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TAUBERIAN PROBLEMS FOR THE n-DIMENSIONAL LAPLACE TRANSFORM II

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Introduction.

In [4] the author estimated the remainder in a Tauberian theorem for the n-dimensional Laplace transform. When applied to the one-dimensional case, this result includes the remainder analogue of the part within parenthesis of the following Tauberian theorem of Hardy and Littlewood [7]. (The first method to prove it originates from Littlewood [9].)

Suppose that α is a non-decreasing function with $\alpha(0)=0$, and that A and μ are real numbers, $\mu \geqq 0$. If

$$F(s) = \int_0^\infty \exp(-st)\, d\alpha(t)$$

converges for all $s>0$ and if

$$F(s) \sim A s^{-\mu}, \quad s \to +\infty \ (s \to +0)\,,$$

then

$$\alpha(s) \sim A s^{\mu} \Gamma(1+\mu)^{-1}, \quad s \to +0 \ (s \to +\infty)\,.$$

In this paper we will consider a Tauberian problem which includes a remainder theorem corresponding to the case outside the parenthesis. From certain restrictions on F for large values of its argument we wish to get information about the behaviour of α near the origin.

The method of proof is similar to that in [4], and since it is applicable to several dimensions, we carry it through in that case. We first prove a general result which, for example, also applies to a convolution kernel associated with the Meijer transform. We then apply this result to the n-dimensional Laplace transform. In Section 3 we discuss the precision of the estimates, and in Section 4 we will consider some results concerning the problem of how fast F may tend to zero as its argument tends to infinity without forcing α to vanish in some neighbourhood of the origin.

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We repeat some of the notation from [4, pp. 41–42].

If $x=(x_1,x_2,\ldots,x_n)\in R^n$ and $y=(y_1,y_2,\ldots,y_n)\in R^n$, then

$$x\cdot y = \sum_{\nu=1}^{n} x_\nu y_\nu, \qquad x^y = \prod_{\nu=1}^{n} x_\nu^{y_\nu}, \qquad |x| = \sum_{\nu=1}^{n} |x_\nu| \,.$$

By $x\leqq y$ we mean that $x_\nu \leqq y_\nu$, $\nu=1,2,\ldots,n$, and by $x\to +0$ and $x\to +\infty$ we mean that $x_\nu\to +0$ and $x_\nu \to +\infty$, respectively, for $\nu = 1,2,\ldots,n$. We let R_+^n be all $x\in R^n$ such that $x\geqq 0$ and we use the abbreviations:

$$\begin{aligned} \exp x &= (\exp x_1, \exp x_2,\ldots, \exp x_n)\\ \log x &= (\log x_1, \log x_2,\ldots, \log x_n)\\ \max(x,y) &= (\max(x_1,y_1), \max(x_2,y_2),\ldots, \max(x_n,y_n))\\ \mathbf{1} &= (1,1,\ldots,1)\end{aligned}$$

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1. A general Tauberian remainder theorem.

We first introduce the class E_1 of convolution kernels, which we will consider.

Definition. By E_1 we denote the set of functions $K\in L(R^n)$ which satisfy the following three conditions:

1° $\hat{K}(t)\neq 0$ for all $t\in R^n$.

2° The function g defined by

$$g(t) = \hat{K}(t)^{-1}$$

can be continued analytically into a region $\operatorname{Im} t > -\varrho$, with $\varrho > 0$.

3° The function g satisfies the inequality

$$|g(t)| \leqq C\exp\left(m|x| + p\cdot y - y\cdot\log(1+y)\right) \tag{1.1}$$

for some real positive m and some $p\in R_+^n$ and for all $t=x+iy$ such that $y=\operatorname{Im} t>\max(-\mathbf{1},-\varrho)$.

Let H be a continuous function from R_+^n to R, which is strictly positive and non-decreasing and such that if r is real, then

$$H(rx) < rH(x), \quad r>1, \quad \text{and} \quad \overline{\lim}_{r\to+\infty} rH(rx)^{-1} = +\infty \quad \text{for all } x\in R_+^n\,. \tag{1.2}$$

We also suppose that $H(x)\to +\infty$ when $x\to +\infty$. If G is the transformation defined by

$$G(x) = xH(x)^{-1},$$

then G has an inverse q, and we introduce H_1 by the condition

$$H_1(x) = H(q(x))^{-1}.$$

It follows from (1.2) and the definition of q that for any $x \in R_+^n$ and $r > 1$ there exists a ν such that $q_\nu(rx) \geqq rq_\nu(x)$. Hence

$$H_1(rx) \leqq H_1(x), \quad r > 1.$$

The following theorem states our general result.

