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# REMARKS ON THE IRRATIONALITY AND TRANSCENDENCE OF CERTAIN SERIES

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## 1. Introduction.

Applying a method of Chowla [1] and Erdös [2], Golomb [4] showed the irrationality of

$$G(t^{-1}) = \sum_{n \geqq 0} (t^{2^n} - 1)^{-1}$$

for $t = 2, 3, \ldots$, by using the representation of this number as a $t$-adic fraction. But the result may be more easily deduced by using the large growth of the numbers $t^{2^n}$ $(n \to \infty)$. We show:

THEOREM 1. *Let $k$, $t$ and $b$ be integers satisfying the inequalities*

$$k \geqq 2, \quad t \geqq 2, \quad \text{and} \quad 0 < b < t^{1-1/k} .$$

*Then the number*

$$G_k(bt^{-1}) = \sum_{n \geqq 0} b^{k^n} (t^{k^n} - b^{k^n})^{-1}$$

*is irrational.*

REMARKS. The irrationality of $G(t^{-1})$ implies the irrationality of

$$\sum_{n \geqq 0} (t^{2^n} + 1)^{-1};$$

compare Golomb [4]. The irrationality of

$$\sum_{n \geqq 0} b^{k^n} (t^{k^n} + b^{k^n})^{-1}$$

for $k > 2$ is unsettled.

Erdös and Strauss [3] proved very general irrationality criterions, which include the special case $b = 1$ of theorem 1.

By using the Thue–Siegel–Roth theorem (Roth [6]), it is not difficult to show the following two theorems.

THEOREM 2. *For $k > 2$, $t \geqq 2$ and $0 < b < t^{1-5/2k}$ the number $G_k(bt^{-1})$ is transcendental,*

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THEOREM 3. *Let* $0 < n_1 < n_2 < \dots$ *be a sequence of integers, such that infinitely often we have* $n_{r+1} - n_r \geqq 2$. *Then the number*

$$H(t^{-1}) = \sum_{r \geqq 1} (t^{2^{n_r}} - 1)^{-1}$$

*is transcendental.*

For instance both of the numbers

$$\textstyle\sum_{n \geqq 0,\, n \text{ odd}} (2^{2^n} - 1)^{-1} \quad \text{and} \quad \sum_{n \geqq 0,\, n \text{ even}} (2^{2^n} - 1)^{-1}$$

are transcendental.

If one uses a sharpened version of Roth's theorem, one gets the transcendency of, say,

$$\sum_{n \geqq 0} (2^{2^n} - 1)^{-1} 2^{-2^n};$$

but the method does not apply to $G_2(t^{-1})$. Of course one should conjecture: $G_2(t^{-1})$ is transcendental for $t \geqq 2$. We are only able to show

THEOREM 4. *For* $t \geqq 2$ *the number* $G_2(t^{-1})$ *is not algebraic of the second degree.*

## 2. Proofs of theorems 1, 2, and 3.

PROOF OF THEOREM 1. We assume that $G_k(bt^{-1}) = a/q$ is rational and derive a contradiction.

For any integer $N \geqq 1$ the numbers

$$D_1 = q(t^{k^N} - b^{k^N}) G_k(bt^{-1})$$

and

$$D_2 = q(t^{k^N} - b^{k^N}) \sum_{n \leqq N} b^{k^n} (t^{k^n} - b^{k^n})^{-1}$$

are integers. For $D_1$ this is obvious from our assumption on $G_k(bt^{-1})$. The number $t^d - b^d$ divides $t^n - b^n$, if $0 < b < t$ and $d$ divides $n$; therefore $t^{k^n} - b^{k^n}$ divides $t^{k^N} - b^{k^N}$ for $n \leqq N$, and thus $D_2$ is an integer. Hence the difference

$$D = D_1 - D_2 = q(t^{k^N} - b^{k^N}) \sum_{n > N} b^{k^n} (t^{k^n} - b^{k^n})^{-1}$$

is also an integer, and obviously we have $D \neq 0$; therefore $|D| \geqq 1$.

But for sufficiently large $N$ we have, since $b < t$,

$$\begin{aligned} (2.1) \qquad \sum_{n > N} b^{k^n} (t^{k^n} - b^{k^n})^{-1} &= \sum_{n > N} (bt^{-1})^{k^n} (1 - (bt^{-1})^{k^n})^{-1} \\ &\leqq 2 \cdot \sum_{n > N} (bt^{-1})^{k^n} \\ &< 2 \sum_{m \geqq k^N + 1} (bt^{-1})^m \leqq 4 (bt^{-1})^{k^N + 1}. \end{aligned}$$

Since $0<b<t^{1-1/k}$, we conclude that

$$0 < D < q t^{k^N} 4(bt^{-1})^{k^{N+1}} < \tfrac{1}{2} \tag{2.2}$$

for sufficiently large $N$. This is a contradiction to $|D| \geqq 1$. Consequently $G_k(bt^{-1})$ is irrational.

