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A CLOSURE PROBLEM FOR SPACES OF INFINITELY DIFFERENTIABLE FUNCTIONS

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1. Introduction.

Let $\{m_\nu\}_0^n$ be a sequence of positive numbers, $m_0=1$, and let C_n be the space of complex-valued, n times differentiable functions f, defined for $x \geqq 0$, for which

$$(1.1) \qquad \|f\|_{(n)}^2 = \sum_{\nu=0}^{n} m_\nu^{-2} \int_0^\infty |f^{(\nu)}(x)|^2\, dx < \infty .$$

Further, let

$$(1.2) \qquad \sum_{\nu=0}^{n} m_\nu^{-2} z^{2\nu} = \prod_{j=1}^{n} (1 + r_j^{-2} z^2), \qquad \operatorname{Re} r_j > 0,$$

and suppose that $r_i \neq r_j$ for $i \neq j$.

THEOREM 1. *Let $\varphi \in C_n$. Then the extremal problem*

$$\min_{b_j} \|\varphi(x) - \sum_{j=1}^{n} b_j e^{-r_j x}\|_{(n)}$$

is solved by

$$\varphi_n(x) = \sum_{j=1}^{n} b_j^{(n)} e^{-r_j x},$$

where

$$\varphi_n^{(\nu)}(0) = \varphi^{(\nu)}(0), \qquad \nu = 0,1,\ldots,n-1 .$$

PROOF. Define in C_n the inner product

$$(f,g) = \sum_{\nu=0}^{n} m_\nu^{-2} \int_0^\infty f^{(\nu)}(x)\, \overline{g^{(\nu)}(x)}\, dx .$$

It is well known that the extremal function φ_n is the unique solution of the system of equations

$$(1.3) \qquad (\varphi - \varphi_n, e^{-r_j x}) = 0, \qquad j = 1,2,\ldots,n .$$

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If f is an arbitrary function in C_n, partial integrations give for $1 \leqq \nu \leqq n$

$$\int_0^\infty f^{(\nu)}(x)\, e^{-\bar{r}_j x}\, dx = -\sum_{k=1}^{\nu} \bar{r}_j^{\,k-1} f^{(\nu-k)}(0) + \bar{r}_j^{\,\nu} \int_0^\infty f(x)\, e^{-\bar{r}_j x}\, dx .$$

Hence

$$\begin{aligned}(f, e^{-r_j x}) &= \sum_{\nu=0}^{n} (-1)^\nu m_\nu^{-2} \bar{r}_j^{\,\nu} \int_0^\infty f^{(\nu)}(x)\, e^{-\bar{r}_j x}\, dx \\ &= \sum_{\nu=1}^{n} (-1)^{\nu-1} m_\nu^{-2} \bar{r}_j^{\,\nu} \sum_{k=1}^{\nu} \bar{r}_j^{\,k-1} f^{(\nu-k)}(0) + \sum_{\nu=0}^{n} (-1)^\nu m_\nu^{-2} \bar{r}_j^{\,2\nu} \int_0^\infty f(x)\, e^{-\bar{r}_j x}\, dx \\ &= \sum_{\mu=0}^{n-1} f^{(\mu)}(0) \sum_{\nu=0}^{\mu} (-1)^\nu m_\nu^{-2} \bar{r}_j^{\,2\nu-\mu-1}\end{aligned}$$

by (1.2). Thus (1.3) is equivalent to

$$\sum_{\mu=0}^{n-1} (\varphi^{(\mu)}(0) - \varphi_n^{(\mu)}(0)) \sum_{\nu=0}^{\mu} (-1)^\nu m_\nu^{-2} \bar{r}_j^{\,2\nu-\mu-1} = 0, \qquad j = 1, 2, \ldots, n .$$

Obviously, this system has the solution

$$\varphi_n^{(\mu)}(0) = \varphi^{(\mu)}(0), \qquad \mu = 0, 1, \ldots, n-1 .$$

Since there is a one-to-one correspondence between φ_n and $\{\varphi_n^{(\mu)}(0)\}_{\mu=0}^{n-1}$ the proof is complete.

On the other hand, the function φ_n has the property

$$\|\varphi_n\|_{(n)} = \min \|f\|_{(n)} \tag{1.4}$$

for all f in C_n satisfying

$$f^{(\mu)}(0) = \varphi^{(\mu)}(0), \qquad \mu = 0, 1, \ldots, n-1 \tag{1.5}$$

(see [5, pp. 127–128]). Thus, the function in the span of $\{e^{-r_j x}\}_{j=1}^{n}$ which is the best approximation of φ in the norm (1.1) at the same time solves the problem of interpolating given values (1.5) by a function f in C_n with minimal norm.

