1]$, which are strongly measurable with respect to the Lebesgue measure, and for which $\int_0^1 \|x(t)\|^2 dt < \infty$.

We observe that $B_2(H)$ becomes a Hilbert space with the norm

$$\|x(\cdot)\| = \left(\int_0^1 \|x(t)\|^2\, dt\right)^{\frac{1}{2}} ,$$

if elements differing only on sets of measure zero are identified (see Hille–Phillips [7, p. 88]).

For a linear operator $T(t)$ we define $T(\cdot): B_2(H) \to B_2(H)$ as the function

$$T(\cdot)x(\cdot):\ t \to T(t)x(t) \quad \text{if} \quad x(t) \in D(T(t))\,.$$

The norm of $T(\cdot)$ is

$$\|T(\cdot)\| = \sup\{\|T(\cdot)x(\cdot)\| :\ \|x(\cdot)\|=1,\, x(\cdot) \in D(T(\cdot))\}.$$

Let K be an invertible operator $H \to H$ and

$$\Gamma_\zeta(t) = K\left(\frac{d}{dt} + (i\zeta I - A(t))\right), \qquad -\pi < \operatorname{Re}\zeta \leqq \pi\,.$$

DEFINITION 3.2. We denote by $\Phi_K(A,\zeta)$ the function

$$\Phi_K(A,\zeta) = \inf\{\|\Gamma_\zeta(\cdot)w(\cdot)\| :\ w(0)=w(1),\, \|w(\cdot)\|=1\}\,.$$

$\Phi_K(A,\zeta)$ is not necessarily defined for all ζ, $-\pi < \operatorname{Re}\zeta \leqq \pi$, when the domain of K is not all of H. When it is not defined, assign $+\infty$ as its value.

PROPOSITION 3.1. *The following properties of $\Phi_K(A,\zeta)$ are valid:*

(i) $\Phi_K(A,\zeta) \geqq 0$.

(ii) *If* $\exp(i\zeta)$ *belongs to the point spectrum of U, where U is the solution operator corresponding to $A(t)$, then $\Phi_K(A,\zeta)=0$. If not, no general conclusion can be drawn.*

(iii) *If $D(K)=H$, then $\Phi_K(A,\zeta)$ is an upper semicontinuous function of ζ; if $K \in [H]$, it is continuous.*

(iv) *Let H be a direct orthogonal sum of subspaces H_1 and H_2 such that K is completely reduced by this decomposition, and $\Phi_K^{(\nu)}(A,\zeta)$ denotes the restriction of $\Phi_K(A,\zeta)$ to H_ν, $\nu=1,2$. Then*

$$\Phi_K(A,\zeta) = \min_\nu \Phi_K^{(\nu)}(A,\zeta), \qquad \nu = 1,2\,.$$

PROOF. The verification of (i) and the first part of (ii) is immediate.

By means of an example (involving the operators Λ and $K(\Lambda)$ to be constructed in Theorem 3.1), we shall verify the second part of (ii), by observing that, in that case 1 does not belong to the point spectrum of U but

$$\Phi_I(\Lambda,0) = 0, \qquad \Phi_{K(\Lambda)}(\Lambda,0) \geqq 1\,.$$

(iii) is proved by direct calculation, and for the proof of (iv) we can apply the method in [2].

A necessary condition that all eigensolutions of the equation (1.1) with the symmetry condition (1.2) are uniformly bounded in t is that $\mathrm{sp}(U)$ is contained in the unit circle S^1 (Almkvist [1]).

The fundamental problem.

For a given operator K we define

$$\mathscr{M}(K) = \{B(t) : B(t) \text{ is Bochner integrable, } B(t+1)=B(t) \text{ and } \|K(\cdot)B(\cdot)\|<1\}$$

and turn to the following problem: Find a constant invertible operator K such that between every two consecutive points on the real axis, ϱ_ν and $\varrho_{\nu+1}$ corresponding to points $\exp(i\varrho_\nu)$, $\exp(i\varrho_{\nu+1})$ of the point spectrum of U, there is at least one real point ϱ', so that $\Phi_K(A,\varrho') \geqq 1$. We remark, that since $U-I$ is compact (see e.g. Almkvist [1]), the point spectrum of U is isolated with $\mu=1$ as the only point of accumulation.

