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DIFFERENTIABILITY OF A FUNCTION AND OF ITS COMPOSITIONS WITH FUNCTIONS OF ONE VARIABLE

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1. Statement of results.

Denote by $C^\infty(\mathsf{R}^d,\mathsf{R}^e)$ the set of infinitely differentiable functions from R^d to R^e.

The following theorem was conjectured by H. Rådström in an unpublished work.

THEOREM 1. *Let f be a function from R^d to R, and assume that the composed function $f \circ u$ belongs to $C^\infty(\mathsf{R},\mathsf{R})$ for every $u \in C^\infty(\mathsf{R},\mathsf{R}^d)$. Then $f \in C^\infty(\mathsf{R}^d,\mathsf{R})$.*

This paper contains a proof of Theorem 1 and of several more precise results which are closely related to Theorem 1.

DEFINITION 1. *If σ is a function from $\mathsf{R}_+=\{t\,;\ t>0\}$ to R_+, such that $\lim_{\varepsilon\to 0}\sigma(\varepsilon)=0$, we denote by $K(\sigma)$ the class of functions f: $\mathsf{R}^d \to \mathsf{R}$ with the following property: to each compact subset $F \subset \mathsf{R}^d$ there exists a constant C such that*

$$|f(x+y)-f(x)| \leqq C\sigma(\varepsilon), \quad \textit{when} \quad |y| \leqq \varepsilon,\ x \in F,\ x+y \in F\,.$$

If $f \in K(\sigma)$ when $\sigma(\varepsilon)=\varepsilon^\varrho$, $0<\varrho \leqq 1$, then we write $f \in \mathrm{Lip}(\varrho)$ and say that f is Lipschitz continuous with exponent ϱ.

DEFINITION 2. *If p is a non-negative integer, we denote by C^p or $C^p(\mathsf{R}^d,\mathsf{R}^e)$, or sometimes by $C^{p,0}$ or $C^{p,0}(\mathsf{R}^d,\mathsf{R}^e)$, the class of functions from R^d to R^e whose derivatives of order $\leqq p$ are continuous. Let $C_\sigma^{\,p}$ or $C_\sigma^{\,p}(\mathsf{R}^d,\mathsf{R}^e)$ denote the class of those functions in $C^p(\mathsf{R}^d,\mathsf{R}^e)$ whose derivatives of order p belong to $K(\sigma)$. Similarly, if $0<\varrho \leqq 1$, then $C^{p,\varrho}$ or $C^{p,\varrho}(\mathsf{R}^d,\mathsf{R}^e)$ denotes the class of functions from R^d to R^e whose derivatives of order p are Lipschitz continuous with exponent ϱ.*

We can now state our first generalization of Theorem 1.

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THEOREM 2. *Let f be a function from* R^d *to* R, *and assume that* $f \circ u \in C^{p,\varrho}(\mathrm{R},\mathrm{R})$ *for every* $u \in C^\infty(\mathrm{R},\mathrm{R}^d)$. *Then*

$$f \in C^{p,\varrho}(\mathrm{R}^d,\mathrm{R}) \qquad \text{if} \quad 0<\varrho \leqq 1 \text{ or } p=\varrho=0\,,$$

and

$$f \in C^{p-1,1}(\mathrm{R}^d,\mathrm{R}) \qquad \text{if} \quad \varrho=0 \text{ and } p \geqq 1\,.$$

The next theorem states that the conclusion of Theorem 2 can not be strengthened to read $f \in C^{p,0}$ in the case $\varrho=0$, even if we strengthen the assumption by taking $u \in C^p$ instead of $u \in C^\infty$.

THEOREM 3. *If $p \geqq 1$ and $d \geqq 2$, there exists a function f from* R^d to R *such that $f \notin C^p$, but $f \circ u \in C^p(\mathrm{R},\mathrm{R})$ for every $u \in C^p(\mathrm{R},\mathrm{R}^d)$.*

In fact, the statement of Theorem 3 is true for every function f which is homogeneous of degree p, belongs to C^p outside the origin and is not a polynomial.

Theorem 2 asserts in particular that f must belong to C^p if there exists $\varrho > 0$ such that $f \circ u \in C^{p,\varrho}(\mathrm{R},\mathrm{R})$ for every $u \in C^\infty(\mathrm{R},\mathrm{R}^d)$. It is natural to try to weaken the assumption $f \circ u \in C^{p,\varrho}(\mathrm{R},\mathrm{R})$ here. The next theorem gives a result of that kind.

THEOREM 4. *Let f be a function from* R^d *to* R, *and assume that there exists a function σ such that $\lim_{\varepsilon \to 0}\sigma(\varepsilon)=0$ and such that $f \circ u \in C_\sigma^{\,p}(\mathrm{R},\mathrm{R})$ for every $u \in C^\infty(\mathrm{R},\mathrm{R}^d)$. Then $f \in C^p(\mathrm{R}^d,\mathrm{R})$.*

From our proof of Theorem 4 one could deduce an estimate (depending on σ) of the modulus of continuity of the p^{th} derivatives of f. However, using Theorem 1 in [1] in place of Lemma 7 below we can obtain a better estimate of that kind. Writing

$$\hat{\sigma}(\varepsilon) = \varepsilon\left(1 + \int_{\min(\varepsilon,1)}^{1} t^{-2}\,\sigma(t)\,dt\right)$$

we obtain in this way $f \in C_{\hat{\sigma}}^{\,p}(\mathrm{R}^d,\mathrm{R})$ in the conclusion of Theorem 4. This application of Theorem 1 in [1] is discussed at the end of Section 3 below.

By letting u vary over a smaller class than C^∞ we can obtain generalizations of Theorem 2 and Theorem 4. Let $L=\{L_k\}_{k=0}^{\infty}$ be an increasing sequence of positive numbers. Denote by C^L or $C^L(\mathrm{R},\mathrm{R}^d)$ the set of all functions $u \in C^\infty(\mathsf{R},\mathsf{R}^d)$ such that to every finite interval $F \subset R$ there exists a constant C such that

$$\sup_F |d^k u/dx^k| \leqq C^{k+1} L_k{}^k \quad \text{for every } k .$$

The function u is said to have a zero of infinite order at the point $t \in R$ if u and its derivatives of all orders vanish at t. The class C^L is called quasianalytic if no non-trivial function in C^L has a zero of infinite order. The Denjoy–Carleman Theorem states that C^L is quasianalytic if and only if $\sum_{k=0}^{\infty} L_k{}^{-1}$ is divergent. This shows in particular that the so-called Gevrey classes G_a — which are defined for $a \geqq 1$ by $G_a = C^{\{1+k^a\}}$ — are non-quasianalytic when $a > 1$. When $a = 1$, the class G_a is the set of real analytic function, which of course is quasianalytic. A quasianalytic function is a function which belongs to some quasianalytic class.

A strengthened version of Theorem 2 can now be stated as follows.

THEOREM 5. *Let $L = \{L_k\}_{k=0}^{\infty}$ be an increasing sequence of positive numbers such that the class C^L is non-quasianalytic, let f be a function from R^d to R, and assume that $f \circ u \in C^{p,\varrho}(\mathsf{R},\mathsf{R})$ for every $u \in C^L(\mathsf{R},\mathsf{R}^d)$. Then the conclusion of Theorem 2 holds.*

The assumption of Theorem 5 that C^L is non-quasianalytic can not be omitted. This is implied by our next theorem.