THEOREM 1. *Suppose that $K \in E_1$, that H and H_1 are as defined above, and that φ is a bounded and measurable function from R^n to R. Let*

$$(1.3) \qquad K*\varphi(x) = O\big(\exp(-H(\exp x))\big), \quad x \to +\infty,$$

and

$$(1.4) \qquad \inf_t(\varphi(t)-\varphi(x)) = O(H_1(\exp x)), \quad x \to +\infty,$$

where the infimum is taken over all t with $x \leqq t \leqq x + \mathbf{1} \cdot H_1(\exp x)$. Then there exists a real positive constant a, such that

$$\varphi(x) = O(H_1(a \exp x)), \quad x \to +\infty.$$

(For the value of the constant a see the end of the proof and the remark following.)

PROOF. The method of proof is similar to that used in [4]. We start with the following inequality.

There exists a constant C such that, if u is an integrable function, then

$$(1.5) \quad \sup_{x \in R^n} |u(x)| \leqq C\{-\inf_{x \leqq t \leqq x+h}(u(t)-u(x)) + \int_{-V \leqq \xi \leqq V} |\hat{u}(\xi)| d\xi\}$$

for all positive $V = (V_1, V_2, \ldots, V_n)$ and $h = (h_1, h_2, \ldots, h_n)$ with

$$h_\nu = V_\nu^{-1}, \quad \nu = 1, 2, \ldots, n.$$

We apply (1.5) to the function u defined by the relation,

$$u(x) = \exp(-\tfrac{1}{2}(x-y)^2\omega^2)\,\varphi(x), \quad y \in R_+^n, \ \omega > 1.$$

Then we have

$$(1.6) \qquad \hat{u}(\xi) = \exp(-i\xi \cdot y)\int \psi(y-x)\, Q(x)\, dx,$$

where an unspecified region of integration is R^n, and where

$$\psi(x) = K*\varphi\,(x)$$

and

$$(1.7)\qquad Q(x) = (2\pi)^{-\frac{1}{2}n}\omega^{-n}\int \exp\left(ix\cdot v - \tfrac{1}{2}(\nu-\xi)^2\omega^{-2}\right) g(v)\, dv\,.$$

(Cf. [3, p. 81] or [4, p. 44].)

In (1.7) we make the substitution $v=t+\xi+is$ with $s>\max(-1,-\varrho)$. After an estimation based on (1.1) we conclude that (Cf. [4, p. 44])

$$(1.8)\quad |Q(x)| \leqq C\exp\left(-s\cdot x+\tfrac{1}{2}s^2\omega^{-2}+p\cdot s-s\cdot\log(1+s)+\tfrac{1}{2}m^2n\omega^2+m|\xi|\right).$$

Now suppose that

$$s = (s_1,s_2,\ldots,s_n),\qquad x = (x_1,x_2,\ldots,x_n)\,,$$
$$p = (p_1,p_2,\ldots,p_n),\qquad \varrho = (\varrho_1,\varrho_2,\ldots,\varrho_n)\,,$$

and $z=(z_1,z_2,\ldots,z_n)$, where z is defined by

$$(1.9)\qquad z_\nu = \log(1+m\omega^2)-p_\nu-\tfrac{1}{2}m(n+1)-\varepsilon m^{-1},\qquad \nu=1,2,\ldots,n\,,$$

and where ε is an arbitrary real positive number, which is supposed to be the same throughout the proof. For each ν we let s_ν depend on x_ν, z_ν and ϱ_ν in the following way

$$s_\nu = \begin{cases} -\tfrac{1}{2}\min(1,\varrho_\nu) & \text{for}\quad -\infty<x_\nu<-z_\nu\,,\\ m\omega^2 & \text{for}\quad -z_\nu\leqq x_\nu\,. \end{cases}$$

Since φ is bounded, we see that ψ is bounded. By (1.3) we have that, if y is large, then

$$(1.10)\qquad |\psi(x)| \leqq C\exp(-H(\exp x))\quad \text{for}\quad x\geqq y+z\,.$$