Now let $A = \{A_\nu\}_0^\infty$ be a sequence of positive numbers satisfying

$$A_0 = 1 , \tag{1.6}$$

$$\log A_\nu \text{ is a convex function of } \nu , \tag{1.7}$$

and

$$\underline{\lim}_{\nu\to\infty} (A_\nu/\nu!)^{1/\nu} > 0 . \tag{1.8}$$

Let $\mathscr{F}_A$ be the Banach space of all complex-valued, infinitely differentiable functions $f(x)$, defined for $x \geqq 0$, for which

$$\|f\|^2 = \sum_{\nu=0}^{\infty} A_\nu^{-2} \int_0^\infty |f^{(\nu)}(x)|^2 \, dx < \infty$$

and suppose that

$$\sum_{\nu=0}^{\infty} A_\nu^{-2} z^{2\nu} = \prod_{j=1}^{\infty} (1 + r_j^{-2} z^2), \qquad r_j > 0 , \tag{1.9}$$

where $r_i < r_j$ for $i < j$. If $f \in \mathscr{F}_A$ it follows that

$$\sup_{x \geq 0} |f^{(\nu)}(x)| \leqq K(A_\nu A_{\nu+1})^{\frac{1}{2}}, \qquad \nu = 0, 1, 2, \ldots, \tag{1.10}$$

where K is a constant (depending on f) (see [6, Lemma 6]).

Let φ be a fixed function in $\mathscr{F}_A$. It follows that $\varphi \in C_n$ for every n and, by (1.4),

$$\|\varphi_n\|_{(n)} \leqq \|\varphi\|_{(n)} \leqq \|\varphi\| . \tag{1.11}$$

Using (1.11) it is easy to prove that there exists a subsequence $\{\varphi_{n_k}(x)\}$ converging to an infinitely differentiable function $\psi(x)$ in such a way that

$$\varphi_{n_k}^{(\nu)}(x) \to \psi^{(\nu)}(x), \qquad \nu = 0, 1, 2, \ldots ,$$

uniformly on every interval $[0, a]$ and $\psi(x)$ satisfies (1.10) (see Mandelbrojt [3, pp. 104–105]). But

$$\psi^{(\nu)}(0) = \varphi^{(\nu)}(0), \qquad \nu = 0, 1, 2, \ldots .$$

If the sequence A has the property that the inequalities (1.10) define a quasi-analytic class, and this is the case if and only if (see Mandelbrojt [3])

$$\int_0^\infty x^{-2} \log \left(\textstyle\sum_0^\infty x^{2\nu}/A_\nu^2\right) dx = \infty , \tag{1.12}$$

then, since φ and ψ both satisfy (1.10), we have $\psi \equiv \varphi$. This also implies that

$$\lim_{k\to\infty} \|\varphi - \varphi_{n_k}\|_{(n_k)} = 0 .$$

In the special case $\varphi(x) = e^{-\alpha x}$, $\operatorname{Re} \alpha > 0$, a calculation yields

$$\|\varphi - \varphi_n\|_{(n)}^2 = \frac{1}{2 \operatorname{Re} \alpha} \prod_{j=1}^{n} |1 - \alpha/r_j|^2 .$$

This tends to zero as $n \to \infty$ if and only if

$$\sum_{j=1}^{n} r_j^{-1} = \pi^{-1} \int_0^\infty x^{-2} \log \left(\sum_{\nu=0}^{n} x^{2\nu}/m_\nu^2\right) dx \to \infty$$

(for the equality, see [5, p. 130]), that is, if and only if (1.12) holds.

In view of the above-mentioned facts it is natural to consider the following closure problem. Let $A = \{A_\nu\}_0^\infty$ be a sequence of positive numbers satisfying (1.6)–(1.8). Let, for a fixed $k \geqq 1$, $\mathscr{F}_{A(k)}$ be the Banach space of all complex-valued, infinitely differentiable functions $f(x)$, defined for $x \geqq 0$, for which

$$\|f\|_k^2 = \sum_{\nu=0}^{\infty} k^{-2\nu} A_\nu^{-2} \int_0^\infty |f^{(\nu)}(x)|^2 \, dx < \infty , \tag{1.13}$$

and let $\mathscr{L}_{A(k)}$ be the normed linear subspace of $\mathscr{F}_{A(k)}$ which consists of all f satisfying

$$\|f\|_1 < \infty . \tag{1.14}$$

Further, let $\lambda = \{\lambda_j\}_1^\infty$ be an increasing sequence of positive numbers tending to infinity and such that

$$\sum_{j=1}^{\infty} \lambda_j^{-1} = \infty . \tag{1.15}$$

Then our problem is to decide whether or not the set

$$\Lambda = \{e^{-\lambda_j x} \,; j = 1, 2, 3, \ldots\}$$

is fundamental in $\mathscr{L}_{A(k)}$, that is, the span of Λ is dense in $\mathscr{L}_{A(k)}$. (For a similar problem, see Korenbljum [2].)

Of course, the answer of this question will depend on the sequences A and λ, and on k. Concerning A we speak of *the quasi-analytic case* if (1.12) holds and *the non-quasi-analytic case* if

$$\int_0^\infty x^{-2} \log \left(\sum_0^\infty x^{2\nu}/A_\nu^2\right) dx < \infty . \tag{1.16}$$

The subject of this paper was suggested by Professor Lennart Carleson and I wish to express my deep gratitude to him for his valuable and generously given advice.

2. Approximation of $e^{-\alpha x}$, Re $\alpha > 0$.

We start by considering the approximation in $\mathscr{F}_{A(1)}$ of $e^{-\alpha x}$, $\mathrm{Re}\,\alpha > 0$, by finite linear combinations of functions of the set Λ.