THEOREM 6. *If $d \geqq 2$, there exists a non-continuous function f from R^d to R such that $f \circ u \in C^\infty(\mathsf{R},\mathsf{R})$ for every quasianalytic function $u \in C^\infty(\mathsf{R},\mathsf{R}^d)$.*

A study related to ours has been made by Rosenthal [2]. However, the latter author studies only the continuity of f (not the differentiability properties of f). A function $u \in C^1(\mathsf{R},\mathsf{R}^d)$ is called non-singular if $|u'(t)| \neq 0$ for every t. Rosenthal proves the following. If $f \circ u$ is continuous for every non-singular function $u \in C^1(\mathsf{R},\mathsf{R}^d)$, then $f\colon \mathsf{R}^d \to \mathsf{R}$ is continuous. He also shows that if $d \geqq 2$ there exist non-continuous functions $f\colon \mathsf{R}^d \to \mathsf{R}$ such that $f \circ u$ is continuous for every non-singular $u \in C^2(\mathsf{R},\mathsf{R}^d)$.

Recently we obtained the following generalization of Rosenthal's result.

Let us say that a function $u \in C^{m+1}$ has a singularity of order m at t ($m \geqq 0$), if $u'(t) = \ldots = u^{(m)}(t) = 0$, and $u^{(m+1)}(t) \neq 0$.

THEOREM 7. *Let $p \geqq 1$ and let f be a function from R^d to R such that $f \circ u$ is continuous for each function $u \in C^p(\mathsf{R},\mathsf{R}^d)$ with singularities of order at most $p-1$. Then f is continuous.*

On the other hand, if $d \geqq 2$ and $p \geqq 1$, there exists a function $f\colon R^d \to \mathsf{R}$

such that $f \notin C^1(\mathsf{R}^d,\mathsf{R})$ *and* $f \circ u \in C^p(\mathsf{R},\mathsf{R})$ *for each* $u \in C^p(\mathsf{R},\mathsf{R}^d)$ *with singularities of order at most* $p-1$.

Moreover, if $d \geqq 2$ *and* $p \geqq 2$, *there exists a non-continuous function* $f\colon \mathsf{R}^d \to \mathsf{R}$ *such that* $f \circ u \in C^p(\mathsf{R},\mathsf{R})$ *for every* $u \in C^p(\mathsf{R},\mathsf{R}^d)$ *with singularities of order at most* $p-2$.

The proofs of these results will be published elsewhere.

A few words should be said also about the situation when u varies over some class of functions of more than one variable. In this case we do not lose one degree of differentiability as we did in the case $\varrho = 0$ of Theorem 2. This fact can be stated as follows.

THEOREM 8. *Let* f *be a function from* R^d *to* R, *and assume that* $f \circ u \in C^p(\mathsf{R}^2,\mathsf{R})$ *for every* $u \in C^\infty(\mathsf{R}^2,\mathsf{R}^d)$. *Then* $f \in C^p(\mathsf{R}^d,\mathsf{R})$.

PROOF. Using Theorem 2 with $p = \varrho = 0$, it is easy to prove Theorem 8. Set

$$u(t) = t_1 e_j + w(t_2) \quad \text{for} \quad t = (t_1,t_2) \in \mathsf{R}^2 ,$$

where e_j is the unit vector in the direction of the j^{th} coordinate axis in R^d, and $w \in C^\infty(\mathsf{R},\mathsf{R}^d)$. Then $u \in C^\infty(\mathsf{R}^2,\mathsf{R}^d)$, and hence by assumption $(\partial^p/\partial t_1^{\,p})(f \circ u)$ exists and is a continuous function of $t \in R^2$. Thus, in particular $(D_j^{\,p} f)(w(t_2))$ is a continuous function of t_2. (Here D_j means $\partial/\partial x_j$.) Since this is true for each $w \in C^\infty(\mathsf{R},\mathsf{R}^d)$, the function $D_j^{\,p} f$ must be continuous by the above mentioned special case of Theorem 2.

Again, we do not get the desired result if u is restricted to the class of analytic functions. In fact, this is so even if u is a function of $d-1$ variables:

THEOREM 9. *There exists a non-continuous function* f *from* R^d *to* R *such that* $f \circ u \in C^\infty(\mathsf{R}^{d-1},\mathsf{R})$ *for every analytic function* u *from* R^{d-1} *to* R^d.

Sections 2 and 3 below contain a complete proof of Theorem 5. The reader who wishes to see the proof of Theorem 2 but not that of the more general Theorem 5 may simply disregard Lemma 3. To obtain a proof of Theorem 1 it suffices to combine the first part of Lemma 1 with Lemma 4 and Lemma 5.

2. Boundedness of the directional derivatives $D_\xi f$.

If $\xi \in \mathsf{R}^p$, $|\xi| \neq 0$, we set $D_\xi f(x) = \lim_{t \to 0} (f(x+t\xi) - f(x))/t$, and similarly we define $D_\xi^{\,k} f(x)$ inductively by $D_\xi^{\,k} f(x) = D_\xi (D_\xi^{\,k-1} f(x))$ when $k > 1$.

If the hypothesis of Theorem 2 (or Theorem 5) is fulfilled, it is obvious that the derivatives $D_\xi^{\,k} f(x)$ exist for each $k \leqq p$, and that the restriction of the function $D_\xi^p f$ to any line parallel with ξ is Lipschitz continuous with exponent ϱ. To see this we need only apply the hypothesis to the function $u(t)=x+t\xi$, $x \in R^d$, $t \in R$. Our first and most important step towards proving that $f \in C^{p,\varrho}$ is to prove that these statements hold with a certain uniformity with respect to the set of lines parallel to ξ. To make the last statement precise we need the following definition.

Definition 3. *Let σ be a function from R_+ to R_+ with $\lim_{\varepsilon\to 0}\sigma(\varepsilon)=0$, and let ξ be a non-zero vector in R^d. Then $K(\xi,\sigma)$ will denote the class of functions from R^d to R with the following property: to each compact set $F \subset \mathsf{R}^p$ there exists a constant C such that*

$$|f(x+t\xi)-f(x)| \leqq C\sigma(\varepsilon), \quad \textit{when} \quad |t| \leqq \varepsilon,\ x \in F,\ x+t\xi \in F .$$

If $\sigma(\varepsilon)=\varepsilon^\varrho$, $0<\varrho\leqq 1$, then $K(\xi,\sigma)$ will also be denoted $\mathrm{Lip}(\xi,\varrho)$.

The following is a simple consequence of the definition: if M is a set of vectors which spans R^d, and f belongs to $K(\xi,\sigma)$ for each $\xi \in M$, then $f \in K(\sigma)$.

We may always assume that σ is subadditive and increasing. In fact, if $f \in K(\sigma)$, then there always exists a function σ_1 such that σ_1 is subadditive and increasing, $\sigma_1 \leqq \sigma$, and $f \in K(\sigma_1)$. This is an obvious consequence of the fact that

$$\sup\ \{|f(x+y)-f(x)|\ ;\ x \in F,\ x+y \in F,\ |y| \leqq \varepsilon\}$$

is a subadditive and increasing function of ε if F is convex. In fact we can always take

$$\sigma_1(\varepsilon) = \inf\ \{\textstyle\sum_{i=1}^n \sigma(\varepsilon_i)\ ;\ \sum_{i=1}^n \varepsilon_i \geqq \varepsilon,\ \varepsilon_i \geqq 0\}\ ,$$

which is the largest increasing and subadditive minorant of σ.

If $\sigma(\varepsilon)$ is subadditive and increasing, we have the following inequality:

$$\sigma(a\varepsilon) \leqq \sigma\big((1+[a])\varepsilon\big) \leqq (1+a)\sigma(\varepsilon), \qquad \varepsilon>0,\ a>0 . \tag{2.1}$$

Here $[a]$ denotes the integral part of a.

