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NOTE ON METRIZATION AND ON THE PARACOMPACT p-SPACES OF ARHANGEL'SKIĬ

PER ROAR ANDENÆS

A completely regular T_1-space X is a *p-space* if there exists in the Stone–Čech-compactification of X a sequence $\{\mathscr{A}_n\}_{n \in Z^+}$ of open covers of X such that

$$\bigcap_{n=1}^{\infty} \mathrm{St}(x,\mathscr{A}_n) \subset X \qquad \text{for every } x \in X .$$

The p-spaces were introduced by Arhangel'skiĭ in [2]. The class of p-spaces contains the metrizable spaces and also the locally compact Hausdorff spaces (more generally, the spaces that are complete in the sense of Čech [2, theorems 7 and 8]). Among the p-spaces the paracompact ones have the most noteworthy properties, e.g. a countable product of paracompact p-spaces is a paracompact p-space. Furthermore, X is a paracompact p-space if and only if it is the inverse image of a metric space Y by a perfect map ("application propre") ([2, theorem 16]). These spaces were also studied by Morita [4] (under the name paracompact M-spaces).

The purpose of this note is to provide characterizations of paracompact p-spaces and metrizable spaces in terms of star refinements of arbitrary open covers. We recall to the reader the following theorem (A. H. Stone, cf. [3, p. 168]):

A T_1-space is paracompact if and only if each open cover has an open star refinement.

We shall prove (the necessary definitions are given below):

A T_1-space X is a paracompact p-space (resp. metrizable) if and only if each open cover has an open star refinement which is regular on some *fixed compact cover $\mathscr{K}$ of X (resp. on* every *compact cover $\mathscr{K}$ of X).*

In the sequel all topological spaces under consideration are assumed to be T_1. For notation not explained here, the reader is referred to Dugundji [3]. Let X be a topological space and K a compact (non-empty) subset of X. An open cover $\mathscr{V}$ of X is called *regular on K* ([2,

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definition 6]) if the following conditions are satisfied for each open subset U of X containing K:

(i) For each $x \in K$ there exists $V \in \mathscr{V}$ such that $x \in V \subset U$.
(ii) Only finitely many members of $\mathscr{V}$ intersect both K and $X \setminus U$.

If $\mathscr{K}$ is a compact cover of X (i.e. each $K \in \mathscr{K}$ is compact), we say that an open cover is *regular on* $\mathscr{K}$ if it is regular on each $K \in \mathscr{K}$. An open cover $\mathscr{V}$ of X which is regular on

$$\mathscr{K}_0 = \{\{x\} \mid x \in X\},$$

is called a *uniform base* for X (cf. [1]). Finally, if $\mathscr{V}$ and $\mathscr{W}$ are two covers of X, then $\mathscr{V} \wedge \mathscr{W}$ is the cover consisting of all sets of the form $V \cap W$, $V \in \mathscr{V}$, $W \in \mathscr{W}$.

According to a theorem of Arhangel'skiĭ [2, theorem 22] *a paracompact space X is a p-space if and only if there exists a compact cover $\mathscr{K}$ of X and an open cover $\mathscr{V}$ of X which is regular on $\mathscr{K}$.*

We shall also need a theorem of Alexandroff [1, theorem IV] by which *a topological space is metrizable if and only if it is paracompact and has a uniform base.* The proof of sufficiency given in [1] is rather lengthy because it involves a characterization of spaces having a uniform base ([1, theorem III]). In an appendix we shall give a simpler direct proof.

Our first theorem is based on the following simple lemma:

Lemma. *Let $\mathscr{K}$ be a compact cover of a topological space X and let $\mathscr{V}$ be an open cover which is regular on $\mathscr{K}$. Then $\mathscr{V} \wedge \mathscr{W}$ is regular on $\mathscr{K}$ for every locally finite (in* [3]: nbd-*finite) open cover $\mathscr{W}$ of X.*

Proof. $\mathscr{V} \wedge \mathscr{W}$ is evidently an open cover of X satisfying condition (i) above. Let K be a member of $\mathscr{K}$ and U an open set such that $K \subset U$. Since K is compact and $\mathscr{W}$ is locally finite, only a finite number of elements in $\mathscr{W}$ intersect K; on the other hand $\mathscr{V}$ satisfies condition (ii), hence only finitely many sets of the form $V \cap W$, $V \in \mathscr{V}$, $W \in \mathscr{W}$, can intersect both K and $X \setminus U$.

Theorem 1. *A topological space X is a paracompact p-space if and only if each open cover $\mathscr{U}$ of X has an open star refinement which is regular on some fixed compact cover $\mathscr{K}$ of X (independent of $\mathscr{U}$).*

Proof. Let X be a paracompact p-space. By Arhangel'skiĭ's characterization of paracompact p-spaces there exists a compact cover $\mathscr{K}$ and an open cover $\mathscr{V}$ of X which is regular on $\mathscr{K}$. Let $\mathscr{U}$ be an arbitrary open cover of X. Since X is paracompact, we can find an open star refinement $\mathscr{U}'$ of $\mathscr{U}$. Let $\mathscr{W}$ be a locally finite open refinement of $\mathscr{U}'$. From the previous lemma it follows that $\mathscr{V} \wedge \mathscr{W}$ is regular on $\mathscr{K}$, and since $\mathscr{V} \wedge \mathscr{W}$ refines $\mathscr{U}'$, it is also an open star refinement of $\mathscr{U}$. This proves necessity. Sufficiency follows trivially from Arhangel'skiĭ's characterization of paracompact p-spaces quoted above.