Assume that an operator K solves the problem. Then, if $B(t)$ is compact a.e. and belongs to $\mathscr{M}(K)$, and if $\exp(i\varrho)$ lies in the point spectrum of U corresponding to the perturbed equation, we obtain

$$\Phi_K(A,\varrho) \leqq \|KB(\cdot)\| < 1 .$$

Consequently ϱ' in the problem above does not belong to the point spectrum of U. This will be of importance in the perturbation theory of Section 4. We also remark that the choice of the constant $=1$ is immaterial.

For the solution of the problem, we proceed as follows: Given two neighbour points ϱ_ν, $\varrho_{\nu+1}$, $\nu=1,2,\ldots$, as before. Put

$$M(K,\nu) = \sup_{\varrho\in I_\nu} \Phi_K(A,\varrho) ,$$

where I_ν denotes the closed interval between the two points. We must suppose that there is no interval I_ν where $\Phi_K(A,\varrho)\equiv 0$, otherwise the problem is not solvable. We then determine K such that $M(K,\nu)\geqq 1$ for $\nu=1,2,\ldots$. We solve the problem in two special cases which are fundamental in perturbation theory.

THEOREM 3.1. *Let $(e_\nu)_{\nu=1}^{\infty}$ be an orthonormal basis in H, and Λ a compact operator, which is represented as a matrix with boxes Λ_ν along the principal diagonal and* 0 *in all other positions,*

$$\Lambda_\nu = \begin{pmatrix} 0 & a_\nu \\ -a_\nu & 0 \end{pmatrix},$$

where $a_\nu > 0$ and $a_\nu \neq n\pi$, n integer, $\nu=1,2,\ldots$. With $A(t)=\Lambda$, the problem stated above has a solution $K(\Lambda)$, and $K^{-1}(\Lambda)$ is compact.

PROOF. We decompose H into a direct sum of subspaces generated by $e_{2\nu-1}$ and $e_{2\nu}$ for $\nu=1,2,\ldots$ and apply the result by Giertz [5] to obtain

$$\Phi_I(\Lambda,\varrho) = \inf_\nu \min_n |a_\nu - |\varrho+2\pi n||, \qquad \nu=1,2,\ldots; n=0, \pm 1,\ldots.$$

It is clearly no restriction to assume that

$$a_1 > a_2 > \ldots > a_\nu > \ldots > 0,$$

because there is only one point of accumulation, and that $a_\nu < \pi$, $\nu = 1,2,\ldots$. Then

$$\Phi_I(\Lambda,\varrho) = \inf_\nu |a_\nu - |\varrho||.$$

By symmetry it is sufficient to consider $\varrho \in [0,\pi]$, and we define

$$f_\nu(\varrho) = |a_\nu - \varrho|.$$

Thus

$$M(I,\nu) = \max_{a_{\nu+1} \leqq \varrho \leqq a_\nu} \min(f_\nu(\varrho), f_{\nu+1}(\varrho)) = \tfrac{1}{2}(a_\nu - a_{\nu+1}).$$

We now determine k_ν, $\nu=1,2,\ldots$, such that the operator K, which is represented as a matrix with boxes

$$K_\nu = k_\nu^{-1}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

along the principal diagonal and 0 in the other positions, solves the problem. We obtain

$$\Phi_K(\Lambda,\varrho) = \inf_\nu |k_\nu|^{-1}|a_\nu - \varrho|.$$

Hence it is sufficient to assume $k_\nu > 0$, $\nu=1,2,\ldots$. Thus

$$M(K,\nu) = \frac{a_\nu - a_{\nu+1}}{k_\nu + k_{\nu+1}},$$

and the solution of the system

$$k_\nu + k_{\nu+1} \leqq a_\nu - a_{\nu+1}, \qquad \nu=1,2,\ldots, \tag{3.4}$$

gives the desired result. Define

$$d_1 = a_1 - a_2,$$
$$d_\nu = \min(a_{\nu-1} - a_\nu, a_\nu - a_{\nu+1}), \qquad \nu = 2,3,\ldots.$$

Then $d_\nu \leqq a_\nu - a_{\nu+1}$ and d_ν tends monotonously to zero, as $\nu \to \infty$. The system

$$k_\nu + k_{\nu+1} = d_\nu, \quad k_\nu > 0, \qquad \nu=1,2,\ldots, \tag{3.5}$$

has the unique solution

$$k_n = \sum_{\nu=n}^{\infty} (-1)^{\nu+n} d_\nu, \qquad n=1,2,\ldots .$$

Hence this is a solution of (3.4).