Lemma 1. *If f is a function from R^d to R such that $f\circ u \in C^p(\mathsf{R},\mathsf{R})$ for every $u \in C^\infty(\mathsf{R},\mathsf{R}^d)$, then f is continuous and each derivative $D_\xi^p f$ is a locally bounded function on R^d.*

If, in addition, $D^p(f\circ u) \in K(\sigma)$ for every $u \in C^\infty(\mathsf{R},\mathsf{R}^d)$, then $D_\xi^p f \in K(\xi,\sigma)$ for every ξ.

For the proof of Lemma 1 we need the following trivial lemma.

Lemma 2. *Let L_q be increasing and $\sum L_q^{-1} < \infty$ (that is, the class C^L non-quasianalytic). Assume that $\psi \in C^\infty(\mathsf{R},\mathsf{R})$, that $\psi(t)=0$ for large t and that*

$$|\psi^{(q)}(t)| \leqq C^{q+1} L_q{}^q$$

for all q and t (that is, $\psi \in C^L$). Set

$$u(t) = (a+tb)\,\psi(t/T)$$

for all real t, where $a,b \in \mathsf{R}^d$ and $0 < T \leqq 1$. Then there exists a constant C_1, which is independent of a, b, q, t and T, such that

$$|u^{(q)}(t)| \leqq (|a|+|b|)\,T^{-q}\,C_1{}^{q+1} L_q{}^q \,. \tag{2.2}$$

Proof. Differentiation gives

$$u^{(q)}(t) = (a+tb)\,T^{-q}\psi^{(q)}(t/T) + q\,b\,T^{-q+1}\psi^{(q-1)}(t/T)\,.$$

Hence, if $\psi(t)=0$ when $|t|>B$, then

$$|u^{(q)}| \leqq (|a|+B|b|)\,T^{-q}C^{q+1}L_q{}^q + q\,|b|\,T^{-q}C^q L_{q-1}^{q-1}\,. \tag{2.3}$$

Since L_q is increasing and $\sum L_q^{-1} < \infty$, we have $C_2 L_q \geqq q$ for some C_2, and hence

$$q L_{q-1}^{q-1} \leqq C_2 L_q L_q{}^{q-1} = C_2 L_q{}^q\,.$$

Combining this with (2.3) gives (2.2).

Proof of Lemma 1. We first prove that $D_\xi{}^p f$ is locally bounded. Let us assume the contrary, i.e. that $D_\xi{}^p f$ is unbounded in each neighbourhood of some point, say the origin. Then there exists a sequence of points $x^n \in R^d$ such that $|x^n| \to 0$ and $D_\xi{}^p f(x^n) = F_n$ tends to $+\infty$ or $-\infty$, say $+\infty$. Assuming this we shall construct a function $u \in C^\infty(\mathsf{R},\mathsf{R}^d)$, which assumes infinitely many of the values x^n, such that $D^p(f\circ u)$ is not locally bounded, which contradicts the assumptions of the lemma.

Let L_q be any sequence satisfying the assumptions of Lemma 2 and take $\psi \in C^\infty$ such that $\psi(t)=1$ when $|t|<\frac{1}{2}$, $\psi(t)=0$ when $|t|>\frac{3}{4}$, and

$$|\psi^{(q)}| \leqq C^{q+1} L_q{}^q \quad \text{for every } q$$

(that is, $\psi \in C^L$). For example, we can take $L_q = q^2$. Let T_j be real numbers such that

$$0 < T_j \leqq 1 \quad \text{and} \quad \textstyle\sum_{j=1}^\infty T_j < \infty\,,$$

and set

$$t_k = 2\textstyle\sum_{j=1}^{k-1} T_j + T_k\,.$$

Let r_j be an increasing sequence of natural numbers and c_j a sequence of positive numbers, both of which will be chosen later. Set

$$(2.4) \qquad u_j(t) = (x^{r_j} + \xi c_j(t-t_j))\, \psi((t-t_j)/T_j), \qquad t \in \mathrm{R}\,,$$

and

$$u(t) = \textstyle\sum_{j=1}^{\infty} u_j(t)\,.$$

Since all the u_j have disjoint supports, the sum is trivially convergent. It is also clear that

$$u \in C^\infty \quad \text{when} \quad t \neq t_\infty = \lim_{k\to\infty} t_k\,.$$

We shall now choose r_j and c_j such that

$$(\mathrm{a}_1) \qquad u \in C^\infty \text{ in a neighbourhood of } t_\infty,$$

$$(\mathrm{a}_2) \qquad D^p(f\circ u)(t_j) \to +\infty \text{ as } j\to\infty\,.$$

To do so we shall first take c_j such that

$$(\mathrm{b}_1) \qquad c_j T_j^{-q} \to 0 \quad \text{as } j\to\infty \quad \text{for every } q$$

(for example $c_j = T_j^{\,j}$), and then choose r_j such that

$$(\mathrm{b}_2) \qquad |x^{r_j}| \leq c_j \quad \text{for every } j$$

and

$$(\mathrm{b}_3) \qquad c_j^{\,p} F_{r_j} \to +\infty \quad \text{as } j\to\infty.$$

That this is possible is obvious since $|x^n| \to 0$ and $F_n \to +\infty$ as $n\to\infty$.

It remains to verify that (a_1) and (a_2) are valid if r_j and c_j are chosen so that (b_1), (b_2) and (b_3) hold. To prove (a_1) it is enough to show that

$$|u^{(q)}(t)| \to 0 \quad \text{as} \quad t \to t_\infty$$

for each q. In the interval $\{t\,;\ |t-t_j| \leq T_j\}$ we have according to Lemma 2 and (b_2):

$$(2.5) \qquad \begin{aligned} |u^{(q)}(t)| &\leq (|x^{r_j}| + |\xi| c_j) T_j^{-q} C_1^{q+1} L_q^{\,q} \\ &\leq (1+|\xi|) c_j T_j^{-q} C_1^{q+1} L_q^{\,q} \leq c_j T_j^{-q} C_2^{q+1} L_q^{\,q}\,, \end{aligned}$$

which tends to zero as $j\to\infty$, for each q, by (b_1). To compute $D^p(f\circ u)(t_j)$ we note that $u(t) = x^{r_j} + \xi c_j(t-t_j)$ in a neighbourhood of t_j. Hence

$$D^p(f\circ u)(t_j) = c_j^{\,p} D_\xi^{\,p} f(x^{r_j}) = c_j^{\,p} F_r\,,$$

which tends to infinity by (b_3).

In the case $p=0$ we have to prove the stronger statement that f is continuous. Assume the contrary, for example that

$$|x^n| \to 0 \quad \text{and} \quad \lim_{n\to\infty} f(x^n) \neq f(0)\,.$$

If we take c_j and r_j according to (b_1) and (b_2) and define $u(t)$ again by (2.4), we obtain

$$\lim_{j\to\infty}(f\circ u)(t_j) = \lim_{n\to\infty} f(x^n) \neq f(0) = (f\circ u)(t_\infty)\,,$$

that is, $f\circ u$ is not continuous at t_∞, contrary to the hypothesis. This completes the proof of the first part of Lemma 1.

