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CLASSIFICATION AND DEFORMATION OF RIEMANNIAN SPACES

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A promising new field of research lies in the classification theory of Riemannian spaces with respect to the existence or nonexistence of harmonic functions with various boundedness properties. For 2-spaces harmonicity is known to be invariant not only under isometries but also under conformal mappings. In contrast, there is no conformal invariance for higher dimensions. This is exemplified by the following rather surprising phenomena:

(a) A ball can be endowed with a Riemannian metric, conformally equivalent to the Euclidean one, which makes the ball parabolic.
(b) A compact locally Euclidean space punctured at a point can be endowed with a conformally equivalent Riemannian metric which makes the resulting space hyperbolic.
(c) The classes O_G, O_{HD}, and O_{KD} are not quasi-conformally, and not even conformally invariant. They are, however, quasi-isometrically invariant.

We start by recalling in Section 1 the definitions of the null classes O_G, O_{HD}, and O_{KD}. Making use of function spaces introduced in Section 2 we characterize in Sections 3–5 these null classes by equivalence relations. In Section 6 we introduce the Dirichlet mappings and show by means of the equivalence relations that the null classes are invariant under Dirichlet mappings. In particular, quasi-conformal mappings of Riemannian 2-spaces are shown to be Dirichlet mappings. That the invariance under quasi-conformal mappings is lost for higher dimensions is established in Section 7, which also contains the proofs of the above statements (a) and (b). The paper closes in Section 8 with the result that quasi-isometries are Dirichlet mappings, hence the quasi-isometric invariance of O_G, O_{HD}, and O_{KD}.

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1. Null classes of Riemannian spaces.

1. Let V be a noncompact *Riemannian space*, i.e., a connected, orientable, and C^∞ n-manifold with a positive definite metric tensor g_{ij}. The inverse matrix of the $n \times n$ matrix (g_{ij}) is denoted by (g^{ij}) and the determinant of (g_{ij}) by g. (For fundamentals of Riemannian spaces see, e.g., de Rham [19], Hodge [10].)

We consider the class $H(V)$ of *harmonic functions* in V, i.e., solutions of the self-adjoint second order elliptic partial differential equation

$$\Delta u = -\frac{1}{g^{\frac{1}{2}}}\frac{\partial}{\partial x^i}\left(g^{\frac{1}{2}} g^{ij}\frac{\partial u}{\partial x^j}\right) = 0 \tag{1}$$

in V. (For basic properties of solutions see, e.g., Miranda [14], Feller [6], Itô [12], Hörmander [11].)

For a set $E \subset V$ we denote by $H(E)$ the class of functions u harmonic in some open sets $\mathcal{O}_u$ containing E. For an open set $\mathcal{O} \subset V$ the symbol $H^c(\mathcal{O})$ will be used for the class of functions harmonic in $\mathcal{O}$ and with continuous extensions to $\overline{\mathcal{O}}$:

$$H^c(\mathcal{O}) = H(\mathcal{O}) \cap C(\overline{\mathcal{O}}) .$$

2. Let $\{V_m\}_0^\infty$ with $\overline{V}_m \subset V_{m+1}$ be an exhaustion of V by regular subregions V_m (Sario–Schiffer–Glasner [24]). For $m > 0$ we consider the functions $u_m \in C(V)$ defined by

$$u_m | \overline{V}_0 = 0, \qquad u_m | V - V_m = 1, \qquad u_m | \overline{V}_m - V_0 \in H^c(V_m - \overline{V}_0) .$$

The sequence $\{u_m\}_1^\infty$ is monotone decreasing and converges uniformly in compact subsets to a nonnegative function $u \in C(V)$ with

$$u | \overline{V}_0 = 0, \qquad u | V - V_0 \in H^c(V - \overline{V}_0) .$$

There are only two cases: either $u \equiv 0$ on V or $u | V - \overline{V}_0 > 0$. In the former case we call V *parabolic*, and denote the class of parabolic Riemannian spaces by O_G. The notation is in reference to the fact that these spaces can also be characterized by the nonexistence of Green's functions of (1) on V (cf. Itô [12], Glasner [9]). The spaces with $u \not\equiv 0$ will be called *hyperbolic*.

3. The *Dirichlet integral* of a function f on V is, by definition,

$$D(f) = \int_V |\text{grad} f|^2 dV = \int_V df \wedge *df ,$$

where

$$|\operatorname{grad} f|^2 = g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}$$

is supposed to exist a.e. and $dV = g^{\frac{1}{2}}dx^1 \wedge \ldots \wedge dx^n$ (cf. Duff [5]).

We denote by $HD(V)$ the class of functions $u \in H(V)$ with finite Dirichlet integrals $D(u)$, and by O_{HD} the class of Riemannian spaces V that do not carry nonconstant functions in $HD(V)$.