THEOREM 2. *Let*

$$S(R) = \sum_{\lambda_j \leqq R} \lambda_j^{-1} \tag{2.1}$$

and

$$I(R) = \pi^{-1} \int_1^R y^{-2} \log \left(\textstyle\sum_0^\infty y^\nu / A_\nu\right) dy . \tag{2.2}$$

Then, a sufficient condition that every function $e^{-\alpha x}$, $\mathrm{Re}\,\alpha > 0$, *can be approximated arbitrarily closely in* $\mathscr{F}_{A(1)}$ *by finite linear combinations of functions of* Λ *is that*

$$\underline{\lim}_{R\to\infty} \frac{I(R)}{S(R)} < 1 . \tag{2.3}$$

COROLLARY. *In the non-quasi-analytic case every function* $e^{-\alpha x}$, $\mathrm{Re}\,\alpha > 0$, *can be approximated arbitrarily closely in* $\mathscr{F}_{A(1)}$ *by finite linear combinations of functions of* Λ.

PROOF OF THE COROLLARY. This is immediate since (1.16) implies that $I(R)$ is bounded.

PROOF OF THEOREM 2. As is well known, it is sufficient to show that for every bounded, linear functional L defined on $\mathscr{F}_{A(1)}$ it holds true that

$$L(e^{-\lambda_j x}) = 0, \qquad j = 1,2,3,\dots ,$$

implies

$$L(e^{-\alpha x}) \equiv 0, \qquad \mathrm{Re}\,\alpha > 0 .$$

For arbitrary f and g in $\mathscr{F}_{A(1)}$ let

$$(f,g) = \sum_{\nu=0}^{\infty} A_\nu^{-2} \int_0^\infty f^{(\nu)}(x)\, \overline{g^{(\nu)}(x)}\, dx ; \tag{2.4}$$

this makes $\mathscr{F}_{A(1)}$ into a complete inner product space. Then, by the theorem of Fréchet–Riesz, we can represent L in the form

$$L(f) = (f,\varphi), \qquad \varphi \in \mathscr{F}_{A(1)} .$$

Let

$$L(e^{-\alpha x}) = \sum_{\nu=0}^{\infty} (-\alpha)^\nu A_\nu^{-2} \int_0^\infty e^{-\alpha x}\, \overline{\varphi^{(\nu)}(x)}\, dx = F(\alpha), \qquad \mathrm{Re}\,\alpha > 0 . \tag{2.5}$$

We observe that $F(\alpha)$ is holomorphic in the half plane $\mathrm{Re}\,\alpha > 0$, and we have to show that

$$F(\lambda_j) = 0, \qquad j = 1,2,3,\dots , \tag{2.6}$$

implies

$$F(\alpha) \equiv 0, \qquad \operatorname{Re}\alpha > 0 .$$

To do this we use Carleman's theorem: If $f(\alpha)$ is holomorphic for $\operatorname{Re}\alpha \geqq 0$ and if $r_j e^{i\theta_j}$, $j=1,2,3,\ldots$, are the zeros of $f(\alpha)$ in this half plane, then as $R \to \infty$

(2.7)

$$\sum_{r_j \leqq R} (r_j^{-1} - r_j R^{-2}) \cos\theta_j = (\pi R)^{-1} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \log|f(Re^{i\theta})| \cos\theta \, d\theta + $$

$$+ (2\pi)^{-1} \int_1^R (y^{-2} - R^{-2}) \log|f(iy)f(-iy)| \, dy + O(1) .$$

We take $f(\alpha) = F(\alpha+\delta)$ for a fixed $\delta > 0$. Since for $\operatorname{Re}\alpha \geqq 0$

$$\left| \int_0^\infty e^{-(\alpha+\delta)x} \, \overline{\varphi^{(\nu)}(x)} \, dx \right| \leqq (2\delta)^{-\frac{1}{2}} \, \|\varphi\|_1 \, A_\nu ,$$

we have for $|\alpha| \leqq R$, $\operatorname{Re}\alpha \geqq 0$

$$|F(\alpha+\delta)| \leqq (2\delta)^{-\frac{1}{2}} \, \|\varphi\|_1 \sum_0^\infty (R+\delta)^\nu / A_\nu \leqq (2\delta)^{-\frac{1}{2}} \, \|\varphi\|_1 \, e^{\varrho(R+\delta)}$$

by (1.8), for some constant $\varrho > 0$. Hence the first term in the right hand member of (2.7) is bounded above as $R \to \infty$. Further,

$$(2\pi)^{-1} \int_1^R (y^{-2} - R^{-2}) \log|F(iy+\delta)F(-iy+\delta)| \, dy$$

$$\leqq \pi^{-1} \int_1^R (y^{-2} - R^{-2}) \log\big((2\delta)^{-\frac{1}{2}} \, \|\varphi\|_1 \textstyle\sum_0^\infty (y+\delta)^\nu / A_\nu\big) \, dy$$

$$\leqq \pi^{-1} \int_1^R y^{-2} \log\big(\textstyle\sum_0^\infty (y+\delta)^\nu / A_\nu\big) \, dy + O(1)$$

$$= (1-\delta)^{-2} \, I(R) + O(1)$$

and, for $0 < \beta < 1$,

$$\textstyle\sum_{\lambda_j - \delta \leqq R} \big((\lambda_j - \delta)^{-1} - (\lambda_j - \delta) R^{-2}\big) \geqq (1-\beta^2) \, S(\beta R) .$$