To prove the second part of the lemma it is enough to prove that $D_\xi^p f \in K(\xi,\sigma)$ in some neighbourhood of an arbitrary point x. As was remarked after Definition 3 we can assume that σ is subadditive and increasing. Assume that for some point x, say $x=(0,\ldots,0)$, the assertion is not true. Then there must exist B_n, x^n and h_n such that

$$B_n \to \infty, \quad |x^n| \to 0, \quad h_n \to 0\,,$$

and

$$|D_\xi^p f(x^n+\xi h_n) - D_\xi^p f(x^n)| > B_n\sigma(|h_n|), \quad n = 1,2,\ldots\,. \tag{2.6}$$

Define the function $u(t)=\sum u_j(t)$ again by (2.4) and let the parameters T_j, t_j and c_j have the same values as above. Choose r_j such that again

$$|x^{r_j}| \leqq c_j \quad \text{for every } j, \tag{b$_2$}$$

and such that

$$c_j^{p+1}B_{r_j} \to \infty \quad \text{as } j\to\infty, \tag{b$_3'$}$$

and

$$|h_{r_j}| < \tfrac{1}{2}c_jT_j. \tag{b$_4$}$$

Then as above $u\in C^\infty$. We shall show that with $\varepsilon_j = h_{r_j}/c_j$ we have, at least for large j,

$$|D^p(f\circ u)(t_j+\varepsilon_j) - D^p(f\circ u)(t_j)| > \tfrac{1}{2}c_j^{p+1}B_{r_j}\sigma(|\varepsilon_j|)\,. \tag{2.7}$$

In view of (b_3') this will complete the proof. To prove (2.7), first note that

$$u(t_j+h) = x^{r_j}+\xi c_j h \quad \text{when} \quad |h|<\tfrac{1}{2}T_j\,.$$

Since by (b_4) we have $|\varepsilon_j|<\frac{1}{2}T_j$, we then obtain from (2.6):

(2.8)

$$\begin{aligned}|D^p(f\circ u)(t_j+\varepsilon_j) - D^p(f\circ u)(t_j)| &= |c_j^p(D_\xi^p f)(x^{r_j}+\xi c_j\varepsilon_j) - c_j^p(D_\xi^p f)(x^{r_j})| \\ &\geqq c_j^p B_{r_j}\sigma(c_j|\varepsilon_j|)\,.\end{aligned}$$

Then by (2.1), if j is so large that $c_j \leqq 1$,

$$\sigma(\varepsilon_j) \leqq \bigl(1+(1/c_j)\bigr)\sigma(c_j\varepsilon_j) \leqq (2/c_j)\sigma(c_j\varepsilon_j)\,.$$

This together with (2.8) proves (2.7).

LEMMA 3. *Let $M=\{M_q\}_{q=0}^\infty$ be an increasing sequence of positive numbers such that $\sum_{q=0}^\infty M_q^{-1}<\infty$, that is, the class C^M is non-quasianalytic.*

Then all the assertions of Lemma 1 *hold if we write* $u \in C^M(\mathsf{R},\mathsf{R}^d)$ *instead of* $u \in C^\infty(\mathsf{R},\mathsf{R}^d)$.

Proof. To prove this lemma we have to improve the construction of the function u in the proof of Lemma 1, so that we get $u \in C^M$ instead of just $u \in C^\infty$. Then first of all we must use the fact that every non-quasianalytic class contains non-trivial functions with compact support. (This is easy to prove using the fact that C^L is closed under multiplication if L_k is increasing.) Since $\sum M_q^{-1} < \infty$, we can take L_q such that L_q is increasing, $\sum L_q^{-1} < \infty$, and $A_q = M_q/L_q$ tends to infinity. Then C^L is non-quasianalytic, and we can take $\psi \in C^L$ such that $\psi(t)=1$ when $|t| < \frac{1}{2}$, and $\psi(t)=0$ when $|t| > \frac{3}{4}$. Define t_j and T_j as above, and choose c_j such that

$$(\mathrm{b}_1') \qquad c_j \leqq (T_j A_q)^q, \qquad j=1,2,\ldots, \qquad q=1,2,\ldots .$$

That this is possible is clear since $A_q \to \infty$ as $q \to \infty$, and hence $(T_j A_q)^q \to \infty$ as $q \to \infty$ for every fixed j. Then choose r_j as above, and define $u = \sum u_j$ again by (2.4). Since (b_1') implies (b_1), u will have all the previous properties, in particular $u \in C^\infty$. Moreover, $u \in C^M$, since by (2.5) and (b_1') and the definition of A_q:

$$|u^{(q)}| \leqq c_j T_j^{-q} C^{q+1} L_q^{\,q} \leqq A_q^{\,q} C^{q+1} L_q^{\,q} = C^{q+1} M_q^{\,q} .$$

This completes the proof of Lemma 3.

3. Continuity of the derivatives.

In this section we will start with the directional continuity properties of the directional derivatives $D_\xi^p f$ and prove that all partial derivatives of f of degree p have the corresponding continuity properties as functions on R^d.

Definition 4. *Let* $\xi \in \mathsf{R}^d$, $|\xi| \neq 0$. *A function* $f\colon \mathsf{R}^d \to \mathsf{R}$ *is said to be continuous in the direction* ξ *if* $f(x+t\xi)$ *tends to* $f(x)$ *uniformly for* x *in compact sets as* t *tends to zero* ($t \in \mathsf{R}$).

If $f \in K(\xi,\sigma)$ and $\lim_{\varepsilon\to 0}\sigma(\varepsilon)=0$, then f is obviously continuous in the direction ξ.

Lemma 4. *Assume that* $f \in C^0(\mathsf{R}^d,\mathsf{R})$ *and that* $D_\xi f$ *is continuous in the direction* ξ. *Then* $D_\xi f$ *is continuous.*

Proof. Write $D_\xi f = f_\xi$. Assume that f_ξ is not continuous at the origin $0 \in \mathsf{R}^d$. We can assume, for example, that

$$\overline{\lim}_{x\to 0} f_\xi(x) > f_\xi(0) .$$

Then there exists a sequence $x^n \in \mathsf{R}^d$ tending to 0 and an $\varepsilon > 0$ such that

$$f_\xi(x^n) - f_\xi(0) > \varepsilon, \qquad n = 1, 2, \ldots . \tag{3.1}$$

Since f_ξ is continuous in the direction ξ, it follows from (3.1) that there exists a positive number δ which is independent of n, such that

$$f_\xi(x^n + t\xi) - f_\xi(t\xi) > \tfrac{1}{2}\varepsilon, \qquad 0 \leqq t \leqq \delta, \quad n = 1, 2, \ldots .$$

Integrating over t from 0 to δ gives

$$f(x^n + \delta\varepsilon) - f(x^n) - f(\delta\varepsilon) + f(0) > \tfrac{1}{2}\delta\varepsilon, \qquad n = 1, 2, \ldots .$$

Letting n tend to infinity we arrive at a contradiction, since f is continuous.

We now have to study the problem of how to obtain information about the mixed derivatives of f, when we know something about the derivatives $D_\xi^p f$.

If $\xi \in \mathsf{R}^d$ and $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index, α_j non-negative integers, write $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_d^{\alpha_d}$, and if $|\alpha| = \sum \alpha_j = p$, let $\binom{p}{\alpha}$ denote $p!/(\alpha_1! \ldots \alpha_d!)$. The number of distinct multi-indices α with length $|\alpha| = p$ is

$$\binom{d+p-1}{p} = N .$$

Let Γ be a collection of N vectors in R^d. If α runs through the set of all multi-indices with length p, and ξ runs through Γ, then the numbers $\binom{p}{\alpha}\xi^\alpha$ form a quadratic matrix, which is defined up to a permutation of rows or columns. Denote the absolute value of the determinant of this matrix by $\Delta(\Gamma)$. Considering $\Delta(\Gamma)$ as a function of the Nd variables ξ_j, $\xi \in \Gamma$, it is easily seen that $\Delta(\Gamma)$ is the absolute value of a polynomial of degree pN.

LEMMA 5. *Let g be a continuous function from R^d to R, let p be a natural number, and let Γ be a collection of*

$$\binom{d+p-1}{p}$$

vectors in R^d such that $\Delta(\Gamma) \neq 0$. Assume that the derivative $D_\xi^p g$ exists and is continuous for each $\xi \in \Gamma$. Then $g \in C^p$.