We are also interested in the class $KD(V)$ of functions u in $HD(V)$ with vanishing flux

$$\int_\gamma *du = 0$$

across all dividing $(n-1)$-cycles γ in V. The corresponding null class is denoted by O_{KD}.

4. It is known that

$$O_G \subset O_{HD} \subset O_{KD}$$

(Sario–Schiffer–Glasner [24]), the inclusions being strict. To investigate the invariance of these classes under various mappings of V we shall characterize them by equivalences of certain function spaces which we shall now introduce.

For further classification problems of Riemannian spaces reference is here made to the doctoral dissertations of Smith [25], Glasner [9], Breazeal [3], and Ow [17].

2. Norms and function spaces.

5. Given an exhaustion $\{V_m\}_0^\infty$ of V, we shall make use of the quasinorm

$$[f]_\infty = \sum_0^\infty \frac{1}{2^m}\sup_{V_m}\frac{|f|}{1+|f|}.$$

Convergence in this norm is clearly equivalent to compact convergence, i.e., convergence in every compact subset of V. Uniform convergence in all of V is given by the norm

$$\|f\|_\infty = \sup_V |f|,$$

whereas the Dirichlet convergence is provided by the seminorm

$$\|f\| = (D(f))^{\frac{1}{2}}.$$

Two combinations will be useful:

$$\langle f\rangle = [f]_\infty + \|f\|,$$

corresponding to both compact and Dirichlet convergence, and

$$\langle\langle f\rangle\rangle = \|f\|_\infty + \|f\|$$

furnishing uniform and Dirichlet convergence.

6. Given a parametric ball B with the coordinate system x, we denote by $ACL(B)$ the class of functions in B that are absolutely continuous on almost all lines parallel to the coordinate axes. Analogously, $ACL(V)$ shall designate the class of functions f on V with $f|B \in ACL(B)$ for *every* parametric ball B and every coordinate system in B.

Of basic importance in our study will be the function space

$$\mathscr{D} = \{f \mid f \in C(V) \cap ACL(V),\ \|f\| < \infty\}.$$

It is easily seen that $\partial f/\partial x^i$, $i=1,\ldots,n$, exists a.e. and $df = (\partial f/\partial x^i)dx^i$ a.e.

The space $\mathscr{D}$ is closed in the norm $\langle\cdot\rangle$, hence also in $\langle\langle\cdot\rangle\rangle$ (see Nakai [15]). It is also closed under the operations $f \cup g = \max(f,g)$ and $f \cap g = \min(f,g)$, with $f,g \in \mathscr{D}$ (Nakai [16]). The subspace $\mathscr{D}^\infty = C^\infty \cap \mathscr{D}$ is dense in terms of $\langle\cdot\rangle$ (cf. [15]).

We are also interested in the subspaces

$$\mathscr{D}_0 = \{f \mid f \in \mathscr{D},\ \operatorname{supp} f \text{ compact}\},$$
$$\mathscr{D}_1 = \{f \mid f \in \mathscr{D},\ \operatorname{supp} df \text{ compact}\}.$$

Again the spaces $\mathscr{D}_i^\infty = C^\infty \cap \mathscr{D}_i$, $i=0,1$, are dense in $\mathscr{D}_i$. As a consequence Stokes' formula can be used freely for functions in $\mathscr{D}$ and $\mathscr{D}_i$.

3. Characterization of O_G.

7. Denote by $\overline{\mathscr{D}}_0$ the closure of $\mathscr{D}_0$ in $\mathscr{D}$ with respect to $\langle\cdot\rangle$. We shall prove (cf. Royden [20], [22], Nakai [15], Glasner [9]):

THEOREM 1. *$V \in O_G$ if and only if $\mathscr{D} = \overline{\mathscr{D}}_0$.*

For functions u_m in No. 2, Stokes' formula gives

$$D(u_m) = D(u_{m+p}) + D(u_{m+p} - u_m) \tag{2}$$

for $p \geqq 0$, and we have $D(u - u_m) \to 0$. Since $[u - u_m]_\infty \to 0$, it follows that $\langle u - u_m\rangle \to 0$.

Suppose $V \in O_G$. We know from No. 2 that $\langle u_m\rangle \to 0$, and we set $v_m = 1 - u_m$. Given $f \in \mathscr{D}$, we are to show the existence of a sequence $\{f_m\} \subset \mathscr{D}_0$ such that $\langle f - f_m\rangle \to 0$.