We adjust the value of $\Phi_K(\Lambda,\pi)$, substituting k_1 by

$$k_1' = \alpha \min(k_1, \pi - a_1), \qquad 0<\alpha<1 .$$

The compactness of K^{-1} is obvious.

We observe that the previous choice of the operator K gives

$$\Phi_K(\Lambda,0) = \inf_\nu \frac{a_\nu}{k_\nu} \geqq \inf_\nu \frac{a_\nu}{d_\nu} \geqq \inf_\nu \frac{a_\nu}{a_\nu - a_{\nu+1}} \geqq 1 .$$

Remark. If $(a_\nu - a_{\nu+1})_{\nu=1}^{\infty}$ is monotonously decreasing sequence, then, with $(k_\nu)_{\nu=1}$ determined by the system (3.5), $M(K,\nu)=1$. In this case we also have another simple solution of the system (3.4), viz. $k_\nu = (a_\nu - a_{\nu+1})^{-1}$, but then $M(K,\nu) > 1$, $\nu=1,2,\ldots$. In the general case (3.5) yields a best possible solution of (3.4) in the sense that $M(K,\nu)=1$ for countably many ν.

Theorem 3.2. *Let $(e_\nu)_{\nu=1}^{\infty}$ be an orthonormal basis in H and Ξ a compact operator, which is represented as a diagonal matrix with entries ia_ν, where a_ν is real and $\neq n\pi$, n integer, $\nu=1,2,\ldots$. With $A(t)=\Xi$, the problem stated above has a solution $K(\Xi)$, and $K^{-1}(\Xi)$ is compact.*

Proof. Either

(i) $\operatorname{card}\{a_\nu : a_\nu > 0\} = \operatorname{card}\{a_\nu : a_\nu < 0\}$

or

(ii) $\operatorname{card}\{a_\nu : a_\nu > 0\} \neq \operatorname{card}\{a_\nu : a_\nu < 0\}$.

In case (i) it is no restriction to assume

$$a_1 \geqq a_3 \geqq a_5 \geqq \ldots > 0$$

and

$$a_2 \leqq a_4 \leqq a_6 \leqq \ldots < 0 .$$

By the same method as in Theorem 3.1 we obtain

$$\Phi_I(\Xi,\varrho) = \inf_\nu \min_n |a_\nu - (\varrho + 2\pi n)| .$$

Now consider the elements with even and odd indices separately. In Theorem 3.1, by the symmetry, we only used the positive elements for the construction, and we might equally well have used only the negative ones. Hence the same method is applicable here, too. In case (ii) assume for the sake of simplicity that

$$a_1 \geqq a_2 \geqq \ldots \geqq a_N > 0, \qquad a_\nu < 0 \text{ for } \nu > N .$$

The infinite part is treated as above, and the finite problem is solved by the system

$$\text{(3.6)} \qquad \begin{cases} k_\nu + k_{\nu+1} = d_\nu, & \nu = 1,2,\ldots, N-1 , \\ k_N = \alpha a_N, & 0 < \alpha < 1 , \end{cases}$$

as before by a final correction of k_1. Here the notation conforms with the one used in Theorem 3.1.

Proposition 3.2. *Let A be the operator Λ or Ξ. Then, for every $\nu = 1,2,\ldots$ there exists an open disc $C_\nu \subset \{\zeta : -\pi < \operatorname{Re}\zeta \leqq \pi\}$ containing a_ν so that*

$$C_\nu \cap C_{\nu'} = \emptyset \quad \textit{for} \quad \nu \neq \nu'$$

and

$$\Phi_K(A,\zeta) \geqq 1 \quad \textit{for} \quad \zeta \notin \textstyle\bigcup_\nu C_\nu .$$

Here $\emptyset$ denotes the empty set.