PROOF. We have to prove that an arbitrary mixed derivative $D^\alpha g$ where $|\alpha| = p$ exists and is continuous. We begin by proving that there is a constant C which depends only on Γ, such that, if $h \in C^p(\mathsf{R}^d, \mathsf{R})$, then

$$\max_{|\alpha|=p} |D^\alpha h(x)| \leqq C \max_{\xi \in \Gamma} |D_\xi{}^p h(x)|, \qquad x \in \mathsf{R}^d . \tag{3.2}$$

In fact, if $h \in C^p(\mathsf{R}^d, \mathsf{R})$, then

$$D_\xi{}^p h(x) = \sum_{|\alpha|=p} \binom{p}{\alpha} \xi^\alpha D^\alpha h(x), \qquad \xi \in \Gamma . \tag{3.3}$$

Since by assumption the determinant of the system (3.3) is different from zero, (3.2) follows. Take $\varphi \in C^\infty(\mathsf{R}^d, \mathsf{R})$ such that $\varphi \geqq 0$, $\varphi = 0$ when $|x| > 1$, and $\int \varphi dx = 1$, and form for $\varepsilon > 0$ the "regularization" of g,

$$g_\varepsilon(x) = \int g(x - \varepsilon y)\, \varphi(y)\, dy .$$

By assumption $D_\xi{}^p g$ exists and is continuous when $\xi \in \Gamma$, and hence for such ε,

$$|D_\xi{}^p g_\varepsilon(x) - D_\xi{}^p g(x)| \to 0 \quad \text{as } \varepsilon \to 0 ,$$

uniformly on compact sets. Applying (3.2) to $g_\varepsilon - g_\delta$ we then find that if $|\alpha| = p$, then $D^\alpha g_\varepsilon - D^\alpha g_\delta$ tends to zero uniformly on compact sets as ε and δ tend to zero, and hence that $D^\alpha g_\varepsilon$ converges uniformly on compact sets to some continuous function v. It is clear that these facts imply that $D^\alpha g$ exists and is equal to v.

LEMMA 6. *Let $g \in C^p(\mathsf{R}^d, \mathsf{R})$ and let Γ be a collection of*

$$\binom{d+p}{p+1}$$

vectors in R^d satisfying $\Delta(\Gamma) \neq 0$. Assume that for each $\xi \in \Gamma$

$$D_\xi{}^p g \in \operatorname{Lip}(\xi, 1) .$$

Then $g \in C^{p,1}$.

PROOF. Let g_ε have the same meaning as in the proof of Lemma 5. Then again if $|\alpha| = p$,

$$D^\alpha g_\varepsilon \to D^\alpha g \quad \text{as } \varepsilon \to 0 , \tag{3.4}$$

with uniform convergence on compact sets. Moreover, we claim that for each $\xi \in \Gamma$ the family of functions

(3.5) $\quad D_\xi{}^{p+1} g_\varepsilon$, $0 < \varepsilon < 1$, is uniformly bounded on compact sets .

In fact, since $D_\xi{}^p g \in \mathrm{Lip}(\xi, 1)$, the functions $D_\xi{}^p g_\varepsilon$ have the same property in a uniform way, that is, on each compact set the Lipschitz constant of the function $D_\xi{}^p g_\varepsilon$ can be estimated by a constant independent of ε. But since $g_\varepsilon \in C^\infty$, the last statement is equivalent to (3.5). Applying (3.2) with $p+1$ instead of p we obtain from (3.5) with $|\alpha| = p$ and $D_j = \partial/\partial x_j$, $j = 1, 2, \ldots, d$,

$$D_j D^\alpha g_\varepsilon,\ 0 < \varepsilon < 1, \text{ is uniformly bounded on compact sets}. \tag{3.6}$$

Now (3.4) and (3.6) imply that $D^\alpha g \in \mathrm{Lip}(\xi, 1)$ if ξ is parallel to any coordinate axis. This shows that $g \in C^{p,1}$.

Proof of Theorem 4. From Lemma 1 we know already that f is continuous and that, if $p \geqq 1$, then $D_\xi{}^k f$ is continuous in the direction ξ for each $k \leqq p$. Next we assert that $D_\xi{}^k f$ is continuous for each ξ and each $k \leqq p$. This follows from Lemma 4 by induction on k. Finally we apply Lemma 5 to conclude that $f \in C^p$.

Remark. By applying Lemma 3 instead of Lemma 1 we can obviously prove the analogous assertion with $u \in C^M$ in place of $u \in C^\infty$, where $M = \{M_q\}$ is an increasing sequence such that the class C^M is non-quasi-analytic.

Proof of Theorem 2 and Theorem 5 in the case $\varrho = 0$ or $\varrho = 1$. Since $C^{p,1} \supset C^{p+1,0}$, it is obviously enough to consider the case $\varrho = 1$. However, since by Theorem 4 and the remark above we already know that $f \in C^p$, we can apply Lemma 1 or 3 and Lemma 6 to conclude that $f \in C^{p,1}$.

The following lemma is needed in the case $0 < \varrho < 1$.

Lemma 7. *Let Λ be a finite set of vectors $\xi \in \mathrm{R}^2$ which are pairwise linearly independent, let there be given for each $\xi \in \Lambda$ a function g_ξ from R^2 to R, and let $0 < \varrho < 1$. Assume that*

$$|g_\xi(x)| \leqq 1, \qquad |x| \leqq A, \tag{3.7}$$

$$|g_\xi(x+t\xi) - g_\xi(x)| \leqq |t|^\varrho, \qquad |x|,\ |x+t\xi| \leqq A, \tag{3.8}$$

$$\left|\textstyle\sum_{\xi\in\Lambda}(g_\xi(x) - g_\xi(y))\right| \leqq b, \qquad |x|,\ |y| \leqq A. \tag{3.9}$$

Then for each $\delta > 0$ there exists a constant C which depends only on ϱ, δ and Λ such that

$$|g_\xi(x) - g_\xi(y)| \leqq C(b + |x-y|^\varrho), \qquad |x|,\ |y| \leqq A - \delta, \quad \xi \in \Lambda. \tag{3.10}$$

Corollary. *Let $\xi^1, \ldots, \xi^n$ be pairwise linearly independent vectors in R^2, let g_k, $k = 1, \ldots, n$, be locally bounded functions from R^2 to R, and let*

$0<\varrho<1$. *Assume that* $g_k \in \mathrm{Lip}(\xi^k,\varrho)$ *for each* k *and that* $\sum_{k=1}^n g_k$ *is identically zero. Then* $g_k \in \mathrm{Lip}(\varrho)$ *for each* k.

PROOF. If c is small enough, the functions cg_k, $k=1,\ldots,n$, satisfy the assumptions of Lemma 7 with $b=0$.

REMARK 1. If Λ consists of one or two elements, the assertion (3.10) with $C=2$ is a trivial consequence of (3.8) and (3.9).

REMARK 2. The assertion of the corollary (and of Lemma 7) is not true if $\varrho=1$ and Λ contains at least three elements. This is seen from the example

$$g_1(x) = x_1 \log|x|, \quad g_2(x) = x_2 \log|x|, \quad \text{and} \quad g_3 = -(g_1+g_2)\,,$$

where $|x|=(x_1{}^2+x_2{}^2)^{\frac{1}{2}}$. In fact,

$$|\partial g_1/\partial x_2| = |x_1x_2|/|x|^2 \leqq 1\,,$$

which shows that g_1, g_2, and g_3 belong to $\mathrm{Lip}(\xi,1)$ with ξ equal to $(0,1)$, $(1,0)$ and $(1,-1)$ respectively. On the other hand, it is obvious that the functions g_k do not belong to $\mathrm{Lip}(1)$.