First we consider the case where f is bounded, $|f| < M < \infty$. Obviously $[f - v_m f]_\infty \to 0$ with $\{v_m f\} \subset \mathscr{D}_0$ and for the Dirichlet norm we have

$$D(f-v_m f) = D(u_m f) = \int_V |\mathrm{grad}\, u_m f|^2\, dV$$
$$\leqq 2\int_V (\sup_V u_m)^2 |\mathrm{grad} f|^2\, dV + 2\int_V |\mathrm{grad}\, u_m|^2 (\sup_V |f|)^2\, dV\, .$$

Here the second term is dominated by $2M^2 D(u_m) \to 0$. We decompose the integral I_V in the first term into I_K and I_{V-K}, with K a compact set, and have

$$\limsup_m I_V \leqq \limsup_m I_K + \limsup_m I_{V-K} \leqq D_{V-K}(f)\, .$$

For a sufficiently large K this is arbitrarily small, and we obtain $\|f-v_m f\| \to 0$, hence $\langle f-v_m f\rangle \to 0$.

If $f \in \mathscr{D}$ is unbounded, we approximate it by the truncated bounded functions

$$f_m = \max(\min(f,m), -m) \in \mathscr{D}\, .$$

Obviously $[f-f_m]_\infty \to 0$ and, by virtue of the continuity of the norm $\|\cdot\|$ in the region, $\|f-f_m\| \to 0$. A fortiori $\langle f-f_m\rangle \to 0$, and since each f_m is a limit in $\langle\cdot\rangle$ of functions in $\mathscr{D}_0$, the same is true of f. The proof of the necessity of $\mathscr{D}=\overline{\mathscr{D}}_0$ is herewith complete.

8. To establish the sufficiency, note that $\mathscr{D}=\overline{\mathscr{D}}_0$ implies $1 \in \mathscr{D}_0$, that is, the existence of a sequence of function $\varphi_m \in \mathscr{D}_0$ with $\langle 1-\varphi_m\rangle \to 0$. It follows that

$$D(u) = \lim_m D(u_m) = -\int_{\partial V_0} {*du} = -\lim_m \int_{\partial V_0} \varphi_m {*du}$$
$$= \lim_m D(\varphi_m, u) \leqq \lim_m (D(\varphi_m))^{\frac{1}{2}} (D(u))^{\frac{1}{2}} = 0\, ,$$

and we have $u \equiv 0$, $V \in O_G$.

4. Characterization of O_{HD}.

9. In view of the inclusion $O_G \subset O_{HD}$ we may assume that V is hyperbolic. We shall need the following auxiliary result (cf. Royden [20], Brelot [4], Nakai [16]):

LEMMA 1. *Assume that $V \notin O_G$. The vector space $\mathscr{D}$ is the direct sum of vector subspaces HD and $\overline{\mathscr{D}}_0$:*

$$\mathscr{D} = HD + \overline{\mathscr{D}}_0\, . \tag{3}$$

Spaces HD and $\overline{\mathscr{D}}_0$ are also orthogonal in terms of the inner product $D(\cdot,\cdot)$.

Given $f \in \mathscr{D}$ we know from No. 6 that $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$ are in $\mathscr{D}$. For this reason we may assume that $f \geqq 0$. Then the function $h_m \in C(V)$ defined by

$$h_m | \overline{V}_m \in H^c(V_m), \qquad h_m | V - V_m = f$$

is nonnegative. On setting $g_m = f - h_m$ we obtain

$$D(f) = D(h_m) + D(g_m) \tag{4}$$

and

$$D(h_m) = D(h_{m+p}) + D(h_m - h_{m+p}) \,. \tag{5}$$

We infer that $\{h_m\}$, and consequently $\{g_m\}$, is a Cauchy sequence in $\|\cdot\|$.

10. We shall show that the same is true in $[\cdot]_\infty$. In fact,

$$\int_{\partial V_0} h_m * du_m = \int_{\partial V_0} f * du_m + \int_{\partial V_m - \partial V_0} g_m * du_m \,,$$

and for $p \in V_0$ we have by Harnack's inequality and by (4)

$$c h_m(p) D(u_m) \leqq \min_{\partial V_0} f D(u_m) + (D(f))^{\frac{1}{2}} (D(u_m))^{\frac{1}{2}} \,,$$

where c is a constant $\neq 0$. In view of $0 < D(u) \leqq D(u_m)$ this implies

$$c h_m(p) \leqq \min_{\partial V_0} f + (D(f)/D(u))^{\frac{1}{2}} \,,$$

and therefore $|h_m(p)| < N$ for some finite N. It follows again by Harnack's inequality that $\{h_m\}$ is uniformly bounded in V, and therefore a normal family. There exists a subsequence, again denoted by $\{h_m\}$, and an $h \in H(V)$ such that $[h - h_m]_\infty \to 0$.

11. In summary we have $\langle h - h_m \rangle \to 0$ and $h \in HD$. The function $g = f - h$ has the corresponding property $\langle g - g_m \rangle = \langle h - h_m \rangle \to 0$, and therefore $g \in \overline{\mathscr{D}}_0$. Moreover, (4) yields for $m \to \infty$

$$D(f) = D(h) + D(g) \,.$$

We thus have $h \perp g$, and it follows easily that $HD \perp \overline{\mathscr{D}}_0$.