PROOFS OF THEOREMS 2 AND 3. The proof of theorem 2 uses the famous approximation theorem of Thue–Siegel–Roth (Roth [6]); we cite a version from Schneider's book ([7, p. 13]):

*Let $\alpha$ be algebraic of degree $s>1$. Let $p_\nu, q_\nu, \nu=1,2,\ldots,$ be such pairs of integers that $0<q_\nu \leqq q_{\nu+1}$ and $q_\nu = q_\nu' q_\nu''$, where $q_\nu''$ is a power of some integer $g$. Define $\eta$ by*

$$\eta = \limsup_{\nu\to\infty} (\log q_\nu')/\log q_\nu ,$$

*and let $\mu>\eta+1$. Then the inequality*

$$|\alpha - p_\nu/q_\nu| < q_\nu^{-\mu} \tag{2.3}$$

*is valid for at most a finite number of the $p_\nu/q_\nu$.*

Roth's theorem is the special case $g=1$, $\eta=1$.

We have $\sum_{n\leqq N} b^{k^n}(t^{k^n}-b^{k^n})^{-1} = a_N q_N^{-1}$ with an integer $a_N$ and with $q_N = t^{k^N} - b^{k^N}$; for $q_N$ is the least common multiple of the numbers $t^{k^n}-b^{k^n}$, $n=1,2,\ldots.N$. For sufficiently large $N$ we have by (2.1), $k>2$ and $0<b\leqq t^{1-5/(2k)}$:

$$|G_k(bt^{-1}) - a_N q_N^{-1}| < 4(bt^{-1})^{k^{N+1}} \leqq 4t^{-5k^N/2} \leqq 4q_N^{-5/2},$$

and by Roth's theorem (with $g=1$, $\eta=1$, $\mu=5/2-\varepsilon$) the number $G_k(bt^{-1})$ is rational or transcendental, and hence theorem 2 is proved.

The proof of theorem 3 is analogous.

The transcendency of $\sum_{n\geqq 0}(2^{2^n}-1)^{-1}2^{-2^n}$ follows (with $g=2$, $\eta=\frac{1}{2}$) in a similar manner; only here one needs the cited sharpened version of Roth's theorem.

## 3. Proof of theorem 4.

First we state two simple lemmas:

LEMMA 1. *If $2^n+2^m > 2^K+2^{K-1}$ with nonnegative integers $n$, $m$, $K$, then $2^n+2^m \geqq 2^{K+1}$.*

LEMMA 2. *If $2^n+2^m \leqq 2^K+2^{K-1}$ with nonnegative integers $n$, $m$, $K$, then $(t^{2^n}-1)(t^{2^m}-1)$ divides $(t^{2^K}-1)(t^{2^{K-1}}-1)$.*

The proofs are left to the reader.

Now assume, that $\beta = G_2(t^{-1})$ is an irrationality of the second degree. Then integers $a$, $h$ and $q$ exist, such that

$$a\beta^2 + q\beta = h .$$

Since $\beta$ is irrational, $a \neq 0$. Define $\delta_N$ by

$$\delta_N = (t^{2^N} - 1)(t^{2^{N-1}} - 1) T_N , \tag{3.1}$$

where

$$T_N = h - a \sum_{\substack{m,n \geq 0 \\ 2^n + 2^m \leq 2^N + 2^{N-1}}} \sum (t^{2^m} - 1)^{-1}(t^{2^n} - 1)^{-1} - q \sum_{n \leq N} (t^{2^n} - 1)^{-1} . \tag{3.2}$$

By lemma 2, the number $\delta_N$ is an integer. On the other hand, since $h = a\beta^2 + q\beta$, and

$$\beta^2 = \sum_{m \geq 0} \sum_{n \geq 0} (t^{2^m} - 1)^{-1}(t^{2^n} - 1)^{-1} ,$$

we have

$$T_N = a \sum_{\substack{m,n \geq 0 \\ 2^n + 2^m > 2^N + 2^{N-1}}} \sum (t^{2^m} - 1)^{-1}(t^{2^n} - 1)^{-1} + q \sum_{n > N} (t^{2^n} - 1)^{-1} . \tag{3.3}$$

Using lemma 1 it follows that

$$\begin{aligned} \sum_{\substack{m,n \geq 0 \\ 2^n + 2^m > 2^N + 2^{N-1}}} \sum (t^{2^m} - 1)^{-1} (t^{2^n} - 1)^{-1} & \leq \sum_{\substack{m,n \geq 0 \\ 2^n + 2^m > 2^N + 2^{N-1}}} \sum 4t^{-2^m - 2^n} \\ & \leq \sum_{m,n \leq 2N} \sum 4t^{-2^{N+1}} + \sum_{\substack{m,n \geq 0 \\ m > 2N \text{ or } n > 2N}} \sum 2t^{-2^m} \cdot 2t^{-2^n} \\ & \leq 16 N^2 t^{-2^{N+1}} + 2\left(4 \sum_{u \geq 2^{2N+1}} t^{-u} \sum_{v \geq 0} t^{-v}\right) \\ & \leq 32 N^2 t^{-2^{N+1}} . \end{aligned}$$