We use this inequality to estimate $\hat{u}$ as given in (1.6). Hence, we see that if $D=\{x: x\in R^n \text{ and } x<-z\}$ then

$$|\hat{u}(\xi)| \leqq C\left\{\int_D |Q(x)\,\psi(y-x)|\,dx+\int_{R^n-D}|Q(x)\,\psi(y-x)|\,dx\right\} = C\{I_1+I_2\}\,.$$

From a direct estimate of the first part we see that

$$(1.11)\qquad I_1 \leqq C\exp\left(\tfrac{1}{2}m^2n\omega^2+m|\xi|-H(\exp(y+z))\right).$$

With our choice of s it follows from the inequality

$$I_2 = \sum_{\nu=1}^{n}\int_{x_\nu\geqq -z_\nu}|Q(x)|\,dx$$

that

$$(1.12)\qquad I_2 \leqq C\exp(m|\xi|-\varepsilon\omega^2)\,.$$

If we now combine y and ω in such a way that

$$(1.13)\qquad H(\exp(y+z)) = \tfrac{1}{2}m^2n\omega^2+\varepsilon\omega^2$$

then it is a consequence of (1.11) and (1.12) that

$$(1.14) \qquad \int\limits_{-V\leqq\xi\leqq V} |\hat{u}(\xi)|\,d\xi \leqq C\exp(m|V|-\varepsilon\omega^2)$$

for large values of y.

If in (1.5) we put

$$mnV = \tfrac{1}{2}\varepsilon\omega^2\mathbf{1} = mn(h_1^{-1},h_2^{-1},\ldots,h_n^{-1}),$$

then from (1.13) and the definition of H_1 we see that

$$\omega^{-2} = (\tfrac{1}{2}m^2n+\varepsilon)\,H(\exp(y+z))^{-1} \leqq CH_1\big(\exp(y+z)\,H(\exp(y+z))^{-1}\big)\,.$$

From this inequality and (1.9) we conclude that

$$b\exp y = b\exp(y+z-z) \leqq \exp(y+z)\,H(\exp(y+z))^{-1},$$

if

$$b \leqq 2(mn+2\varepsilon m^{-1})^{-1}\exp\big(-p_\nu-\tfrac{1}{2}m(n+1)-\varepsilon m^{-1}\big) \quad \text{for all } \nu=1,2,\ldots,n.$$

Hence

$$\omega^{-2} \leqq C\,H_1(b\exp x),$$

where b is as above.

By modification of the estimate used in [4, p. 46] and by (1.4) we find that if $0<h<\mathbf{1}$ then

$$(1.15) \qquad \inf_{x\leqq t\leqq x+h}(u(t)-u(x)) \geqq A - C\exp(-\tfrac{1}{2}\varepsilon^2\omega^2) - C\sum_{\nu=1}^{n}h_\nu,$$

where

$$A = \inf_{\substack{x\leqq t\leqq x+h\\ -\mathbf{1}\leqq\varepsilon^{-1}(x-y)\leqq\mathbf{1}}}(\varphi(t)-\varphi(x)) - C\omega^{-1}\sup_{-\mathbf{1}\leqq\frac{1}{2}\varepsilon^{-1}(x-y)\leqq\mathbf{1}}|\varphi(x)|\,.$$

If we put (1.14) and (1.15) into (1.5) we obtain

$$|\varphi(y)| \leqq \sup_{x\in R^n}|u(x)| \leqq C\left\{H_1(\exp(y-\varepsilon)) + H_1(b\exp y)^{\frac{1}{2}}\sup_{-\mathbf{1}\leqq\frac{1}{2}\varepsilon^{-1}(x-y)\leqq\mathbf{1}}|\varphi(x)|+\omega^{-2}\right\}$$

for large values of y.

Since ε is arbitrary it follows by iteration that

$$\varphi(y) = O(H_1(a\exp y)), \quad y\to+\infty,$$

for any a such that $a<2(mn)^{-1}\exp\big(-p_\nu-\tfrac{1}{2}m(n+1)\big)$ for all $\nu=1,2,\ldots,n$.

Hence Theorem 1 is proved.

Remark. It is possible to improve the result by a larger a, but if for example $n=1$ this constant is of no importance in all those cases where H is of the type

$$H(x) = x^\varepsilon L(x), \qquad 0\leqq\varepsilon<1,$$

if L is continuous and satisfies

$$L(cx)\,L(x)^{-1} \to 1, \quad x \to +\infty\,,$$

for any fixed $c > 0$.