Then, by (2.7) and since $I(R) = I(\beta R) + O(1)$,

$$(1-\beta^2) \, S(\beta R) \leqq (1-\delta)^{-2} I(\beta R) + O(1) .$$

Here $S(\beta R) \to \infty$ as $R \to \infty$ by (1.15), and choosing β and δ sufficiently small this leads to a contradiction if (2.3) holds. Hence $F(\alpha+\delta)\equiv 0$ for $\mathrm{Re}\,\alpha \geqq 0$ and thus, since we can take $\delta>0$ arbitrarily small, $F(\alpha)\equiv 0$ for $\mathrm{Re}\,\alpha>0$. This completes the proof of Theorem 2.

3. The non-quasi-analytic case.

In this section we consider the closure porblem in the non-quasi-analytic case (1.16). We prove the following theorem which can be considered as an extension of the well-known theorem of Müntz (see Schwartz [4]).

THEOREM 3. *In the non-quasi-analytic case, Λ is fundamental in $\mathscr{L}_{A(k)}$, $k>1$.*

To prove this theorem we need some lemmas. For a function f, defined in a set containing $[0,\infty)$, we always denote its restriction to $[0,\infty)$ by the same symbol f.

LEMMA 1. *Let $f\in\mathscr{F}_{A(k)}$, and define, for $\eta>0$,*

$$f_\eta(x) = f(x+\eta), \qquad x\geqq -\eta\,.$$

Then

$$\lim_{\eta\to 0+}\|f-f_\eta\|_k = 0, \qquad k\geqq 1\,.$$

PROOF. Obviously

$$\sum_{\nu=N}^{\infty} k^{-2\nu}A_\nu^{-2}\int_0^\infty |f^{(\nu)}(x)-f^{(\nu)}(x+\eta)|^2\,dx$$

becomes arbitrarily small for N sufficiently large, independent of η. Having fixed N, we can make

$$\sum_{\nu=0}^{N-1} k^{-2\nu}A_\nu^{-2}\int_0^\infty |f^{(\nu)}(x)-f^{(\nu)}(x+\eta)|^2\,dx$$

arbitrarily small by choosing η sufficiently small, since

$$\lim_{\eta\to 0+}\int_0^\infty |f^{(\nu)}(x)-f^{(\nu)}(x+\eta)|^2\,dx = 0\,.$$

LEMMA 2. *If* (1.16) *holds, there exists for arbitrary constants $\eta>0$ and $k_0>0$ an infinitely differentiable function $\varphi(x)$, defined on $(-\infty,\infty)$, such that*

(3.1) $$\varphi(x) = \begin{cases} 0 & \text{for } x \leqq 0 \\ 1 & \text{for } x \geqq \eta \end{cases}$$

and for some constant C

(3.2) $$\sup_x |\varphi^{(\nu)}(x)| \leqq C k_0^{\nu} A_\nu, \quad \nu = 0, 1, 2, \ldots .$$

PROOF. By (1.7), the sequence $\{A_{\nu-1}/A_\nu\}_1^\infty$ is decreasing. Further, (1.16) is equivalent to (see Mandelbrojt [3])

$$\sum_1^\infty A_{\nu-1}/A_\nu < \infty .$$

Then we can construct a sequence $\{B_\nu\}_0^\infty$ such that

(3.3) $$B_{\nu-1}/B_\nu = q_\nu A_{\nu-1}/A_\nu, \quad \nu = 1, 2, 3, \ldots ,$$

where $q_\nu \to \infty$ as $\nu \to \infty$, the sequence $\{B_{\nu-1}/B_\nu\}_1^\infty$ is decreasing and

$$\sum_1^\infty B_{\nu-1}/B_\nu < \infty .$$

It is well known that there exists an infinitely differentiable function $\varphi(x)$, defined on $(-\infty, \infty)$, satisfying (3.1) and, for constants C_η and K_η depending on η,

$$\sup_x |\varphi^{(\nu)}(x)| \leqq C_\eta K_\eta^{\nu} B_\nu, \quad \nu = 0, 1, 2, \ldots .$$

Hence, by (3.3), the inequalities (3.2) are fulfilled if C is sufficiently large.