Proof. Between any two neighbour points $a_\nu, a_{\nu+1}$, there exists a real ϱ_ν such that $\Phi_K(A,\varrho_\nu) \geqq 1$ as proved in the preceding two theorems. We choose

$$C_\nu = \{\zeta : |\zeta - \tfrac{1}{2}(\varrho_\nu + \varrho_{\nu+1})| < \tfrac{1}{2}|\varrho_\nu - \varrho_{\nu+1}|\} .$$

According to the determination of K we can also apply this construction to the points nearest π or $-\pi$, respectively 0, by taking the neighbour points used above as π or $-\pi$, respectively 0. Now $\Phi_K(A,\zeta)$ can be calculated by the same methods as were used for $\Phi_K(A,\zeta)$, and we obtain

$$\Phi_K(\Lambda,\zeta) = \inf_\nu k_\nu^{-1} \min(|\zeta + a_\nu|, |\zeta - a_\nu|)$$

and

$$\Phi_K(\Xi,\zeta) = \inf_\nu k_\nu^{-1} |\zeta - a_\nu| ,$$

$\nu = 1,2,\ldots$.

Let $\zeta \notin \bigcup_\nu C_\nu$, and for an arbitrary C_ν, let ϱ be the real point in ∂C_ν which has minimal distance from a_ν. Then

$$k_\nu^{-1}|\zeta - a| \geqq k_\nu^{-1}|\varrho - a_\nu| = \Phi_K(A,\varrho) \geqq 1 ,$$

and consequently $\Phi_K(A,\zeta) \geqq 1$. Here ∂C_ν denotes the boundary of C_ν.

4. Stability and weak G-stability.

Definition 4.1. The equation (1.1) is *stable* if all solutions are uniformly bounded on R^+.

For a discussion of consequences of this property in connection with indefinite metric we refer to Kreĭn [9].

LEMMA 1. *Suppose that iA is self-adjoint and $B(t)$ is periodic with period 1, Bochner integrable and has domain H. Then $U(\theta,t)$, satisfying the equation*

$$U'(t)-AU(t) = \theta B(t)U(t), \quad U(0)=I, \quad 0 \leqq \theta \leqq 1,$$

is an analytic function of θ.

PROOF. Standard methods show that

$$U(\theta,t) = \sum_{n=0}^{\infty} \theta^n U_n(t),$$

where

$$U_0(t) = \exp(tA),$$

$$U_n(t) = \int_0^t [\exp(t-s)A]\, B(s)\, U_{n-1}(s)\, ds, \qquad n \geqq 1,$$

and the series is absolutely convergent in the uniform operator topology.

LEMMA 2. *Assume that $-1 \notin \mathrm{sp}(U)$, where U corresponds to the equation* (1.1) *with the symmetry condition* (1.2), *then its Cayley transform $V=i(U-I)(U+I)^{-1}$ is compact and D*-s.a.

PROOF. Since $U-I$ is compact and $(U+I)^{-1}$ is continuous the first assertion is clear. The second one is obvious by the D-isometry of U.

PROPOSITION 4.1. *Let $A(t)=A+B(t)$, where either*

$$A = \Lambda \quad \text{and} \quad B(t) \in \mathcal{M}(K(\Lambda))$$

or

$$A = \Xi \quad \text{and} \quad B(t) \in \mathcal{M}(K(\Xi)).$$

Further assume that A has simple eigenvalues,

$$DA = -A^*D,$$

and $B(t)$ is compact with domain H, satisfying

$$DB(t) = -B^*(t)D \quad \text{a.e.}\,.$$

Denote $U(\theta,1)$ by U_θ and its Cayley transform by V_θ.

Then the spectrum of V_1 with zero excluded consists of simple real eigenvalues, and V_θ^{-1} exists, $0 \leqq \theta \leqq 1$.

REMARK. With $\|B(\cdot)\|_K = \|KB(\cdot)\|$ we obtain $\mathcal{M}(K(A))$ as the interior of the unit sphere in the normed linear space of operators on H, satisfying the previous regularity conditions and $\|KB(\cdot)\| < \infty$.