2. Application to the Laplace transform.

Let H and H_1 be as in section 1, and suppose that α is a measure on $R_+{}^n$ and that $\mu \in R_+{}^n$. We define H_2 by

$$H_2(s_1,s_2,\ldots,s_n) = H_1(s_1{}^{-1},s_2{}^{-1},\ldots,s_n{}^{-1})\,.$$

As in [4, p. 47] we introduce regions $\Omega_{x,s}$ by

$$\Omega_{x,s} = \{t:\ t \in R_+{}^n \text{ and } t \leqq s\} - \{t:\ t \in R_+{}^n \text{ and } t \leqq x\}\,.$$

Our first result for the n-dimensional Laplace transform is the following. (For a result with weaker conditions in case $\mu = 0$ or $n = 1$, see Theorem 3 below.)

THEOREM 2. *If*

$$F(s) = \int_{R_+{}^n} \exp(-s\cdot t)\,d\alpha(t) = O\big(s^{-\mu}\exp(-H(s))\big), \quad s \to +\infty\,, \tag{2.1}$$

where the integral is boundedly convergent for all $s > 0$, if

$$|F(s)| \leqq C s^{-\mu} \quad \textit{for all } s > 0\,, \tag{2.2}$$

and if

$$\sup_x\Big(\int_{\Omega_{x,s}} d\alpha(t)\Big) \leqq C s^{\mu} H_2(s), \quad s > 0\,, \tag{2.3}$$

where the supremum is taken over all x with $0 \leqq x \leqq s \leqq x \exp H_2(s)$, then there exists a positive real number a such that

$$\int_{0 \leqq t \leqq s} d\alpha(t) = O(s^{\mu} H_2(as)), \quad s \to +0\,.$$

PROOF. As in [4, p. 48] we let

$$\beta(t) = \begin{cases} \displaystyle\int_{0 \leqq x \leqq t} d\alpha(x) & \text{when } t > 0\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Then it follows that

$$F(s) = s^{1} \int_{R_+{}^n} \exp(-s\cdot t)\,\beta(t)\,dt\,, \tag{2.4}$$

where the integral is absolutely convergent.

With the substitutions

$$s = \exp x \quad \text{and} \quad t = \exp(-v)\,,$$

formula (2.1) implies that

$$K*\varphi\,(x) = O\big(\exp(-H(\exp x))\big), \quad x \to +\infty\,,$$

where

$$K(x) = \exp(-|\exp x| + (1+\mu)\cdot x)$$

and

$$\varphi(x) = \exp(\mu\cdot x)\,\beta(\exp(-x))\,.$$

We shall apply Theorem 1 and have to prove that the conditions of this theorem are satisfied. It is easy to see that $K \in E_1$. We have $K \in L(R^n)$ and if

$$t = (t_1, t_2, \ldots, t_n), \qquad \mu = (\mu_1, \mu_2, \ldots, \mu_n)\,,$$

then

$$\hat{K}(t) = \prod_{\nu=1}^{n} \Gamma(1+\mu_\nu - it_\nu)$$

Here $\Gamma(\cdot)^{-1}$ is analytic in the whole complex plane, and Stirlings formula shows that

$$|\Gamma(1+\mu_\nu - iz_\nu)^{-1}| \leqq C\exp\big(m|x_\nu| + y_\nu - y_\nu\log(1+y_\nu)\big)$$

where

$$z_\nu = x_\nu + iy_\nu \quad \text{with} \quad y_\nu \geqq -\tfrac{1}{2} \text{ and } m \geqq \tfrac{1}{2}\pi\,.$$

To prove that φ satisfies the Tauberian condition, we write

$$\text{(2.5)} \qquad \begin{aligned}\varphi(t) - \varphi(x) = {} & \exp(\mu\cdot t)[\beta(\exp(-t)) - \beta(\exp(-x))] + \\ & + \exp(\mu\cdot x)\beta(\exp(-x))[\exp(\mu\cdot(t-x)) - 1]\end{aligned}$$

and observe that (1.4) follows from (2.3) if we have

$$\text{(2.6)} \qquad |\beta(s)| \leqq Cs^{\mu}, \quad s > 0\,.$$

Then we also see that φ is bounded. To prove (2.6) we suppose that $0 \leqq s \leqq x \leqq 2s$ and we write

$$\beta(x) - \beta(s) = \sum_{k=1}^{m} \big(\beta(s^{(k)}) - \beta(s^{(k-1)})\big)\,,$$

where the $s^{(k)}$ are chosen on the line segment from s to x, so that

$$s = s^{(0)} < s^{(1)} < \ldots < s^{(m)} = s$$

and

$$\text{(2.7)} \qquad s^{(k-1)} \leqq s^{(k)} \leqq s^{(k-1)}\exp H_2(s^{(k)})\,.$$