LEMMA 3. *If* (1.16) *holds and if* $f \in \mathscr{L}_{A(k)}$, $k > 1$, *then for every* $\varepsilon > 0$ *there exist constants* $\eta > 0$ (*small*) *and* $L > 0$ (*large*) *and an infinitely differentiable function* $g(x)$, *defined on* $(-\infty, \infty)$, *such that* $g(x) \equiv 0$ *outside* $(-\eta, L)$,

(3.4) $$\sum_{\nu=0}^\infty k^{-2\nu} A_\nu^{-2} \int_{-\infty}^{\infty} |g^{(\nu)}(x)|^2 \, dx < \infty$$

and

(3.5) $$\|f - g\|_k < \varepsilon .$$

PROOF. First, by Lemma 1, we can choose $\eta > 0$ so small that

$$\|f - f_\eta\|_k < \tfrac{1}{2}\varepsilon .$$

By Lemma 2, for arbitrary numbers $0 < L_1 < L$ there exists for an arbitrary $k_0 > 0$ an infinitely differentiable function $\psi(x)$ such that

$$\psi(x) = \begin{cases} 1, & -\tfrac{1}{2}\eta \leqq x \leqq L_1, \\ 0, & \text{outside } (-\eta, L), \end{cases}$$

and

$$(3.6) \qquad \sup_x |\psi^{(\nu)}(x)| \leqq C k_0^{\nu} A_\nu, \qquad \nu = 0,1,2,\ldots .$$

Define

$$g(x) = \begin{cases} f_\eta(x)\,\psi(x), & x \geqq -\eta\,, \\ 0 & , \quad x < -\eta\,. \end{cases}$$

By (1.7) and (1.6)

$$(3.7) \qquad A_{\nu-j} A_j \leqq A_\nu, \qquad j = 0,1,\ldots,\nu\,.$$

We also need the simple inequality

$$(3.8) \qquad \binom{\nu}{j}^2 \leqq \binom{2\nu}{2j}, \qquad j = 0,1,\ldots,\nu\,.$$

For $x \geqq -\eta$ and $\nu \geqq 0$ we have, by (3.6) and (3.7),

$$|g^{(\nu)}(x)| = \left| \sum_{j=0}^{\nu} \binom{\nu}{j} f^{(j)}(x+\eta)\,\psi^{(\nu-j)}(x) \right| \leqq C A_\nu \sum_{j=0}^{\nu} \binom{\nu}{j} k_0^{\nu-j} A_j^{-1}\, |f^{(j)}(x+\eta)|\,.$$

Hence, by Cauchy's inequality,

$$|g^{(\nu)}(x)|^2 \leqq C^2 A_\nu^2 \sum_{j=0}^{\nu} \binom{\nu}{j}^2 k_0^{2\nu-2j} \sum_{j=0}^{\nu} A_j^{-2}\, |f^{(j)}(x+\eta)|^2\,.$$

But, by (3.8),

$$\sum_{j=0}^{\nu} \binom{\nu}{j}^2 k_0^{2\nu-2j} \leqq \sum_{i=0}^{2\nu} \binom{2\nu}{i} k_0^{2\nu-i} = (1+k_0)^{2\nu}\,.$$

Thus

$$\int_{-\infty}^{\infty} |g^{(\nu)}(x)|^2\, dx \leqq C^2 A_\nu^2 (1+k_0)^{2\nu}\, \|f\|_1^2$$

and finally

$$\sum_{\nu=0}^{\infty} k^{-2\nu} A_\nu^{-2} \int_{-\infty}^{\infty} |g^{(\nu)}(x)|^2\, dx \leqq C^2 \|f\|_1^2 \sum_{\nu=0}^{\infty} (k^{-1}(1+k_0))^{2\nu} < \infty$$

if we choose k_0 small enough.

By similar estimates

$$\|f_\eta - g\|_k^2 \leqq (C+1)^2 \sum_{j=0}^{\infty} A_j^{-2} \int_{L_1}^{\infty} |f^{(j)}(x)|^2\, dx \sum_{\nu=0}^{\infty} (k^{-1}(1+k_0))^{2\nu}\,.$$

Since $\|f\|_1 < \infty$ this can be made arbitrarily small by taking L_1 sufficiently large. Thus

$$\|f_\eta - g\|_k < \tfrac{1}{2}\varepsilon$$

and (3.5) is proved.

Proof of Theorem 3. By the Corollary of Theorem 2, every function $e^{-\alpha x}$, $\mathrm{Re}\,\alpha > 0$, can be approximated arbitrarily closely in $\mathscr{F}_{A(1)}$, hence a fortiori in $\mathscr{F}_{A(k)}$, $k>1$, by finite linear combinations of functions of Λ. Then, by Lemma 3, it is sufficient to show that if g is a function of the type described in Lemma 3, there exists for every $\varepsilon>0$ a function of the form

$$Q(x) = \sum_{\nu=1}^{m} c_\nu e^{-\alpha_\nu x}\,, \tag{3.9}$$

where α_ν are complex numbers with $\mathrm{Re}\,\alpha_\nu>0$, such that

$$\|g-Q\|_k < \varepsilon\,.$$

For an arbitrary positive integer n, let

$$P_n(x) = a_n^{-1}\big(1-(1-2^{-x/L})^2\big)^n\, 2^{-x/L}\,,$$

where a_n is chosen so that

$$\int_{-L}^{\infty} P_n(x)\,dx = 1; \tag{3.10}$$

we find

$$a_n = \frac{L}{\log 2}\int_{-1}^{1}(1-t^2)^n\,dt = \frac{2L}{\log 2}\,\frac{(2n)!!}{(2n+1)!!}.$$