REMARK 3. The assertion of the corollary (and of Lemma 7) is not true if we omit the assumption that the g_ξ are locally bounded. This is seen from the example

$$g_1(x_1,x_2) = \varphi(x_1), \quad g_2(x_1,x_2) = \varphi(x_2), \quad \text{and} \quad g_3(x_1,x_2) = -\varphi(x_1+x_2)\,,$$

where φ is a non-measurable solution of the equation

$$\varphi(s+t) = \varphi(s)+\varphi(t), \quad s,t \in \mathsf{R}\,.$$

For the proof of Lemma 7 we need the following lemma.

LEMMA 8. *Let* w *be a real-valued function of one variable, such that* $w(0)=0$ *and* $|w(t)|\leqq 1$ *when* $|t|\leqq 2a$, *let* $0<\varrho<1$, *and assume that*

$$|w(2t)-2w(t)| \leqq b+C_1|t|^\varrho, \qquad |t|\leqq a\,. \tag{3.11}$$

Then

$$|w(t)| \leqq b+C_2|t|^\varrho, \qquad |t|\leqq 2a\,, \tag{3.12}$$

where $C_2=(1/a^\varrho)+C_1/(2-2^\varrho)$.

PROOF. When $a\leqq|t|\leqq 2a$, it is obvious that (3.12) holds, since $C_2a^\varrho\geqq 1$. Now assume that $0<|s|<a$ and that (3.12) holds when $t=2s$. Then by (3.11) and the triangle inequality

$$\begin{aligned} 2|w(s)| &\leqq |w(2s)| + b + C_1|s|^\varrho \\ &\leqq b + C_2|2s|^\varrho + b + C_1|s|^\varrho \\ &\leqq 2b + |s|^\varrho(2^\varrho C_2 + C_1) \leqq 2b + 2C_2|s|^\varrho . \end{aligned}$$

The last inequality follows from the fact that $C_2 \geqq C_1/(2-2^\varrho)$, that is, $2^\varrho C_2 + C_1 \leqq 2C_2$. Thus we have proved that (3.12) holds when $t=s$. The proof is completed by an obvious induction argument.

PROOF OF LEMMA 7. We prove the lemma by induction on the number of elements of Λ. As remarked above the assertion is true when Λ consists of one or two elements, and in this case we can take $C=2$. Let

$$\Lambda = \Lambda_0 \cup \{\eta\}, \quad \text{where } \eta \notin \Lambda_0 ,$$

and assume that all the vectors in Λ are pairwise linearly independent. We assume that the assertion of Lemma 7 is true for Λ_0, and we shall prove that it is true for Λ. Assume that $\{g_\xi\}_{\xi\in\Lambda}$ satisfies (3.7), (3.8) and (3.9). For $\xi \in \Lambda_0$ set

$$(3.13) \qquad h_\xi(x,s) = h_\xi(x) = \tfrac{1}{2}(g_\xi(x+s\eta) - g_\xi(x)), \qquad |x| \leqq A - \delta_1,\ |s\eta| \leqq \delta_1 ,$$

where $\delta_1 = \frac{1}{3}\delta$. Then it is obvious that h_ξ satisfies (3.7) and (3.8) when

$$|x| \leqq A - \delta_1 \quad \text{and} \quad |x + t\xi| \leqq A - \delta_1 .$$

Moreover, by (3.8) for $\xi = \eta$ and (3.9)

$$\begin{aligned} |\Sigma_{\xi\in\Lambda_0}(h_\xi(x) - h_\xi(y))| &\leqq \tfrac{1}{2}|\Sigma_{\xi\in\Lambda}(g_\xi(x+s\eta) - g_\xi(x))| + \\ &\quad + \tfrac{1}{2}|\Sigma_{\xi\in\Lambda}(g_\xi(y+s\eta) - g_\xi(y))| + \\ &\quad + \tfrac{1}{2}|g_\eta(x+s\eta) - g_\eta(x)| + \tfrac{1}{2}|g_\eta(y+s\eta) - g_\eta(y)| \\ &\leqq b + |s|^\varrho, \qquad |x|, |y| \leqq A - \delta_1,\ |s\eta| \leqq \delta_1 . \end{aligned}$$

This means that $\{h_\xi\}_{\xi\in\Lambda_0}$ satisfies (3.9) with $b + |s|^\varrho$ instead of b. By the induction hypothesis there exists a constant C_1, which depends only on Λ_0, ϱ and δ_1, such that

$$|h_\xi(x) - h_\xi(y)| \leqq C_1(b + |s|^\varrho + |x-y|^\varrho), \qquad |x|, |y| \leqq A - 2\delta_1,\ |s\eta| \leqq \delta_1,\ \xi \in \Lambda_0 .$$

If we take $y - x = s\eta$ and use (3.13), this becomes

$$(3.14) \quad |g_\xi(x+2s\eta) - 2g_\xi(x+s\eta) + g_\xi(x)| \leqq 2C_1(b + |s|^\varrho + |s\eta|^\varrho) \leqq C_2(b + |s|^\varrho) ,$$

if

$$|x| \leqq A - 3\delta_1,\ |s\eta| \leqq \delta_1,\ \xi \in \Lambda_0 .$$

Here C_2 is any constant $\geqq 2C_1(1 + |\eta|^\varrho)$. Set

$$w(s) = g_\xi(x+s\eta) - g_\xi(x) \qquad \text{when } \xi \in \Lambda_0 .$$

Then (3.14) means that

$$|w(2s)-2w(s)| \leqq C_2(b+|s|^\varrho), \qquad |s\eta| \leqq \delta_1 .$$

Lemma 8 then shows that

$$(3.15) \qquad |w(s)| = |g_\xi(x+s\eta)-g_\xi(x)| \leqq C_3(b+|s|^\varrho),$$
$$|s\eta| \leqq \delta_1,\ |x| \leqq A-3\delta_1,\ \xi \in \Lambda_0 ,$$

where C_3 depends only on C_2, ϱ and $\delta_1/|\eta|$, that is on Λ, ϱ and δ. If we take C_3 so large that $C_3(\delta_1/|\eta|)^\varrho \geqq 2$ and use the assumption (3.7), we obtain (3.15) for all s such that $|x+s\eta| \leqq A-3\delta_1$. Now it is easy to prove (3.10) for an arbitrary $\xi \in \Lambda_0$ by combining (3.8) and (3.15). In fact we obtain (3.10) with a constant $C=C_3C_4$, where C_4 depends only on ξ and η. Finally, by interchanging the roles of the vector η and one of the vectors $\xi \in \Lambda_0$ we obtain (3.10) with η in place of ξ, that is, we obtain (3.10) for all $\xi \in \Lambda$. The proof is complete.

Conclusion of proof of Theorem 2 and Theorem 5. Let $0<\varrho<1$, and assume that f satisfies the assumptions of Theorem 2 or Theorem 5. By Theorem 4 we know that $f \in C^p$. It remains to prove that $f \in C^{p,\varrho}$. If $p=0$ this assertion trivially follows from Lemma 1, respectively Lemma 3. Let H be an arbitrary two-dimensional plane in R^d. Let $p \geqq 1$ and choose a finite set Λ of vectors ξ in R^d, pairwise linearly independent and all parallel to H, and choose for each $\xi \in \Lambda$ a constant $c_\xi \neq 0$ such that the polynomial

$$\textstyle\sum_{\xi\in\Lambda} c_\xi \langle \xi, x\rangle^p$$

is identically zero. A moment of reflection shows that Λ must contain at least $p+2$ vectors, and also that any set of $p+2$ pairwise linearly independent vectors can be chosen for Λ. Since $f \in C^p$ we then have

$$\textstyle\sum_{\xi\in\Lambda} c_\xi D_\xi{}^p f = 0 .$$

By Lemma 1, respectively Lemma 3, the functions $D_\xi{}^p f$ belong to $\mathrm{Lip}(\xi,\varrho)$. The corollary of Lemma 7 then shows that the restrictions to H of the functions $D_\xi{}^p f$, $\xi \in \Lambda$, belong to $\mathrm{Lip}(\varrho)$. Moreover, applying Lemma 7 with $b=0$, which is a quantitative counterpart of the corollary, we find that if $\xi \in \Lambda$,

$$\sup\{|D_\xi{}^p f(x)-D_\xi{}^p f(y)|/|x-y|^\varrho\ ;\ |x|,|y| \leqq A,\ x,y \in H\} .$$

depends only on ϱ, Λ, and on bounds for

$$\sup\{|D_\xi{}^p f(x)|\ ;\ |x| \leqq 2A,\ \xi \in \Lambda\}$$

and

$$\sup\{|D_\xi^p f(x+t\xi)-D_\xi^p f(x)|/|t|^\varrho\,;\ |x|,\ |x+t\xi|\leqq 2A,\ x,x+t\xi\in H,\ \xi\in\Lambda\}.$$