From (2.3) we see that

$$\beta(x)-\beta(s) \leqq Cs^{\mu}\sum_{k=1}^{m} H_2(s^{(k)}) ,$$

and by (2.7) the division points can be chosen so that

$$\sum_{k=1}^{m} H_2(s^{(k)}) \leqq C .$$

Hence we see that

$$\beta(x)-\beta(s) \leqq Cs^{\mu} \quad \text{for} \quad 0\leqq s\leqq x\leqq 2s . \tag{2.8}$$

To complete the proof of (2.6) we now apply the same method by which a corresponding inequality was derived in [4, p. 49–51]. Since all details are nearly the same this time, the proof is left to the reader.

By (2.6) we have both that φ is bounded and that the Tauberian condition is satisfied. Hence Theorem 2 is proved.

THEOREM 3. *Suppose that $\mu=0$ or that $n=1$. If*

$$F(s) = \int_{R_+^n} \exp(-s\cdot t)\, d\alpha(t) = O\big(s^{-\mu}\exp(-H(s))\big), \quad s \to +\infty , \tag{2.9}$$

where the integral is boundedly convergent for some $s_0>0$, and if

$$\sup_x\Bigg(\int_{\Omega_{x,s}} d\alpha(t)\Bigg) = O(s^{\mu}H_2(s)), \quad s \to +0 , \tag{2.10}$$

where the supremum is taken over all x with, $0\leqq x\leqq s\leqq x\exp H_2(s)$ then there exists a positive real number a such that

$$\int_{0\leqq t\leqq s} d\alpha(t) = O(s^{\mu}H_2(as)), \quad s \to +0 .$$

PROOF. We start as in the proof of Theorem 2. This time we know only that (2.4) is true for $s>s_0$, since the Laplace transform is boundedly convergent for some $s=s_0$, and hence for all $s>s_0$ (cf. [1, p. 473]). Let $q\in R_+^n$, $q>s_0$; then we suppose that $\beta(s)=0$ if we do not have $s\leqq q$. The bounded convergence implies that

$$|\beta(s)| \leqq C\exp(s_0\cdot s) \quad \text{for} \quad s \geqq 0$$

(cf. [1, p. 474]), and therefore we have, as $x\to+\infty$,

$$\begin{aligned} \psi(x) &= K*\varphi\,(x) \\ &= O\Big(\exp(-H(\exp x))+\exp(\mu\cdot x)\sum_{\nu=1}^{n}\exp(-q_\nu\exp x_\nu)\Big). \end{aligned} \tag{2.11}$$

From the boundedness of β we conclude that the integral in (2.4) is absolutely convergent for all $s>0$. Hence ψ exists for all x.

From (2.10) and (2.5) we have that, if φ is bounded, then

$$(2.12) \qquad \inf_t (\varphi(t)-\varphi(x)) = O(H_1(\exp x)), \quad x \to +\infty,$$

where the infimum is taken over all t with $x \leqq t \leqq x+1H_1(\exp x)$. If $\mu=0$ this is trivially true. Otherwise $n=1$ and we see as in (2.8) that

$$(2.13) \qquad \beta(x)-\beta(s) \leqq Cs^{\mu} \quad \text{for} \quad 0 \leqq s \leqq x \leqq 2s$$

when s is small. From (2.13) we derive that if $\mu>0$ and s is small then

$$\beta(s)-\beta(s2^{-m}) \leqq Cs^{\mu}\sum_{k=1}^{m} 2^{-k\mu} \leqq Cs^{\mu}.$$

Here we let $m \to +\infty$ and hence, for small values of s,

$$\beta(s) \leqq Cs^{\mu},$$

since $\beta(s) \to 0$ as $s \to +0$. (This follows from Theorem 3 applied to the case $\mu=0$, since the proof in that case does not depend on this statement.) Since β is bounded the inequality is true for all $s>0$.

We also have

$$(2.14) \qquad |\beta(s)| \leqq Cs^{\mu} \quad \text{for all } s>0,$$

since if we did not have this inequality, then by use of (2.13) it is easy to see that we would not have (2.9).