The function

$$Q(x) = \int_{-\infty}^{\infty} P_n(x-y)\,g(y)\,dy$$

is of the form (3.9) and

$$Q^{(\nu)}(x) = \int_{-\infty}^{\infty} P_n(y)\,g^{(\nu)}(x-y)\,dy\,.$$

Hence for $x \geqq 0$, by (3.10) and Schwarz's inequality,

$$|g^{(\nu)}(x)-Q^{(\nu)}(x)|^2 = \left|\int_{-L}^{\infty} P_n(y)\,\big(g^{(\nu)}(x)-g^{(\nu)}(x-y)\big)\,dy\right|^2$$

$$\leqq \int_{-L}^{\infty} P_n(y)\,|g^{(\nu)}(x)-g^{(\nu)}(x-y)|^2\,dy\,,$$

and so

$$\|g-Q\|_k^2 \leqq \int_{-L}^{\infty} P_n(y)\,dy \sum_{\nu=0}^{\infty} k^{-2\nu}A_\nu^{-2}\int_{0}^{\infty} |g^{(\nu)}(x)-g^{(\nu)}(x-y)|^2\,dx\,. \tag{3.11}$$

By (3.4), we can make

$$\sum_{\nu=N}^{\infty} k^{-2\nu} A_{\nu}^{-2} \int_{0}^{\infty} |g^{(\nu)}(x) - g^{(\nu)}(x-y)|^2 \, dx$$

arbitrarily small by taking N sufficiently large, independent of y. Having fixed N, since

$$\lim_{y \to 0} \int_{-\infty}^{\infty} |g^{(\nu)}(x) - g^{(\nu)}(x-y)|^2 \, dx = 0$$

also

$$\sum_{\nu=0}^{N-1} k^{-2\nu} A_{\nu}^{-2} \int_{0}^{\infty} |g^{(\nu)}(x) - g^{(\nu)}(x-y)|^2 \, dx$$

is arbitrarily small if $|y|$ is sufficiently small. Thus, there exists a $\delta > 0$ such that, independent of n,

$$I_1 = \int_{-\delta}^{\delta} P_n(y) \, dy \sum_{\nu=0}^{\infty} k^{-2\nu} A_{\nu}^{-2} \int_{0}^{\infty} |g^{(\nu)}(x) - g^{(\nu)}(x-y)|^2 \, dx < \tfrac{1}{2}\varepsilon^2 .$$

Finally, having fixed δ, we have to consider that part I_2 of the integral with respect to y in the right hand member of (3.11), where $|y| \geqq \delta$. By (3.4) it is sufficient to show that

$$\lim_{n \to \infty} \left(\int_{-L}^{-\delta} P_n(y) dy + \int_{\delta}^{\infty} P_n(y) \, dy \right) = 0 .$$

But that follows from

$$\int_{-L}^{-\delta} P_n(y) \, dy = \frac{(2n+1)!!}{2\,(2n)!!} \int_{-1}^{1-2^{\delta/L}} (1-t^2)^n \, dt \leqq \text{const.}\, n^{\frac{1}{2}} \left(1-(1-2^{\delta/L})^2\right)^n$$

and an analogous estimate of the second integral. Hence, for sufficiently large n, $I_2 < \frac{1}{2}\varepsilon^2$.

This completes the proof of Theorem 3.

4. The quasi-analytic case.

In the quasi-analytic case, (1.15) is not sufficient for a set Λ to be fundamental in $\mathscr{L}_{A(k)}$.

Starting with an increasing sequence λ of positive numbers such that (1.15) holds, suppose that a sequence A satisfying (1.6)–(1.8) and (1.12) is defined by

$$\prod_{j=1}^{\infty}(1+\lambda_j^{-2}z^2) = \sum_{\nu=0}^{\infty} A_\nu^{-2} z^{2\nu} \tag{4.1}$$

((1.7) is automatically fulfilled; see Boas [1, p. 24]). Then, as will appear below, there are reasons to expect the following to be true: The set

$$\Lambda_a = \{e^{-a\lambda_j x}\,; j=1,2,3,\ldots\}, \qquad a>0\,,$$

is fundamental in $\mathscr{L}_{A(k)}$, $k>1$, if $a<2$; but if $a>2k$, $k\geqq 1$, there exists, for instance, an exponential function which does not belong to the closure in $\mathscr{F}_{A(k)}$ of the span of Λ_a. To motivate this conjecture we prove the following two theorems.

THEOREM 4. *Suppose that the sequence* A, *defined by* (4.1), *fulfils* (1.12). *Then if* $a<2$ *every function* $e^{-\alpha x}$, $\operatorname{Re}\alpha>0$, *belongs to the closure in* $\mathscr{F}_{A(1)}$ *of the span of* Λ_a.