Applying this result to a collection of planes parallel to H we find that, for any η parallel to H, the function $D_\xi^p f$ belongs to $\mathrm{Lip}(\eta,\varrho)$ as a function on R^d. Since H was arbitrary, the same statement must be true for any $\eta\in\mathsf{R}^d$, hence in particular for a set of η which forms a basis for R^d. This shows that, on R^d, the function $D_\xi^p f$ is Lipschitz continuous with exponent ϱ. For reasons of symmetry the same statement must be true for any non-zero $\xi\in\mathsf{R}^d$. Since an arbitrary mixed derivative of order p of f is a linear combination of derivatives $D_\xi^p f$, $\xi\in\mathsf{R}^d$, this shows that $f\in C^{p,\varrho}$.

As was pointed out in Section 1, one could use Theorem 1 in [1] instead of Lemma 7 above in proving that the derivatives of order p of f belong to $\mathrm{Lip}(\varrho)$. We will now indicate briefly how this could be done. Let J be the set of all multi-indices $\alpha=(\alpha_1,\ldots,\alpha_d)$ of length $|\alpha|=p$. Then J contains

$$N = \binom{p+d-1}{p}$$

elements. We will consider N-tuples $(c_\alpha)_{\alpha\in J}$ of real numbers as elements of R^N. The usual inner products in R^d and R^N will be denoted $\langle\cdot,\cdot\rangle$. If $\xi\in\mathsf{R}^d$, the numbers $(\xi^\alpha)_{\alpha\in J}$ define an element of R^N which we denote $\mathsf{V}^p\xi$. Since we know that $f\in C^p$, we can define a continuous function $g=(g_\alpha)_{\alpha\in J}$ from R^d to R^N by

$$g_\alpha(x) = \binom{p}{\alpha} D^\alpha f(x)\,.$$

Then $D_\xi^p f(x)=\langle \mathsf{V}^p\xi, g(x)\rangle$. By Lemma 1 we know that

$$\langle \mathsf{V}^p\xi, g\rangle \in K(\xi,\sigma) \quad \text{for each non-zero } \xi\in\mathsf{R}^d\,. \tag{3.16}$$

Now, taking

$$\Lambda = \{(\xi,\mathsf{V}^p\xi)\,;\ 0\neq\xi\in\mathsf{R}^d\}\subset \mathsf{R}^d\times\mathsf{R}^N\,,$$

we can apply Theorem 1 in [1]. It is easy to see that Λ satisfies condition $(\hat{A})$ of [1], i.e. that

$$a\in\mathsf{R}^d,\quad b\in\mathsf{R}^N,\quad \text{and}\quad \langle a,\xi\rangle\langle b,\mathsf{V}^p\xi\rangle=0 \text{ for every } \xi\in\mathsf{R}^d$$

implies that $a=0$ or $b=0$. On the other hand, Λ does not satisfy condition (A), since every element of the linear hull L of the set of tensor products $\xi\otimes\mathsf{V}^p\xi$ has an obvious symmetry property, and hence L must be a proper subset of the tensor product $\mathsf{R}^d\otimes\mathsf{R}^N$. The cited theorem then shows that $g\in K(\hat{\sigma})$, where

$$\hat{\sigma}(\varepsilon) = \varepsilon\left(1 + \int_{\min(1,\varepsilon)}^{1} t^{-2}\,\sigma(t)\,dt\right),$$

and that this is the strongest result that follows from (3.16). If $\sigma(\varepsilon) = \varepsilon^{\varrho}$ and $\varrho < 1$, we have $\hat{\sigma}(\varepsilon) \approx \sigma(\varepsilon)$, hence

$$K(\hat{\sigma}) = K(\sigma) = \mathrm{Lip}(\varrho),$$

and thus (3.16) implies $g \in \mathrm{Lip}(\varrho)$ in this case. On the other hand, if $\sigma(\varepsilon) = \varepsilon$ we have

$$\hat{\sigma}(\varepsilon) \approx \varepsilon \log(1/\varepsilon),$$

and hence the theorem shows that the assumption

$$\langle \nabla^p \xi, g\rangle \in \mathrm{Lip}(\xi, 1) \quad \text{for each non-zero } \xi \in \mathsf{R}^d$$

does not imply $g \in \mathrm{Lip}(1)$. This explains why we had to use a separate method (Lemma 6) in the case $\varrho = 1$.

4. The converse theorems.

We now pass to the proofs of the counterexamples Theorem 3, 6 and 9. The following theorem gives sufficient conditions for a function to have the property of Theorem 3.

THEOREM 10. *Let f be a function from R^d to R, and assume that f belongs to C^p outside the origin and that f is homogeneous of degree p, where p is a natural number. Then $f \circ u \in C^p(\mathsf{R},\mathsf{R})$ for every $u \in C^p(\mathsf{R},\mathsf{R}^d)$.*

Of course, any function which satisfies the hypotheses of Theorem 10 and is not a polynomial must have a discontinuity in some derivative of degree p at the origin. In fact, if $f \in C^p$ and f is homogeneous of degree p, then each derivative of degree p is constant, and hence f is a polynomial.

PROOF OF THEOREM 10. If $\xi \in \mathsf{R}^d$, let $\langle \xi, D\rangle f$ denote $\sum \xi_j \partial f/\partial x_j$, which has been denoted by $D_\xi f$ above. Set $E = \{t\,;\ t \in \mathsf{R},\ |u(t)| \neq 0\}$. In the open set E we have

$$h = f \circ u \in C^p$$

and

$$h^{(p)}(t) = \big(\langle u'(t), D\rangle^p f\big)(u(t)) + \ldots, \tag{4.1}$$

where the dots indicate terms containing derivatives of f of order at most $p-1$. On the other hand, if $t^0 \notin E$,

$$h^{(p)}(t^0) = p!\,f(u'(t^0))\,. \tag{4.2}$$

In fact, in this case we can write $u(t)=(t-t^0)v(t)$, where $v\in C^{p-1}$ and $v(t^0)=u'(t^0)$, and hence in view of the homogeneity,

$$h(t) = (t-t^0)^p f(v(t)), \quad \text{where} \quad f(v(t))\in C^{p-1}\,.$$

These facts imply (4.2). Thus it only remains to prove that if $t^0 \notin E$,

$$h^{(p)}(t) \to h^{(p)}(t^0) \quad \text{as } t\to t^0, \qquad t\in E\,.$$

Since $D^\alpha f$ is homogeneous of degree $p-|\alpha|$, the omitted terms in (4.1) tend to zero as $t\to t^0$. Moreover,

$$\begin{aligned}&\big(\langle u'(t),D\rangle^p f\big)(u(t)) - \big(\langle u'(t^0),D\rangle^p f\big)(u(t))\\ &\quad = \textstyle\sum_{|\alpha|=p}\binom{p}{\alpha}\big(u'(t)^\alpha - u'(t^0)^\alpha\big)\big(D^\alpha f\big)(u(t)) \to 0 \quad \text{as } t\to t^0, \quad t\in E\,,\end{aligned}$$