To prove the uniqueness of the orthogonal decomposition (3) so established suppose $f \equiv 0$. By the above relation $h = c$, a constant, and $g = -c$. If $c \neq 0$, then $c \in \overline{\mathscr{D}}_0$ implies $1 \in \mathscr{D}_0$, and we have proved in No. 8 that this entails $V \in O_G$, contrary to our assumption. We conclude that $c = 0$, and the uniqueness is established.

12. Let R be the real number field. We are ready to prove (cf. Royden [20], [22], Nakai [15], [16]):

THEOREM 2. *$V \in O_{HD}$ if and only if $\mathscr{D} = \overline{\mathscr{D}}_0 \pmod R$.*

If $V \in O_G$, there is nothing to prove since $\mathscr{D} = \overline{\mathscr{D}}_0$. We therefore assume that $V \notin O_G$.

Suppose $V \in O_{HD}$. Then $\mathscr{D} = R + \overline{\mathscr{D}}_0$, as maintained. Conversely, if this equality holds, then $HD = R$.

5. Characterization of O_{KD}.

13. Let $\overline{\mathscr{D}}_1$ be the closure in $\mathscr{D}$ of $\mathscr{D}_1$ with respect to $\|\cdot\|$. Lemma 1 has the following counterpart for KD (cf. Ahlfors [1], Royden [22]):

LEMMA 2. *The space $\mathscr{D}$ has the orthogonal decomposition*

$$\mathscr{D} = KD + \overline{\mathscr{D}}_1, \qquad KD \cap \overline{\mathscr{D}}_1 = R \tag{6}$$

in terms of $D(\cdot,\cdot)$.

We shall first show that $\overline{\mathscr{D}}_1^{\perp} \subset KD$. Let $\varphi \perp \overline{\mathscr{D}}_1$. Then by virtue of $\mathscr{D}_0 \subset \mathscr{D}_1 \subset \overline{\mathscr{D}}_1$ we have $\varphi \perp \mathscr{D}_0$ and, by continuity, $\varphi \perp \overline{\mathscr{D}}_0$. By (3) this implies $\varphi \in HD$.

Let γ be a dividing $(n-1)$-cycle in V. There exists an open regular subset Ω of V and a function $\psi \in C^\infty(V)$ with the following properties:

(α) $\partial\Omega = \gamma \cup \gamma'$,

(β) γ is homologous to γ',

(γ) ψ is constant in each component of $V - \Omega$,

(δ) $\psi | \gamma = 1$, $\psi | \gamma' = 0$.

Clearly $\psi \in \mathscr{D}_1$, and

$$0 = D(\varphi,\psi) = \int_\gamma *d\varphi \,.$$

Consequently $\varphi \in KD$.

14. To prove the orthogonality $KD \perp \overline{\mathscr{D}}_1$ we observe that $D(\varphi,\psi) = 0$ for $\varphi \in KD$, $\psi \in \mathscr{D}_1$, and the statement follows by continuity.

15. It remains to prove that $KD + \overline{\mathscr{D}}_1$ actually is $\mathscr{D}$. To this end let $\mathscr{D}^{\sim}$ and $\mathscr{D}_1^{\sim}$ be the completions of $\mathscr{D}$ and $\mathscr{D}_1$ in $\|\cdot\|$. Then $\mathscr{D}^{\sim} \cap C = \mathscr{D}$ and $\mathscr{D}_1^{\sim} \cap C = \mathscr{D}_1$. For the Hilbert space $\mathscr{D}^{\sim}$ considered $\operatorname{mod} R$ we have the decomposition

$$\mathscr{D}^{\sim} = \mathscr{D}_1^{\sim\perp} + \mathscr{D}_1^{\sim} \,.$$

We repeat the argument on KD in Nos. 13, 14 by replacing $\mathscr{D}$ by $\mathscr{D}^{\sim}$ and $\overline{\mathscr{D}}_1$ by $\mathscr{D}_1^{\sim}$ to obtain $\mathscr{D}_1^{\sim\perp} = KD$, and consequently

$$\mathscr{D}^{\sim} = KD + \mathscr{D}_1^{\sim} \,.$$

On taking intersections with C we arrive at

$$\mathscr{D} = KD + \overline{\mathscr{D}}_1 .$$

The proof of Lemma 2 is herewith complete.

16. We are ready to characterize the class O_{KD}.

THEOREM 3. *$V \in O_{KD}$ if and only if $\mathscr{D} = \overline{\mathscr{D}}_1$.*

Since the orthogonality $KD \perp \overline{\mathscr{D}}_1$ is in $D(\cdot, \cdot)$, the null space is R. If $KD = R$, then by (6) $\mathscr{D} = \overline{\mathscr{D}}_1$, and conversely.