PROOF. By Theorem 2 it is sufficient to prove that (2.3) holds. Applying Carleman's theorem to the entire function of exponential type

$$\prod_{j=1}^{\infty}(1-\lambda_j^{-2}z^2) = \sum_{\nu=0}^{\infty}(-1)^\nu A_\nu^{-2} z^{2\nu}$$

we get, since the first term in the right hand member of (2.7) is bounded in this case (see Boas [1, p. 31]),

$$\begin{aligned} aS(aR) &= \sum_{\lambda_j\leqq R}\lambda_j^{-1} \geqq \sum_{\lambda_j\leqq R}(\lambda_j^{-1}-\lambda_j R^{-2}) \\ &= \pi^{-1}\int_1^R y^{-2}\log\left(\sum_{\nu=0}^{\infty}A_\nu^{-2}y^{2\nu}\right)dy + O(1)\,. \end{aligned}$$

But for an arbitrary β, $0<\beta<1$,

$$(1-\beta^2)\left(\sum_{\nu=0}^{\infty}A_\nu^{-1}(\beta y)^\nu\right)^2 \leqq \sum_{\nu=0}^{\infty}A_\nu^{-2}y^{2\nu}$$

and hence

$$\int_1^R y^{-2}\log\left(\sum_{\nu=0}^{\infty}A_\nu^{-2}y^{2\nu}\right)dy \geqq 2\beta\int_1^{aR} y^{-2}\log(\textstyle\sum_{\nu=0}^{\infty}A_\nu^{-1}y^\nu)\,dy + O(1)\,.$$

Thus

$$a\,S(R) \geqq 2\beta I(R) + O(1)$$

and so

$$\underline{\lim}_{R\to\infty}\frac{I(R)}{S(R)} \leqq \frac{a}{2\beta} < 1$$

if we choose $\beta>\frac{1}{2}a$.

THEOREM 5. *Let* $\lambda_j = \frac{1}{2}\pi j$, $j = 1, 2, 3, \ldots$. *Then the corresponding set* Λ_a^0 *is fundamental in* $\mathscr{L}_{A(k)}$, $k > 1$, *if* $a \leqq 2$, *but if* $a > 2k$ *there exists, for instance, an exponential function which does not belong to the closure in* $\mathscr{F}_{A(k)}$, $k \geqq 1$, *of the span of* Λ_a^0.

PROOF. We have in this case

$$A_\nu = 2^{-\nu}\{(2\nu+1)!\}^{\frac{1}{2}} = \nu!\,(4\pi^{-1}\nu)^{\frac{1}{4}}\,(1+\varepsilon_\nu)\,, \tag{4.3}$$

where $\varepsilon_\nu \to 0$ as $\nu \to \infty$. If $f \in \mathscr{L}_{A(k)}$ it follows that

$$\sup_{x \geqq 0} |f^{(\nu)}(x)| \leqq \text{const.}\, \gamma^\nu \nu! \tag{4.4}$$

for an arbitrary $\gamma > 1$. Hence $f(x)$ is the restriction to $[0, \infty)$ of a function $f(z)$, holomorphic in the domain

$$D: \begin{cases} |\mathrm{Im}\, z| < 1, & \mathrm{Re}\, z \geqq 0, \\ \quad |z| < 1, & \mathrm{Re}\, z < 0, \end{cases}$$

and bounded in every set

$$\bar{D}_b: \begin{cases} |\mathrm{Im}\, z| \leqq b < 1, & \mathrm{Re}\, z \geqq 0\,, \\ \quad |z| \leqq b < 1, & \mathrm{Re}\, z < 0\,. \end{cases}$$

a. The case $a \leqq 2$, $k > 1$.

We need the following lemma.

LEMMA 4. *For* $f \in \mathscr{L}_{A(k)}$,

$$\lim_{\delta \to 0+} \|f(x) - e^{-\delta x} f(x)\|_k = 0\,.$$

PROOF. For

$$g_\delta(x) = f(x)(1 - e^{-\delta x}), \qquad x \geqq 0,\ \delta > 0\,,$$

we find

$$|g_\delta^{(\nu)}(x)| \leqq \sum_{j=0}^{\nu-1} \binom{\nu}{j} \delta^{\nu-j}\, |f^{(j)}(x)| + (1 - e^{-\delta x})\, |f^{(\nu)}(x)|\,.$$

By Cauchy's inequality,

$$\tfrac{1}{2}|g_\delta^{(\nu)}(x)|^2 \leqq \sum_{j=0}^{\nu-1} \binom{\nu}{j}^2 (2\delta)^{2\nu-2j}\, |f^{(j)}(x)|^2 + (1 - e^{-\delta x})^2\, |f^{(\nu)}(x)|^2,$$

and thus

$$\tfrac{1}{2}\int_0^\infty |g_\delta^{(\nu)}(x)|^2\, dx \leqq \|f\|_1^2 \sum_{j=0}^{\nu-1} \binom{\nu}{j}^2 (2\delta)^{2\nu-2j} A_j^2 + \int_0^\infty (1 - e^{-\delta x})^2\, |f^{(\nu)}(x)|^2\, dx\,.$$

Finally,

$$\tfrac{1}{2}\|g_\delta\|_k^2 \leqq \|f\|_1^2 \sum_{\nu=0}^{\infty} k^{-2\nu} A_\nu^{-2} \sum_{j=0}^{\nu-1} \binom{\nu}{j}^2 (2\delta)^{2\nu-2j} A_j^2 +$$

$$+ \sum_{\nu=0}^{\infty} k^{-2\nu} A_\nu^{-2} \int_0^{\infty} (1-e^{-\delta x})^2 \, |f^{(\nu)}(x)|^2 \, dx$$

$$= S_1(\delta) + S_2(\delta) .$$

For $\delta < \frac{1}{4}$, by (3.8) and (4.3),

$$\|f\|_1^{-2} S_1(\delta) \leqq (4\delta)^2 \sum_{\nu=0}^{\infty} k^{-2\nu} \sum_{j=0}^{\infty} 1/(2j)!\,,$$

and so

$$\lim_{\delta\to 0+} S_1(\delta) = 0 .$$

By dominated convergence,

$$\lim_{\delta\to 0+} S_2(\delta) = 0 ,$$

and thus the proof of Lemma 4 is complete.