By (2.1) with $b = 1$, $k = 2$ we obtain from this estimate and (3.3) that

$$|T_N| \leq 32 N^2 t^{-2^{N+1}} (|a| + |q|) ,$$

and

$$|\delta_N| \leq 32 N^2 t^{-2^{N+1}} (|a| + |q|) t^{2^N + 2^{N-1}} .$$

Since $\delta_N$ is an integer, we have $\delta_N = 0$ for all sufficiently large $N$, say for $N \geq N_0$. Hence we have by (3.1) for all $N \geq N_0$

$$T_N = 0 . \tag{3.4}$$

Consequently $T_{N+1}-T_N=0$ for $N \geqq N_0$, and by (3.2) we obtain

$$-a\left\{\sum_{m,n\geqq 0}^{*} (t^{2^m}-1)^{-1}(t^{2^n}-1)^{-1}\right\} = q(t^{2^{N+1}}-1)^{-1}. \tag{3.5}$$

The * indicates the condition $2^N+2^{N-1}<2^m+2^n \leqq 2^{N+1}+2^N$; therefore $n$ and $m$ take the values $n=N+1$ and $m=0,1,\ldots,N$, or $n=N$ and $m=N$, or $n=0,1,\ldots,N$ and $m=N+1$. Hence we obtain from (3.5) for $N \geqq N_0+1$:

$$a\left\{2\sum_{0\leqq m\leqq N}(t^{2^m}-1)^{-1} + (t^{2^{N+1}}-1)(t^{2^N}-1)^{-2}\right\} = -q. \tag{3.6}$$

Using (3.6) twice (with $N$ and $N+1$) we infer by subtraction:

$$a\{2(t^{2^{N+1}}-1)^{-1}+(t^{2^{N+2}}-1)(t^{2^{N+1}}-1)^{-2}-(t^{2^{N+1}}-1)(t^{2^N}-1)^{-2}\} = 0,$$

hence, after a bit calculation,

$$1-t^{2^N} = 0.$$

This contradicts $t>1$. Hence $\beta$ cannot be a quadratic irrationality.

## 4. $p$-adic analogues.

Theorem 1a. *Let $k \geqq 2$, $t \geqq 2$ be integers, $p$ a prime dividing $t$. Then the $p$-adic number*

$$\varrho = \sum_{n\geqq 0} t^{k^n}(t^{k^n}-1)^{-1}$$

*is irrational.*

Remark. The series for $\varrho$ is ($p$-adic) convergent, since $|t|_p<1$, $|t^{k^n}-1|_p=1$. — Some knowledge of $p$-adic numbers is presupposed.

Proof. Without loss of generality, we take $t$ squarefree, $t=\prod_{1\leqq\lambda\leqq l} p_\lambda$ with different primes $p_\lambda$ (the details are a bit more complicated if $t$ is not squarefree). We assume, that $\varrho=a/q$ (with integers $a$, $q$) is rational. Let $N \geqq 1$, $Q=t^{k^N}-1$. The $p$-adic number

$$A = q\varrho Q - q\sum_{n\leqq N} t^{k^n}Q(t^{k^n}-1)^{-1}$$
$$= qQ\sum_{n>N} t^{k^n}(t^{k^n}-1)^{-1} \neq 0$$

is a rational integer. Further the $p_\lambda$-adic value of $A$ is

$$|A|_{p_\lambda} = |q|_{p_\lambda}\left|\sum_{n>N} t^{k^n}(t^{k^n}-1)^{-1}\right|_{p_\lambda} = |q|_{p_\lambda}\, p_\lambda^{-k^{N+1}}. \tag{5.1}$$

The absolute value of $A$ is obviously estimated by

$$|A| \leqq |a|t^{k^N} + |q|(N+1)2t^{k^N} \leqq \gamma\, N t^{k^N},$$

with a constant $\gamma > 0$, independent of $N$. Since (by the product formula)

$$\prod_{1 \leqq \lambda \leqq l} |A|_{p_\lambda} \geqq |A|^{-1},$$

we obtain

$$\prod_{1 \leqq \lambda \leqq l} |A|_{p_\lambda} \geqq (\gamma N t^{k^N})^{-1}. \tag{5.2}$$

But by (5.1) we have

$$\prod_{1 \leqq \lambda \leqq l} |A|_{p_\lambda} = \prod_{1 \leqq \lambda \leqq l} |q|_{p_\lambda} t^{-k^{N+1}}, \tag{5.3}$$

a contradiction to (5.2) for sufficiently large $N$. Hence $\varrho$ cannot be rational.

Using the $p$-adic analogue of Roth's theorem (Ridout [5]), one proves easily

THEOREM 2a. *Let $k > 2$, $t \geqq 2$, $p$ be a prime dividing $t$. Then the $p$-adic number*

$$\varrho = \sum_{n \geqq 0} t^{k^n}(t^{k^n} - 1)^{-1}$$

*is transcendental.*

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