We now modify the proof of Theorem 1 to take account of the weaker conditions (2.11) and (2.12). The proof is the same as in Theorem 1 to (1.10), to which we now have to add

$$C\exp(\mu\cdot x)\sum_{\nu=1}^{n}\exp(-q_{\nu}\exp x_{\nu})$$

to the right-hand member. Hence there corresponds a modification of (1.11), but (1.9) takes care of that if y is large enough and if at the same time (1.13) is true. When estimating I_2 we observe that ψ is still bounded and hence (1.12) is true. Then (1.14) follows as in Theorem 1.

Since φ is bounded, (1.15) is still true. Hence the conclusion follows as in Theorem 1, and Theorem 3 is proved.

REMARK 1. Conditions (2.2) and (2.3) in Theorem 1 can be weakened somewhat. We only need (2.2) when $|s|$ is large and (2.3) when $\min(s_1,s_2,\ldots,s_n)$ is small.

REMARK 2. Let ε be an arbitrary real, positive number. If for example D_{ε} is a region of form

$$\{s:\ s=(s_1,s_2,\ldots,s_n) \text{ and } s_{\nu} \geqq \varepsilon s_k>0,\ \nu \neq k,\ \nu,k=1,2,\ldots,n\},$$

then by the method used in Theorem 3, (2.1) and (2.3) implies that if $\mu \neq 0$, then

$$\int\limits_{0\leq t\leq s} d\alpha(t) = O(s^{\mu} H_2(s)), \quad s \to +0 \quad (s \in D_\varepsilon)\,.$$

3. The precision in Theorems 2 and 3.

If the function R defined by

$$R(x) = H(\exp x)$$

is sub-additiv, that is, if

$$R(x+y) \leq R(x)+R(y) \qquad \text{for all } x \in R^n \text{ and } y \in R^n\,,$$

then it follows from Theorems 2 and 3 that

$$H_1(s) = O(H(s)^{-1}), \quad s \to +\infty\,.$$

For all such functions H, the result is best possible (cf. [5]).

If we have

$$H(s) = (s_1^{-1}+s_2^{-1}+\ldots+s_n^{-1})^{-\varepsilon}, \qquad 0<\varepsilon<1\,,$$

then

(3.1)
$$H_2(s) = (s_1+s_2+\ldots+s_n)^{\varepsilon/(1-\varepsilon)}\,.$$

In the one-dimensional case the example

$$\beta(s) = \pi^{-\frac{1}{2}} \int\limits_0^s t^{-\frac{1}{2}} \sin(\tfrac{1}{2}t^{-1})\,dt$$

shows that (3.1) is best possible for $\varepsilon = \frac{1}{2}$, since if

$$F(s) = \int\limits_0^\infty \exp(-st)\,d\beta(t)\,,$$

then

$$F(s) = s^{-\frac{1}{2}} \exp(-s^{\frac{1}{2}}) \sin(s^{\frac{1}{2}})$$

(cf. [2, p. 254]), and it is easy to see that there exists a sequence of s tending to zero, for which $\beta(s)$ is of order $s^{3/2}$.

Formula (3.1) is in fact generally best possible in the one-dimensional case. If for example

(3.2)
$$\beta(s) = \int\limits_0^s t^{-1} \sin(t^{-k})\,dt$$

for some positive real number k, then by the saddle-point method one can show that

$$F(s) = O(\exp(-\gamma s^{k/(1+k)})), \quad s \to +\infty, \tag{3.3}$$

for some $\gamma > 0$. Since there exists a sequence of s tending to zero, for which $\beta(s)$ is of order s^k, this example shows that (3.1) is precise.

We give a short outline of how to prove (3.3). We shall estimate

$$F(s) = \int_0^\infty t^{-1} \exp(-st - it^{-k})\, dt, \quad k > 0,$$

for large values of s. If we make the change of variables

$$t = us^{-1/(1+k)} \quad \text{and} \quad s^{k/(1+k)} = r,$$

we obtain that

$$F(s) = \int_0^\infty u^{-1} \exp(-rf(u))\, du,$$

where $f(u) = u + iu^{-k}$. Take a branch of the analytic function f such that u^{-k} is positive for positive u. Now, we may deform the path of integration so that it passes through the saddle point

$$u_0 = k^{1/(1+k)} \exp\left(\frac{i\pi}{2(1+k)}\right).$$

Let ε and R be two real positive numbers, $\varepsilon < \operatorname{Re} u_0 < R$. We shall use a broken-line path from 0 to ε, from ε along the line $\operatorname{Re} u = \varepsilon$ up to u_1 such that $\arg u_1 = \arg u_0$, then from u_1 through the saddle-point u_0 to a point u_2 with $\operatorname{Re} u_2 = R$. From u_2 we go along $\operatorname{Re} u = R$ down to R and then along the axis to infinity. We estimate separately the integral along each segment, and with properly chosen ε and R we have the only important contribution from the segment passing through the saddle-point.