6. Dirichlet mappings.

17. Let $V, \tilde{V}$ be Riemannian n-spaces. To study invariances of our null classes we introduce:

DEFINITION. *A homeomorphism $T: V \to \tilde{V}$ is a Dirichlet mapping provided that*

$$f \circ T \in \mathscr{D}(V) \quad \textit{if and only if} \quad f \in \mathscr{D}(\tilde{V}) \tag{7}$$

and there exists a constant $K \geqq 1$ with the property

$$K^{-1} D(f) \leqq D(f \circ T) \leqq K D(f) \tag{8}$$

for all $f \in \mathscr{D}(\tilde{V})$.

From Theorems 1–3 we conclude immediately:

THEOREM 4. *The classes O_G, O_{HD}, and O_{KD} are invariant under Dirichlet mappings.*

In fact, the induced mapping $T^*: \mathscr{D}(\tilde{V}) \to \mathscr{D}(V)$ given by $T^* f = f \circ T$ is a homeomorphism in $\|\cdot\|$ and also in $[\cdot]_\infty$. For this reason

$$T^*(\overline{\mathscr{D}}_0(\tilde{V})) = \overline{\mathscr{D}}_0(V), \qquad T^*(\overline{\mathscr{D}}_1(\tilde{V})) = \overline{\mathscr{D}}_1(V) .$$

18. After these preparations we are ready to turn to quasi-conformal mappings. For $p, q \in V$ we set

$$d(p,q) = \inf \int_\alpha (g_{ij} dx^i dx^j)^{\frac{1}{2}} ,$$

where the infimum is taken among all rectifiable arcs α from p to q. In terms of d and similarly defined $\tilde{d}$ in $\tilde{V}$ we write

$$l(p,r) = \inf_{d(p,q)=r} \tilde{d}(T(p), T(q)) ,$$

$$L(p,r) = \sup_{d(p,q)=r} \tilde{d}(T(p),T(q)) .$$

DEFINITION. *A homeomorphism T: $V \to \tilde{V}$ is quasi-conformal if there exists a constant $K \geqq 1$ such that*

$$\limsup_{r\to 0} \frac{L(p,r)}{l(p,r)} \leqq K \tag{9}$$

for all $p \in V$.

By a simple generalization of the results of Gehring [7], [8] and Väisälä [26] we see that a quasi-conformal mapping is ACL in terms of local parameters of V and $\tilde{V}$ and that it is totally differentiable a.e. Moreover at any $p \in V$,

$$I(p)^n \leqq K^{n-1} J(p) ,$$

where

$$I(p) = \limsup_{r\to 0} L(p,r)/r$$

and

$$J(p) = \limsup_{r\to 0} \tilde{V}(TB_r)/V(B_r) .$$

Here B_r is the geodesic r-ball at p, and $\tilde{V}(TB_r)$, $V(B_r)$ are the volumes of TB_r and B_r respectively. At a point p where T is totally differentiable, J is the Jacobian and we have $J > 0$ a.e. Moreover if $E \subset V$ is measurable, then so is TE, and its volume is

$$\tilde{V}(TE) = \int_E J\, dV .$$

Corresponding properties, with the same K, are possessed by the inverse mapping T^{-1}.

19. In the remainder of Section 6 we restrict our attention to spaces of dimension 2. We shall show (Pfluger [18], Royden [21, 22], Nakai [15]):

THEOREM 5. *The classes O_G, O_{HD}, and O_{KD} of Riemannian 2-spaces are invariant under quasi-conformal mappings.*

For the proof we shall first consider functions $f \in \mathscr{D}^\infty(\tilde{V})$ and show that they satisfy conditions (7) and (8) (cf. Künzi [13]). Clearly $f \circ T$ is in $\mathscr{D}(V)$ and totally differentiable a.e. We consider only points $p \in V$ with this property. We shall show that

$$|\mathrm{grad}\, F|^2 \leqq |\mathrm{grad} f|^2\, I^2(p) , \tag{10}$$

where $F = f \circ T$. If $\mathrm{grad}\, F = 0$, then this is trivially true. We shall assume that $|\mathrm{grad}\, F| \neq 0$.

At p and $\tilde{p}=T(p)$ choose coordinate systems x and $\tilde{x}$ with $g_{ij}=\delta_{ij}$ and $\tilde{g}_{ij}=\delta_{ij}$. For $F_i=\partial F/\partial x^i$ at p we have $F_i dx^i=f_i d\tilde{x}^i$. Since $\operatorname{grad} F \neq 0$ we can choose the dx^i such that $F_1/dx^1=F_2/dx^2$. Then by the Schwarz inequality

$$(\Sigma F_i^2)\,(\Sigma dx^{i^2}) = (F_i dx^i)^2 = (f_i d\tilde{x}^i)^2 \leqq (\Sigma f_i)^2\,(\Sigma d\tilde{x}^i)^2 .$$

Since $g_{ij}=\delta_{ij}$, we have $\sum f_i^2=|\operatorname{grad} f|^2$ and $\sum F_i^2=|\operatorname{grad} F|^2$. Similarly $\sum(dx^i)^2=ds^2$ and $\sum(d\tilde{x}^i)^2=d\tilde{s}^2$. But $d\tilde{s}^2/ds^2 \leqq I^2(p)$, and (10) follows.