To prove Theorem 5 in the case $a \leqq 2$ we perform the conformal mapping $w = e^{-\frac{1}{2}\pi a z}$. Then D_b is mapped onto a Jordan region Ω_b in the w-plane with 0 on the boundary. Let for $0 < \delta < 1$

$$F(w) = \begin{cases} w^\delta f(z), & w \neq 0 , \\ 0 \quad , & w = 0 . \end{cases}$$

Since $F(w)$ is holomorphic in Ω_b and continuous in $\bar{\Omega}_b$, $F(w)$ can be approximated uniformly and arbitrarily closely in $\bar{\Omega}_b$ by a polynomial in w without constant term. But every function $w^{m-1+\delta}$, where m is a positive integer, can be approximated in the same way and this implies that, for an arbitrary $\varepsilon > 0$, there exists a function

$$P(w) = \sum_{j=1}^{N} a_j w^{j-\delta}$$

such that

$$\max_{\bar{\Omega}_b} |F(w) - P(w)| < \varepsilon .$$

This yields, for

$$h(z) = e^{-\pi a \delta z} f(z) \qquad \text{and} \qquad Q(z) = \sum_{j=1}^{N} a_j e^{-j\frac{1}{2}\pi a z} ,$$

the inequality

$$\max_{\bar{D}_b} |h(z) - Q(z)| < C\varepsilon$$

and, for some constant M independent of ε,

$$|Q(z)| < M e^{-\frac{1}{4}\pi a\delta x}, \qquad z \in \bar{D}_b .$$

By Lemma 4 it is now sufficient to prove that $\|h-Q\|_k$ can be made arbitrarily small.

By Cauchy's estimates,

$$|Q^{(\nu)}(x)| \leqq M\, \nu!\, b^{-\nu}\, e^{-\frac{1}{4}\pi a\delta(x-1)}, \qquad x \geqq 0 ,$$

and hence, choosing b so that $kb > 1$,

$$\sum_{\nu=0}^{\infty} k^{-2\nu} A_\nu^{-2} \int_R^\infty |Q^{(\nu)}(x)|^2\, dx \leqq \frac{M^2}{\pi a\delta} e^{-\pi a\delta(R-1)} \sum_{\nu=0}^{\infty} (kb)^{-2\nu} .$$

This, and also

$$\sum_{\nu=0}^{\infty} k^{-2\nu} A_\nu^{-2} \int_R^\infty |h^{(\nu)}(x)|^2\, dx ,$$

is arbitrarily small if R is sufficiently large. Finally, for $x \geqq 0$,

$$|h^{(\nu)}(x) - Q^{(\nu)}(x)| \leqq C b^{-\nu} \nu!\, \varepsilon ,$$

and hence

$$\sum_{\nu=0}^{\infty} k^{-2\nu} A_\nu^{-2} \int_0^R |h^{(\nu)}(x) - Q^{(\nu)}(x)|^2\, dx < R C^2 \varepsilon^2 \sum_{\nu=0}^{\infty} (kb)^{-2\nu} ,$$

which is arbitrarily small if ε is sufficiently small. This proves Theorem 5 in the case $a \leqq 2$.

b. The case $a > 2k$, $k \geqq 1$.

Suppose that for a certain function f in $\mathscr{L}_{A(k)}$ there exists a sequence $\{Q_n(z)\}_{n=1}^\infty$, where $Q_n(x)$ belongs to the span of $\Lambda_a{}^0$, such that

$$\lim_{n\to\infty} \|f - Q_n\|_k = 0 . \tag{4.5}$$

Hence

$$\|Q_n\|_k \leqq \text{const.}$$

uniformly in n and so for an arbitrary $\gamma > 1$

$$\sup_{x \geqq 0} |Q_n{}^{(\nu)}(x)| \leqq \text{const.}\,(k\gamma)^\nu\, \nu! .$$

Thus, on every compact subset of the half strip

$$|\mathrm{Im}\, z| < 1/k, \qquad \mathrm{Re}\, z > 0 , \tag{4.6}$$

we have

$$\sup |Q_n(z)| \leqq \text{const.}$$

uniformly in n. The family $\{Q_n(z)\}_{n=1}^{\infty}$ is thus normal in the half strip (4.6). Since, by (4.5),

$$\lim_{n\to\infty} Q_n(x) = f(x)$$

uniformly on $[0,\infty)$, we infer that

$$\lim_{n\to\infty} Q_n(z) = f(z)$$

uniformly on every compact subset of the half strip (4.6). But $Q_n(z)$ is periodic with period $4i/a$ and then this is true also for $f(z)$, if the width of the half strip is greater than $4/a$, that is, if $a > 2k$. However, there are for instance exponential functions for which this is not true. This completes the proof of Theorem 5.

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