In several dimensions formula (30) on p. 165 in [10], which can easily be generalized to the n-dimensional case, shows that (3.1) is essentially best possible as soon as it is best possible in the one-dimensional case.

4. The rate at which a convolution transform, with kernel in class E_1, can tend to zero in the non-trivial case.

The class of functions that H belongs to in Theorems 1, 2 and 3 is rather restricted, but the results below shows that these restrictions are natural. It is known that a Laplace transform cannot tend to zero too rapidly when its argument tends to infinity, without forcing the original measure (α in Theorem 2) to be without any mass in some neighbourhood of the origin (cf. e.g. [8]). Here we prove a corresponding result for the class E_1, which applied to the Laplace case gives a sometimes weaker result than that obtained in [8], see Corollary 1, but also a stronger result, see Corollary 2.

We need the concept of conjugate Young functions. We say that F and G are conjugate if they are defined in the following way:

$$F(x) = \int_0^x f(t)\,dt \quad \text{and} \quad G(x) = \int_0^x g(t)\,dt\,,$$

where $f(0)=g(0)=0$, and where f and g are non-decreasing functions which satisfy the relation

$$f(g(x)) = g(f(x)) = x, \qquad 0 \leqq x < \infty\,.$$

It is well known that for such functions

$$xy \leqq F(x)+G(y), \qquad 0 \leqq x < +\infty,\ 0 \leqq y < +\infty\,. \tag{4.1}$$

We need a simple lemma (cf. [6, p. 25]).

Lemma. *If F, G and F_1, G_1 are two pairs of coujugate Young functions, then*

$$\overline{\lim}_{x\to+\infty}\big(F(x)-F_1(x)\big) > 0 \tag{4.2}$$

implies that

$$\underline{\lim}_{x\to+\infty}\big(G(x)-G_1(x)\big) < 1\,. \tag{4.3}$$

Proof. By (4.2) there exists a sequence $(x_k)_{k=1}^{\infty}$, such that $F(x_k) \geqq F_1(x_k)$ and $x_k \to +\infty$ as $k \to +\infty$. If now $(y_k)_1^{\infty}$ is defined by $y_k=f(x_k)$, $k=1,2,\ldots$, then $y_k \to +\infty$ as $x \to +\infty$ and

$$x_k y_k = F(x_k)+G(y_k), \qquad k=1,2,\ldots\,.$$

We also know that

$$x_k y_k \leqq F_1(x_k)+G_1(y_k), \qquad k=1,2,\ldots\,.$$

Hence

$$G_1(y_k)-G(y_k) \geqq F(x_k)-F_1(x_k)\,,$$

and the lemma follows from this.

Theorem 4. *Suppose that $K \in E_1$, that φ is a bounded, measurable function, that F is a Young function and that γ is a positive number.* Let

$$K*\varphi\,(x) = O\big(\exp(-F(x))\big), \qquad x \to +\infty\,, \tag{4.4}$$

where

$$\overline{\lim}_{x\to+\infty} F(x)\exp(-x) > \gamma\,. \tag{4.5}$$

Then it follows that

$$\varphi(x) = 0$$

for every point x,

$$x > -\log\gamma + (p-1) + m ,$$

at which φ *is continuous.*

PROOF. The proof is the same as in Theorem 1 up to formula (1.8), except that we now use the same s for all values of x. Combining (1.8) and (1.6), we find that

$$|\hat{u}(\xi)| \leqq C \exp\left(m|\xi| + \tfrac{1}{2}m^2\omega^2 + \tfrac{1}{2}s^2\omega^{-2} + ps - s\log(1+s)\right) \cdot \int \exp(-sx)\, |\psi(y-x)|\, dx . \tag{4.6}$$

Let

$$I = \int \exp(-sx)\, |\psi(y-x)|\, dx .$$

We make the change of variable $x = y - t$. Then from (4.4) and from the fact that ψ is bounded, we conclude that

$$I \leqq C \exp(-sy) \left(s^{-1} + \int_0^\infty \exp\left(st - F(t)\right) dt\right). \tag{4.7}$$

Now introduce F_0 such that

$$F(t) = F_0(t) + t - C ,$$

where C is a positive constant. We suppose that C is properly chosen so that F_0 is a Young function.