PROOF OF PROPOSITION 4.1. Direct calculation yields, for $A = \Lambda$, U_0 represented as a matrix with boxes

$$\begin{pmatrix} \cos a_\nu & \sin a_\nu \\ -\sin a_\nu & \cos a_\nu \end{pmatrix}, \qquad \nu = 1, 2, \dots ,$$

along the principal diagonal and 0 in all other positions, and for $A = \Xi$ as a diagonal matrix with entries $\exp(ia_\nu)$, $\nu = 1, 2, \dots$. Hence V_0 is represented as a matrix with boxes

$$i \begin{pmatrix} 0 & \tan \frac{1}{2}a_\nu \\ -\tan \frac{1}{2}a_\nu & 0 \end{pmatrix}$$

along the principal diagonal and 0 in all other positions, respectively as a diagonal matrix with entries $-\tan\frac{1}{2}\alpha_\nu$. We observe that the spectrum of V_0 with zero excluded consists of simple real eigenvalues. By the construction of C_ν, $\nu = 1, 2, \dots$, it is immediately seen that $\zeta \notin \bigcup_\nu C_\nu$ implies $e^{i\zeta} \notin \mathrm{sp}(U_\theta)$, that is,

$$f(\zeta) = i\,\frac{\exp(i\zeta) - 1}{\exp(i\varrho) + 1} = -\tan \tfrac{1}{2}\zeta \notin \mathrm{sp}(V_\theta)\,,$$

where $0 \leqq \theta \leqq 1$. We observe that $\Omega_\nu = f(C_\nu)$ are symmetric with respect to the real axis. Consequently $\partial\Omega_k \cap \mathrm{sp}(V_\theta) = \emptyset$, $k = 1, 2, \dots$, and for $\lambda \in \partial\Omega_k$

$$R_\lambda(\theta) = (\lambda I - V_\theta)^{-1}$$

is an analytic function of λ and θ separately. Thus the projections

$$E_k(\theta) = \frac{1}{2\pi i} \int_{\partial\Omega_k} R_\lambda(\theta)\, d\lambda$$

are continuous for $0 \leqq \theta \leqq 1$, and we can use the same argument as Almkvist [1] (which depends on [4, Lemma VII.6.7]), to conclude that there is exactly one simple eigenvalue λ_k of V_1 in each Ω_k. Since V_1 is D-s.a. and Ω_k is symmetric with respect to the real axis for all k, the eigenvalues λ_k are real. By the construction of $K(A)$, $\Phi_K(A, 0) \geqq 1$. Hence, according to Proposition 3.1 (ii), $\lambda = 0$ is no eigenvalue of V_θ, $0 \leqq \theta \leqq 1$, which proves the last assertion.

REMARK. A self-adjoint operator D satisfying

$$DA = -A^*D, \qquad A = \Lambda \text{ respectively } A = \Xi,$$

is represented as a matrix with boxes

$$\begin{pmatrix} \alpha_\nu & i\beta_\nu \\ -i\beta_\nu & \alpha_\nu \end{pmatrix}, \qquad \alpha_\nu, \beta_\nu \text{ real},$$

along the principal diagonal, and 0 in all other positions, respectively as a diagonal matrix with real entries γ_ν, where $\nu = 1, 2, \ldots$. D-positiveness of V_0 is equivalent to the conditions

$$\min(\beta_\nu + \alpha_\nu, \beta_\nu - \alpha_\nu) > 0,$$

respectively $$a_\nu \gamma_\nu < 0, \quad \nu = 1, 2, \ldots .$$

Let $(e_\nu)_{\nu=1}^\infty$ be the same orthonormal basis of H as in theorems 3.1 and 3.2. For $n = 1, 2, \ldots$ the linear subspace which is generated by $\{e_1, e_2, \ldots, e_n\}$ is denoted by H_n; the space $\mathscr{H}_n$ is H_n regarded as an n-dimensional space and $\iota_n : \mathscr{H}_n \to H_n$ denotes the corresponding isomorphic and isometric embedding. The same symbols are used for the inner product and norm in $\mathscr{H}_n$ as in H. The projection $P_n : H \to H_n$ is defined by

$$P_n\left(\sum_{i=1}^\infty \alpha_i e_i\right) = \sum_{i=1}^n \alpha_i e_i .$$

For operators $T : H \to H$ we set

$$\iota_n^{-1} P_n T \iota_n = \mathscr{T}_n : \mathscr{H}_n \to \mathscr{H}_n ,$$

and the corresponding embedded operator is

$$T_n = P_n T P_n = \iota_n \mathscr{T}_n \iota_n^{-1} P_n ,$$

mapping $H \to H_n$.