20. By virtue of $I^2 \leqq KJ$ we obtain

$$|\operatorname{grad} f\circ T|^2 \leqq K\,|\operatorname{grad} f|^2\, J(p) .$$

An integration with respect to dV yields

$$D(f\circ T) \leqq KD(f) .$$

By a similar reasoning we conclude that

$$K^{-1}D(f) \leqq D(f\circ T) .$$

Properties (7) and (8) have thus been established for $f\in\mathscr{D}^\infty(\tilde{V})$.

21. Suppose now $f\in\mathscr{D}(\tilde{V})$. Then there exists a sequence of functions $f_m\in\mathscr{D}^\infty(\tilde{V})$ such that $\langle f-f_m\rangle \to 0$ if $m\to\infty$. Since $f_m\circ T\in\mathscr{D}(V)$,

$$K^{-1}D(f_m) \leqq D(f_m\circ T) \leqq KD(f_m) , \tag{11}$$

and

$$K^{-1}D(f_m-f_{m+p}) \leqq D(f_m\circ T-f_{m+p}\circ T) \leqq KD(f_m-f_{m+p}) ,$$

we see that

$$\langle f\circ T-f_m\circ T\rangle \to 0 .$$

Thus (7) holds: $f\circ T\in\mathscr{D}(V)$. On letting $m\to\infty$ we obtain (8).

7. Conformal noninvariance.

22. Broadly speaking, the reason for the invariance under a quasiconformal mapping T in the case $n=2$ is that $|\operatorname{grad} f\circ T|^2$ is, in essence, divided by I^2, while dV is multiplied by J, and by virtue of the relation $I^n \leqq K^{n-1}J$ these changes compensate in $D(f\circ T)$. For $n>2$ this compensation no longer takes place and the above proof breaks down. We shall now show that the difficulty is in the very nature of the phenomenon: there does not exist even conformal invariance.

We start by constructing two special Riemannian spaces. The first one provides us with a proof of statement (a) of the introduction.

In Euclidean n-space with $n > 2$, set $|x|^2 = \sum_1^n (x^i)^2$.

EXAMPLE 1. *The ball $B: |x| < 1$ becomes parabolic when endowed with the metric $g_{ij}(x) = \lambda(x)\delta_{ij}$, where $\lambda \in C^\infty$, $\lambda > 0$ in B and*

$$\lambda(x) = |x|^{(2-2n)/(n-2)}\,(1-|x|)^{4/(n-2)} \tag{12}$$

in $\frac{1}{2} \leqq |x| < 1$.

For the proof note first that equation (1) takes the form

$$\Delta u = -\lambda^{\frac{1}{2}n} \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \lambda^{\frac{1}{2}(n-2)} \frac{\partial u}{\partial x^i}. \tag{13}$$

We exhaust $V = B$ by

$$V_m:\ |z| < 1 - \frac{1}{2+m}$$

and set

$$u_m(x) = \frac{1}{m(1-|x|)} - \frac{2}{m} \tag{14}$$

in $\bar{V}_m - V_0$. Then

$$\lambda^{\frac{1}{2}(n-2)} \frac{\partial u_m}{\partial x^i} = \frac{x_i}{m\,|x|^n}$$

and

$$-\lambda^{\frac{1}{2}n} \Delta u_m = \sum_{i=1}^{n} \frac{-n x_i{}^2 + |x|^2}{m\,|x|^{n+2}} = 0\,.$$

Thus u_m is harmonic, the limiting function is $u \equiv 0$, and therefore $V \in O_G$.

23. We turn to the proof of statement (b) of the introduction.

EXAMPLE 2. *Let B be a parametric ball $|x| < 1$ of a compact locally Euclidean space V_0 with $n > 2$. The space $V = V_0 - p$, with p the center of B, becomes hyperbolic when endowed with a Riemannian C^∞ metric in V such that $g_{ij}(x) = \lambda(x)\delta_{ij}$ in V with $\lambda > 0$ in V and*

$$\lambda(x) = |x|^{(2-2n)/(n-2)} \tag{15}$$

in $B - p$.