If we let F_1 be defined by

$$F_1(x) = \gamma \exp x - \gamma x - \gamma ,$$

then F_1 is a Young function and its conjugate function G_1 satisfies the inequality

$$G_1(x) \leqq x\log(1+x) - x - x\log\gamma + C\log(1+x) + C, \quad 0 \leqq x < \infty .$$

It follows from (4.5) and the lemma that there exists a sequence $(s_k)_1^\infty$ tending to infinity as $k \to \infty$, such that G_0, the conjugate Young function of F_0, satisfies the inequality

$$G_0(s_k) \leqq s_k\log(1+s_k) - s_k - s_k\log\gamma + C\log(1+s_k) + C, \quad k = 1,2,\ldots .$$

If we combine (4.1) for the functions F_0 and G_0 with (4.7) we conclude that for $s = s_k$ we have

$$(4.8)\quad I \leqq C\exp\Big(-s_k y + s_k\log(1+s_k) - s_k - s_k\log\gamma + C\log(1+s_k)\Big)\int_0^\infty \exp(-x)\,dx\,.$$

It follows from this estimate and (4.6) that

$$(4.9)\quad \int_{-V\leqq\xi\leqq V} |u(\xi)|\,d\xi$$
$$\leqq C\exp\big(mV+\tfrac12 m^2\omega^2+\tfrac12 s_k{}^2\omega^{-2}+(p-1)s_k-s_k\log\gamma-s_k y+C\log(1+s_k)\big)\,.$$

The first part in the right-hand member of (1.5) we estimate as in (1.15), but with $\varepsilon=\omega^{-\frac12}$. Hence we have the inequality

$$|\varphi(y)| \leqq C\Big\{-\inf_{\substack{x\leqq t\leqq x+h\\ -1\leqq\omega^{\frac12}(x-y)\leqq 1}}\big(\varphi(t)-\varphi(x)\big)+\omega^{-1}+h+$$
$$+\exp\big(mV+\tfrac12 m^2\omega^2+\tfrac12 s_k{}^2\omega^{-2}+s_k(p-1)-s_k\log\gamma-s_k y+C\log(1+s_k)\big)\Big\}.$$

Now suppose that

$$y = -\log\gamma+p-1+m+\delta \quad \text{for some } \delta>0\,.$$

Let $\omega=\omega_k$, $V=V_k$, and let

$$(4.10)\qquad 4mV_k = s_k\delta \quad\text{and}\quad s_k = m\omega_k{}^2\,.$$

Hence for large values of k we have the inequality

$$|\varphi(y)| \leqq C\Big\{-\inf_{\substack{x\leqq t\leqq x+\omega_k^{-1}\\ -1\leqq\omega_k^{\frac12}(x-y)\leqq 1}}\big(\varphi(t)-\varphi(x)\big)+\omega_k{}^{-1}\Big\}.$$

If we let $k\to+\infty$ in this inequality, then from the fact that $\omega_k\to+\infty$ when $k\to+\infty$, and if φ is continuous at y, we conclude that

$$\varphi(y) = 0\,.$$

Hence, Theorem 4 is proved.

COROLLARY 1. *Let F be the same as in Theorem 4. If α is a measure on R_+ such that*

$$\int_0^\infty \exp(-st)\,d\alpha(t) = O\big(\exp(-F(\log s))\big),\quad s\to+\infty\,,$$

where the integral converges for all large s, then (4.5) *implies that*

$$\int_{0\leqq t\leqq s} d\alpha(t) = 0 \quad\text{a.e.}\quad \textit{for } s<\gamma\exp(-\tfrac12\pi)\,.$$

PROOF. Introduce β like in Theorem 2, and suppose that β is identically equal to zero when its argument is greater than some constant larger than γ (like in Theorem 3 this is no restriction). Since β is of bounded variation we conclude that φ is continuous almost everywhere. Since the Laplace kernel satisfies condition (1.1), with $p=1$ and $m=\frac{1}{2}\pi$, the result follows from Theorem 4.

COROLLARY 2. *If*

$$\overline{\lim}_{x\to+\infty} F(x)\exp(-x) = +\infty ,$$

then

$$\int\limits_{0\leqq t\leqq s} d\alpha(t) = 0 \quad \text{a.e.}$$

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