LEMMA 3. *Consider the equation* (1.1) *and denote the solution operator corresponding to* $\mathscr{A}_n(t)$ *by* $\mathscr{U}_n$. *The operator* $U_n = \iota_n \mathscr{U}_n \iota_n^{-1} P_n$ *is the embedded operator. Then, as* $n \to \infty$,

(i) $A_n(t) \to A(t)$ *a.e. in the uniform operator topology,*

(ii) $U_n - I_n \to U - I$ *in the same topology as above, and hence* $U_n x \to Ux$ *in norm,* $x \in H$.

PROOF. (i) is clear by the compactness of $A(t)$ (see e.g. [11, p. 189]). For the proof of (ii) we use the standard methods of Lemma 1.

For $x \in H$ we define

$$x_n = \iota_n^{-1} P_n x \in \mathscr{H}_n .$$

If the operator D is chosen according to the remark after Proposition 4.1 we obtain

$$DP_n = P_nD \quad \text{for } n=0,2,4,\ldots, \text{ respectively } n=0,1,2,\ldots\,,$$

hence $D_nx=DP_nx$. The definition

$$[x_n,y_n] = (\mathscr{D}_nx_n,y_n)$$

yields

$$[x_n,y_n] = [P_nx,P_ny]\,.$$

Let A be an operator with domain H. Then, for $x,y \in H$

$$[\mathscr{A}_nx_n,y_n] = [A_nx,y]\,.$$

Proposition 4.2. *Consider the equation* (1.1) *with the symmetry condition* (1.2), *where* $A(t)$ *satisfies the assumptions of Proposition* 4.1. *We denote the solution operator* $\mathscr{U}_n(\theta,1)$ *corresponding to* $\mathscr{A}_n+\theta\mathscr{B}_n(t)$ *by* $\mathscr{U}_{\theta n}$, *where* $0\leqq\theta\leqq 1$ *and* $n=1,2,\ldots$. *Further*

$$\begin{aligned} U_{\theta n} &= \iota_n\mathscr{U}_{\theta n}\iota_n^{-1}P_n\,,\\ \mathscr{V}_{\theta n} &= i(\mathscr{U}_{\theta n}-\mathscr{I}_n)(\mathscr{U}_{\theta n}+\mathscr{I}_n)^{-1}\,,\\ V_{\theta n} &= \iota_n\mathscr{V}_{\theta n}\iota_n^{-1}P_n\,. \end{aligned}$$

Then

(i) $DP_n=P_nD$, *and* $\mathscr{V}_{\theta n}$ *exists and is* $\mathscr{D}_n$-s.a., *and* $\operatorname{sp}(\mathscr{V}_{\theta n})$ *consists of simple eigenvalues* $\neq 0$;

(ii) $V_{\theta n}x \to V_\theta x$ *in norm as* $n\to\infty$, *where* n *is even in the case* $A=\Lambda$;

(iii) *if* V_0 *is D-positive, then* $\mathscr{V}_{1n}$ *is* $\mathscr{D}_n$*-positive and* V_1 *is D-positive.*

Proof. The first property in (i) is clear from the choice of the operator D. Thus

$$A_n^*(t)D_n = -D_nA_n(t)$$

and hence

$$\mathscr{D}_n\mathscr{A}_n(t) = -\mathscr{A}_n^*(t)\mathscr{D}_n\,.$$

We observe that $B(t)\in\mathscr{M}(K(A))$ implies that $B_n(t)\in\mathscr{M}(K(A))$. With $\Phi_K^{(n)}(A,\zeta)$ defined as $\Phi_K(A,\zeta)$ extended only over the numbers a_ν corresponding to $\mathscr{H}_n$ we use Proposition 4.1 and the finite dimensionality to conclude that the second part of (i) holds true.