In $B - p$, equation (13) is again valid. We choose

$$B_m:\ |x| < \frac{1}{1+m}, \qquad V_m = V - \bar{B}_m\,,$$

and

$$u_m = \frac{1-|x|}{1-(1+m)^{-1}} \tag{16}$$

in $\bar{V}_m - V_0$. Then

$$\lambda^{\frac{1}{2}(n-2)} \frac{\partial u_m}{\partial x^i} = \frac{-cx^i}{|x|^m}$$

with c a constant, and

$$-\lambda^{\frac{1}{2}n} \Delta u_m = c \sum_{i=1}^{n} \frac{-nx_i^2 + |x|^2}{|x|^{n+2}} = 0$$

The limiting function is $u = 1 - |x| \not\equiv 0$, and $V \notin O_G$.

24. We are ready to prove that Theorem 5 is no longer true for higher dimensions:

Theorem 6. *The classes O_G, O_{HD}, and O_{KD} are not conformally invariant.*

Let V be the ball B: $|x| < 1$ with the Euclidean metric $g_{ij} = \delta_{ij}$. For $\tilde{V}$ take the same ball with the metric $g_{ij} = \lambda \delta_{ij}$ of Example 1. We shall show that the mapping T: $V \to \tilde{V}$ induced by the identity mapping of the base space B is conformal.

Let $p, q \in V$ and, in the notation of No. 18, $d(p,q) = r$. Then

$$\tilde{d}(T(p), T(q)) = \lambda(p) r + \varepsilon(p,q) ,$$

where $\varepsilon(p,q) \to 0$ as $d(p,q) \to 0$. We have

$$L(p,q) = \lambda(p) r + \sup_{d(p,q)=r} \varepsilon(p,q) ,$$

$$l(p,q) = \lambda(p) r + \inf_{d(p,q)=r} \varepsilon(p,q) ,$$

and consequently

$$\lim_{r \to 0} \frac{L(p,r)}{l(p,r)} = 1 ,$$

as claimed.

Clearly $V \notin O_G$, O_{HD}, O_{KD}, whereas $\tilde{V} \in O_G \subset O_{HD} \subset O_{KD}$. Theorem 6 is herewith established.

8. Quasi-isometric invariance.

25. Although our null classes for $n > 2$ lack quasi-conformal invariance, they have another important invariance property, associated in a perhaps more natural manner with the Riemannian metric.

Definition. *A homeomorphism T: $V \to \tilde{V}$ is a quasi-isometry if there exists a constant $K > 1$ such that*

$$K^{-1}r \leqq l(p,r) \leqq L(p,r) \leqq Kr \tag{17}$$

for all $p \in V$.

A quasi-isometry is clearly a quasi-conformal mapping and therefore enjoys the properties of the latter given in No. 18.

THEOREM 7. *The classes* O_G, O_{HD}, *and* O_{KD} *are preserved under quasi-isometries.*

26. By way of preparation we insert here, for the convenience of the reader, two elementary properties of nonnegative matrices. We recall that a matrix A is nonnegative (positive) if the form (Ax,x) is positive semidefinite (definite resp.).

Let A and B be two $n \times n$ symmetric matrices. The inequality $A \leqq B$ $(A < B)$ means that $B - A$ is a nonnegative (positive, resp.) matrix. The standard notations A^{-1}, A^t, and $|A|$ are used for the inverse matrix, the transposed matrix, and the determinant of A.

LEMMA 3. *Let* $A \geqq 0$ and $B > 0$. *Then the inequality* $A \leqq B$ *implies that* $|A| \leqq |B|$.

To see this take the orthogonal matrix P such that P^tBP is a diagonal matrix with elements $\lambda_1, \lambda_2, \ldots, \lambda_n$, say. Since $B > 0$ implies $\lambda_i > 0$, $i = 1, 2, \ldots, n$, we can consider the diagonal matrix C with elements $\lambda_1^{-\frac{1}{2}}, \lambda_2^{-\frac{1}{2}}, \ldots, \lambda_n^{-\frac{1}{2}}$. Let Q be the orthogonal matrix such that $Q^t(PC)^tA(PC)Q$ is diagonal with elements $\mu_1, \mu_2, \ldots, \mu_n$. Then $A \leqq B$ implies for $S = PCQ$

$$S^tAS \leqq S^tBS\,. \tag{18}$$

In terms of the $n \times n$ unit matrix E and the $1 \times n$ matrix $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ we have

$$S^tAS = \mu E, \qquad S^tBS = E\,, \tag{19}$$

and therefore

$$\mu E \leqq E\,. \tag{20}$$

This in turn implies that $\mu_i \leqq 1$. Since $A \geqq 0$, we obtain $S^tAS = \mu E \geqq 0$, and thus $\mu_i \geqq 0$. Therefore $\prod_1^n \mu_i \leqq 1$, i.e., $|\mu E| \leqq |E|$. Substitution by (19) gives $|A|\,|S|^2 \leqq |B|\,|S|^2$, and the assertion $|A| \leqq |B|$ follows.