since $(D^\alpha f)(u(t))$ is bounded ($D^\alpha f$ is homogeneous of degree zero) and u' is continuous. Thus it suffices to prove that

$$\big(\langle u'(t^0),D\rangle^p f\big)(u(t)) \to p!\,f(u'(t^0)) \quad \text{as } t\to t^0, \qquad t\in E\,, \tag{4.3}$$

for an arbitrary $t^0\notin E$. If $|u'(t^0)|=0$, it is obvious that (4.3) holds. If $|u'(t^0)|\neq 0$, we again write $u(t)=(t-t^0)v(t)$. Since $\langle u'(t^0),D\rangle^p f$ is homogeneous of degree zero, the left-hand side of (4.3) is equal to

$$\big(\langle u'(t^0),D\rangle^p f\big)(v(t))\,,$$

which tends to

$$\big(\langle u'(t^0),D\rangle^p f\big)(u'(t^0))$$

as $t\to t^0$. Applying Euler's equality for homogeneous functions we obtain the desired result.

Proof of Theorem 6. It is enough to prove the theorem for $d=2$. Take a non-constant function $F\colon \mathsf{R}^2\to\mathsf{R}$, such that F is homogeneous of degree zero, $F(x_1,x_2)=0$ when $x_1x_2=0$, and F is infinitely differentiable in the complement of the origin. Define $e^{-1/|x_2|}$ as zero for $x_2=0$ (then $e^{-1/|x_2|}\in C^\infty$), and set

$$f(x_1,x_2) = F(x_1,e^{-1/|x_2|}), \qquad (x_1,x_2)\in\mathsf{R}^2\,.$$

Then f is obviously not continuous at the origin. We claim that $f\circ u\in C^\infty$ for each quasianalytic function $u=(u_1,u_2)\in C^\infty(\mathsf{R},\mathsf{R}^2)$. Since $f\in C^\infty$ outside the origin, it will be enough to show that $f\circ u\in C^\infty$ in a neighbourhood of $t=0$, if $|u(0)|=0$. If u_1 is identically zero, then f is identically zero. If $u_1(0)=0$ and u_1 is not identically zero, then by virtue of the quasianalyticity there exists an integer $k\geqq 1$, such that

$$u_1(t) = t^k v_1(t), \qquad v_1 \in C^\infty \text{ and } v_1(0) \neq 0 .$$

Since $u_2(0)=0$, we have

$$e^{-1/|u_2(t)|} = t^k v_2(t) ,$$

where $v_2 \in C^\infty$ and $v_2(0)=0$. Using the fact that F is homogeneous and that $F(x_1,0)=0$, we obtain

$$f(u_1(t), u_2(t)) = F(t^k v_1(t), t^k v_2(t)) = F(v_1(t), v_2(t)), \qquad t \in \mathsf{R} .$$

Since $v_1(0) \neq 0$, this proves that $f \circ u \in C^\infty$ in a neighbourhood of zero.

REMARK. We have actually proved that $f \circ u \in C^\infty$ for each $u \in C^\infty$ which has no zero of infinite order. This shows in particular that one can not prove Theorem 1, Theorem 2 or Theorem 5 without applying the assumption to functions u with singularities of infinite order.

The following result implies Theorem 9.

THEOREM 11. *Let $r_j(s)$, $j=0,1,\ldots,d$, be continuous functions defined when $s \geqq 0$, vanishing at the origin, positive and infinitely differentiable when $s>0$ and satisfying*

$$\lim_{s\to 0} r_{j-1}(s)/r_j(s)^B = 0, \qquad j=1,\ldots,d ,$$

and

$$\lim_{s\to 0} r_d(s)/s^B = 0 \quad \textit{for every } B .$$

Set $r(s)=(r_1(s),\ldots,r_d(s))$. Let $\varphi \in C^\infty(\mathsf{R}^d,\mathsf{R})$, $\varphi \not\equiv 0$ and $\varphi(x)=0$ when $|x|>1$. Define the function f: $\mathsf{R}^{d+1} \to \mathsf{R}$ by

$$f(x,s) = \begin{cases} \varphi\big((x-r(s))/r_0(s)\big), & s>0,\ x \in \mathsf{R}^d \\ 0, & s \leqq 0,\ x \in \mathsf{R}^d . \end{cases}$$

Then $f \circ u \in C^\infty(\mathsf{R}^d,\mathsf{R})$ for every analytic function u from R^d to R^{d+1}, but f is not continuous.

It is obvious that there exist functions $r_j(s)$ with the properties mentioned in the theorem; for example, we can take $r_j(s)=\exp(-x^{j-d-1})$ when $s>0$, $r_j(0)=0$, $j=0,1,\ldots,d$.

PROOF OF THEOREM 11. It is obvious that f is not continuous at the origin, and that f is infinitely differentiable outside the origin. Thus it is sufficient to prove that $f \circ u = 0$ in a neighbourhood of each point $t^0 \in \mathsf{R}^d$ such that $|u(t^0)|=0$. Take a non-trivial analytic function $U=U(x,s)$ from a neighbourhood of $0 \in \mathsf{R}^{d+1}$ to R such that

$$(U \circ u)(t) = 0 \quad \text{in a neighbourhood of } t^0 .$$

We shall show that if $r_j(s)$ has the properties described in Theorem 10, then there exist $c>0$, $\delta>0$ and B such that

$$(4.4) \qquad |U(x,s)| \geqq cr_0(s)^B, \quad \text{if} \quad |x-r(s)| \leqq r_0(s), \qquad 0<s<\delta .$$

This will complete the proof, since if $(x,s)=u(t)$ for some t sufficiently close to t^0, then $U(x,s)=0$, and hence by virtue of (4.4) we have $|x-r(s)|>r_0(s)$, that is, $f(x,s)=0$. To prove (4.4) we begin by showing that there exist $c>0$, $\delta>0$ and B such that

$$(4.5) \qquad |U(r(s),s)| \geqq cr_1(s)^B, \qquad 0<s<\delta ,$$

or more generally that

$$|V(r(s))| \geqq cr_1(s)^B, \qquad 0<s<\delta ,$$

for an arbitrary non-trivial analytic function V from a neighbourhood of 0 in R^d to R if the functions $r_j(s)$, $j=1,\ldots,d$, satisfy the assumptions of Theorem 10. This is easily proved by induction on the dimension d if we write V in the form

$$V(x) = x_1^k\, v(x_2,\ldots,x_d) + x_1^{k+1} R(x) ,$$

where $k\geqq 0$, v and R are analytic and $v\not\equiv 0$, and use the estimate

$$\begin{aligned} |V(r(s))| &\geqq r_1(s)^k \big(|v(r_2(s),\ldots,r_d(s))| - Cr_1(s)\big) \\ &\geqq r_1(s)^k \big(cr_2(s)^B - Cr_1(s)\big) \\ &\geqq \tfrac{1}{2}cr_1(s)^{k+B}, \qquad 0<s<\delta . \end{aligned}$$

It remains to show that (4.5) implies (4.4). Take C so large that

$$|\mathrm{grad}_x U(x,s)| \leqq C$$

in some neighbourhood of 0, and assume that

$$|x-r(s)| \leqq r_0(s) .$$

With a new δ we then have

$$\begin{aligned} |U(x,s)| &\geqq |U(r(s),s)| - C\,|x-r(s)| \\ &\geqq cr_1(s)^B - Cr_0(s) \\ &\geqq \tfrac{1}{2}c\, r_1(s)^B, \qquad 0<s<\delta . \end{aligned}$$

This completes the proof of Theorem 11.

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