(ii) Direct calculation yields

$$(U_{\theta n}+I_n)(V_{\theta n}-V_\theta) = i[(U_{\theta n}-I_n)-(U_\theta-I)]+ \\ +[(U_\theta+I)-(U_{\theta n}+I_n)]V_\theta\,,$$

hence for all $x\in H$

$$\|(U_{\theta n}+I_n)(V_{\theta n}-V_\theta)x\| \to 0, \quad n\to\infty\,.$$

Now

$$I_n = \iota_n(\mathscr{U}_{\theta n}+\mathscr{I}_n)^{-1}\iota_n^{-1}P_n(U_{\theta n}+I_n)\,,$$

which implies

$$\begin{aligned}\|P_n(V_{\theta n}-V_\theta)x\| &= \|I_n(V_{\theta n}-V_\theta)x\| \\ &\leqq \|\iota_n(\mathscr{U}_{\theta n}+\mathscr{I}_n)^{-1}\iota_n^{-1}\|\,\|U_{\theta n}+I_n)(V_{\theta n}-V_\theta)x\| \\ &= \|(\mathscr{U}_{\theta n}+\mathscr{I}_n)^{-1}\|\,\varepsilon_n\,,\end{aligned}$$

where $\lim_{n\to\infty}\varepsilon_n=0$. Since there is a neighbourhood of $\zeta=-1$ which for all n has void intersections with $\operatorname{sp}(\mathscr{U}_{\theta n})$, as is seen from the construction, the resolvent is uniformly bounded in n, and consequently

$$\|P_n(V_{\theta n}-V_\theta)x\| \to 0 \quad \text{as} \quad n\to\infty\,.$$

Hence

$$\|V_{\theta n}x-V_\theta x\| \leqq \|P_n(V_{\theta n}-V_\theta)x\|+\|(P_n-I)V_\theta x\| \to 0, \quad n\to\infty\,.$$

(iii) Since $V_0P_n=P_nV_0$, it follows that

$$(D_nV_{0n}P_nx,P_nx) = (DV_0P_nx,P_nx) \geqq 0\,,$$

and hence $\mathscr{V}_{0n}$ is $\mathscr{D}_n$-positive. We define

$$f_n(\theta) = \inf_{\|x_n\|=1}(\mathscr{D}_n\mathscr{V}_{\theta n}x_n,x_n), \qquad 0\leqq\theta\leqq1\,,$$

that is, $f_n(\theta)$ is the least eigenvalue of $\mathscr{D}_n\mathscr{V}_{\theta n}$. Since $\mathscr{V}_{\theta n}$ is an analytic function of θ, $f_n(\theta)$ is continuous. If $\mathscr{V}_{1n}$ were not $\mathscr{D}_n$-positive, then $f_n(1)<0$. Now $f_n(0)>0$, and hence $f_n(\eta)=0$ for some $\eta\in(0,1)$, which contradicts (i). Thus for all $x\in H$

$$0 \leqq (\mathscr{D}_n\mathscr{V}_{1n}x_n,x_n) = (DV_{1n}x,x)\,.$$

The continuity of the inner product and property (ii) imply that $(DV_1x,x)\geqq0$.

REMARK. The D-positiveness of V_1 implies that $\operatorname{sp}(V_1)$ is real and we can obtain a proof of Proposition 4.1 if V_0 is D-positive without extending $\Phi_K(A,\varrho)$.

DEFINITION 4.2. The equation (1.1) is *weakly G-stable* if for its solutions $u(t)$ the function $[u(t),v]$ is bounded uniformly in t for $v\in G\subset H$. This is equivalent to the condition

$$|[U^mx,y]| \leqq M(x,y) < \infty\,,$$

where $x\in H$, $y\in G\subset H$, $m=0,1,2,\ldots$, and $M(x,y)$ is independent of m.

We can now state our main theorem.

THEOREM 4.1. *Let $A(t)=A+B(t)$ satisfy the assumptions of Proposition*

4.1 *and assume further that* V_0 *is* D*-positive. Then the equation* (1.1) *with the symmetry condition* (1.2) *is weakly* G*-stable for* $G=R(V_1)$.

REMARK. See the remark after Proposition 4.1 for the conditions on A implying D-positiveness of V_0.

PROOF. According to the propositions 4.1 and 4.2 the corollary after Proposition 2.3 is applicable for $V=V_1$ and $U=U_1$.

REMARK 1. We observe that the weak $R(V_1)$-stability implies stability with respect to the norm $\|\cdot\|=\{\cdot,\cdot\}^{\frac{1}{2}}$.

REMARK 2. If $\dim H<\infty$, then $R(V_1)=H$ and weak $R(V_1)$-stability implies stability.

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