27. For positive matrices we shall show:

LEMMA 4. *The inequality* $A \leqq B$ *with* $A > 0$ *and* $B > 0$ *implies that* $B^{-1} \leqq A^{-1}$.

Using the notations of No. 26 we observe that $A>0$ implies $S^tAS>0$ and a fortiori $\mu_i>0$. This together with (20) gives $0<\mu_i\leqq 1$ and consequently $1\leqq \mu_i^{-1}$. Therefore

$$E^{-1} \leqq (\mu E)^{-1},$$

which in terms of (19) is

$$(S^tBS)^{-1} \leqq (S^tAS)^{-1}.$$

Hence $S(S^tBS)^{-1}S^t \leqq S(S^tAS)^{-1}S^t$, and this is equivalent to $B^{-1}\leqq A^{-1}$.

28. We turn to the proof of Theorem 7. To show that quasi-isometries are Dirichlet mappings we are to compare Dirichlet integrals over V and $\tilde{V}$. To this end we compare volume elements in the present No., and gradients in No. 29.

Choose a set $E\subset V$ such that (i) meas $E=0$, (ii) T is totally differentiable in $V-E$, (iii) T^{-1} is totally differentiable in $\tilde{V}-T(E)$, and (iv) $J>0$ in $V-E$. All considerations below will be in $V-E$ and $\tilde{V}-T(E)$.

Let $\tilde{x}=T(x)$. Then by (12),

$$K^{-1}ds \leqq d\tilde{s} \leqq K ds,$$

or equivalently,

$$K^{-2}g_{ij}dx^i dx^j \leqq \tilde{g}_{ij}\frac{\partial\tilde{x}^i}{\partial x^k}\frac{\partial\tilde{x}^j}{\partial x^l}dx^k dx^l \leqq K^2 g_{ij}\, dx^i dx^j. \tag{21}$$

In terms of the matrix $M=(\partial\tilde{x}^i/\partial x^j)$, with i indicating the row, j the column, this can be written

$$K^{-2}(g_{ij}) \leqq M^t(\tilde{g}_{ij})M \leqq K^2(g_{ij}). \tag{22}$$

On taking the determinants we obtain by Lemma 3

$$K^{-2n}g \leqq \tilde{g}|M|^2 \leqq K^{2n}g.$$

For volume elements $dV=g^{\frac{1}{2}}\,dx^1\ldots dx^n$ and $d\tilde{V}=\tilde{g}^{\frac{1}{2}}J\,dx^1\ldots dx^n$ this takes the form

$$K^{-n}dV \leqq d\tilde{V} \leqq K^n dV. \tag{23}$$

29. To compare gradients on V and $\tilde{V}$ we obtain from Lemma 4 and (22)

$$K^{-2}(g^{ij}) \leqq M^{-1}(\tilde{g}^{ij})(M^t)^{-1} \leqq K^2(g^{ij}).$$

A left and right multiplication by M and M^t respectively gives

$$K^{-2}(\tilde{g}^{ij}) \leqq M(g^{ij})M^t \leqq K^2(\tilde{g}^{ij}). \tag{24}$$

First consider functions $f \in \mathscr{D}^\infty(\tilde{V})$. Then clearly $f \circ T \in \mathscr{D}(V)$. On applying the $n \times 1$ matrix $(\partial f/\partial \tilde{x}^i)$ on the right and its transpose on the left to (24) we obtain

$$K^{-2}\,\tilde{g}^{ij}\frac{\partial f}{\partial \tilde{x}^i}\frac{\partial f}{\partial \tilde{x}^j} \leqq g^{kl}\frac{\partial \tilde{x}^i}{\partial x^k}\frac{\partial \tilde{x}^j}{\partial x^l}\frac{\partial f}{\partial \tilde{x}^i}\frac{\partial f}{\partial \tilde{x}^j} \leqq K^2\,\tilde{g}^{ij}\frac{\partial f}{\partial \tilde{x}^i}\frac{\partial f}{\partial \tilde{x}^j}$$

a.e. in $V-E$. The expression in the middle is

$$g^{kl}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial x^l} = |\mathrm{grad} f \circ T|^2\,,$$

and we obtain

$$K^{-2}|\mathrm{grad} f|^2 \leqq |\mathrm{grad} f \circ T|^2 \leqq K^2|\mathrm{grad} f|^2 \tag{25}$$

a.e. in $V-E$.

On combining (23) and (25) we obtain the desired relations

$$K_1^{-1}D(f) \leqq D(f \circ T) \leqq K_1 D(f)\,, \tag{26}$$

with $K_1 = K^{n+2}$.

In the general case of $f \in \mathscr{D}(\tilde{V})$ we argue as in No. 21 to conclude that $f \circ T$ is in $\mathscr{D}(V)$ and satisfies (26). This completes the proof of Theorem 7.

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