THEOREM 2. *A topological space is metrizable if and only if each open cover has an open star refinement which is regular on* every *compact cover* $\mathscr{K}$ *of* X.

PROOF. Recalling that an open cover which is regular on $\mathscr{K}_0 = \{\{x\} \mid x \in X\}$, is a uniform base, we easily obtain sufficiency from Alexandroff's metrization theorem quoted above. To prove necessity, let X be metrizable with metric d. For each $n \in Z^+$ let $\mathscr{V}_n$ be a locally finite open refinement of the cover consisting of all open spheres with d-radius $1/n$. Let K be a compact subset of X and U an open set containing K. Since K is compact, the d-distance between K and $X \setminus U$ is strictly positive, i.e. there exists $n_0 \in Z^+$ such that for $n \geqq n_0$ no member of $\mathscr{V}_n$ intersects both K and $X \setminus U$. It easily follows that $\mathscr{V} = \bigcup_{n=1}^{\infty} \mathscr{V}_n$ is regular on K. Since K was arbitrary, $\mathscr{V}$ is regular on every compact cover $\mathscr{K}$ of X; in particular, $\mathscr{V}$ is a uniform base. Let $\mathscr{U}$ be an arbitrary open cover of X and select an open star refinement $\mathscr{U}'$ of $\mathscr{U}$ and a locally finite open refinement $\mathscr{W}$ of $\mathscr{U}'$. It is now easily verified (cf. the proof of theorem 1) that $\mathscr{V} \wedge \mathscr{W}$ is an open star refinement of $\mathscr{U}$ and that it is regular on every compact cover $\mathscr{K}$ of X. This completes the proof.

Appendix.

PROOF OF ALEXANDROFF'S METRIZATION THEOREM. 1) Let X be paracompact and let $\mathscr{A}$ be a uniform base for X. We put

$$\mathscr{I} = \{\{x\} \mid x \text{ is an isolated point in } X\}$$

and $\mathscr{A}_1 = \mathscr{A} \setminus \mathscr{I}$. Let $\mathscr{W}_1$ be a locally finite open refinement of $\mathscr{A}_1 \cup \mathscr{I}$, and let $\mathscr{V}_1$ be an irreducible subcover of $\mathscr{W}_1$, i.e. no proper subfamily of $\mathscr{V}_1$ covers X (cf. [3, p. 160]). We define

$$\mathscr{B}_1 = \{A \mid A \in \mathscr{A}_1, A \text{ is properly contained in no } V \in \mathscr{V}_1\}$$

and

$$\mathscr{A}_2 = \mathscr{A}_1 \setminus \mathscr{B}_1 .$$

Then $\mathscr{A}_2 \cup \mathscr{I}$ is a base for X: Let x be a non-isolated point and U an open neighbourhood of x. For some $V \in \mathscr{V}_1$ we have $x \in V$. Since X is T_1, there exists $A_x \in \mathscr{A}_1$ such that

$$x \in A_x \subset V \cap U, \qquad A_x \neq V \cap U .$$

Then $A_x \notin \mathscr{B}_1$, that is, $A_x \in \mathscr{A}_2$. Proceeding by induction, we obtain sequences $\{\mathscr{A}_n\}$, $\{\mathscr{V}_n\}$ and $\{\mathscr{B}_n\}$, $n \in Z^+$, such that

a) $\mathscr{A}_n \cup \mathscr{I}$ is a base for X,
b) $\mathscr{V}_n$ is a locally finite irreducible open refinement of $\mathscr{A}_n \cup \mathscr{I}$,
c) $\mathscr{B}_n = \{A \mid A \in \mathscr{A}_n,\ A$ is properly contained in no $V \in \mathscr{V}_n\}$,
d) $\mathscr{A}_{n+1} = \mathscr{A}_n \setminus \mathscr{B}_n$.

The union $\mathscr{V} = \bigcup_{n=1}^{\infty} \mathscr{V}_n$ is a σ-locally finite open cover of X; we claim that it is also a base for X. Let $x \in X$ be arbitrary and for each $n \in Z^+$ select

$$V_{x,n} \in \mathscr{V}_n \quad \text{and} \quad A_{x,n} \in \mathscr{A}_n \cup \mathscr{I}$$

such that

$$x \in V_{x,n} \subset A_{x,n} .$$

If

$$A_{x,n_0} \in \mathscr{I} \quad \text{for some } n_0 \in Z^+ ,$$

then we have

$$\{x\} = V_{x,n_0} = A_{x,n_0} ,$$

and $\{V_{x,n}\}$ is a neighbourhood base at x. Assume that $A_{x,n} \in \mathscr{A}_n$ for every $n \in Z^+$. Then we also have $A_{x,n} \in \mathscr{B}_n$: If

$$A_{x,n} \subset V, \qquad A_{x,n} \neq V \quad \text{for some } V \in \mathscr{V}_n ,$$

$\mathscr{V}_n$ would not be irreducible, which is a contradiction. It follows that $A_{x,n} \notin \mathscr{A}_{n+1}$, hence the sequence $\{A_{x,n}\}$ consists of distinct elements. From the regularity of $\mathscr{A}$ it follows that $\{A_{x,n}\}$, and therefore also $\{V_{x,n}\}$, is a neighbourhood base at x, and X is metrizable by the Nagata–Smirnov theorem.

2) The reverse implication is clear from the proof of our theorem 2.

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UNIVERSITY OF OSLO, NORWAY