MATH. SCAND. 20 (1967), 275–280

ON THE CONCEPT OF CONGRUENCE RELATION IN PARTIAL ALGEBRAS

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1. Introduction.

There are two concepts of congruence relations on partial algebras which are usually used in the literature. The first is simply the concept of "congruence relation", the second is the concept of "strong congruence relation". The two concepts coincide whenever the partial algebras in consideration are universal algebras. Some mathematicians seem to be convinced that the first concept is the preferable generalization of congruence relations on universal algebras, others prefer the second one. The authors of this paper believe that neither of the convictions is particularly justified, but that rather the individual nature of a problem in question determines which concept should be used. Applications of congruence relations can be found e.g. in [3], applications of strong congruence relations can be found e.g. in [4]. A systematic treatment of both concepts is contained in [1].

The concept of a partial algebra is the natural one if we want to talk about subsets of universal algebras rather than about subalgebras. If we take any subset B of a universal algebra $\mathfrak{A}$ which is not necessarily closed under the operations, then we just delete all n_γ-tupels in the domain of each n_γ-ary operation f_γ which would yield a value outside of B and thus arrive at a partial algebra.

From here comes the main motivation for the theory of partial algebras. So it would be highly desirable that both the congruence relations and the strong congruence relations on partial algebras could be characterized as restrictions of congruence relations on universal algebras. Theorems 1 and 2 of Section 3 yield precisely those characterizations. A stronger version of Theorem 1 can be found in [3]. The authors construct there an embedding of any partial algebra into a universal algebra $\mathfrak{A}$ such that all the congruence relations on the embedded partial algebra can simul-

Received April 24, 1966.

* Research supported by the U. S. National Science Foundation under grant number GP-4221.

taneously be extended to $\mathfrak{A}$. A still stronger version of Theorem 1 can be found in [1] where the author gives an explicite description of $\mathfrak{A}$ as a free algebra over the class of all algebras of a given type generated by the partial algebra in consideration. The same stronger version of Theorem 2 due to the first author of this paper is given in [1]. The stronger results mentioned are used by the first author and E. T. Schmidt in order to characterize algebraic lattices as congruence lattices of universal algebras (see [3]). From our point of view these strengthenings are irrelevant, and a different approach of the proofs of Theorems 1 and 2 avoiding the rather technical proofs for the strengthenings mentioned above seems to be justified. Theorem 3 and its proof are due to the second author of this paper. Theorem 3 yields Theorems 1 and 2 and permits a unifying proof of the same. Theorem 2 was announced in [2] by the first author.

2. Preliminaries.

In this section we collect the concepts needed in our results.

DEFINITION 1. $\mathfrak{B}=\langle B; F\rangle$ is a *partial algebra* if $B\neq\emptyset$ and F consists of finitary partial operations on B. If f_γ is a n_γ-ary partial operation in F, then $D(f_\gamma,\mathfrak{B})$ denotes the domain of f_γ in $\mathfrak{B}$, that is, $D(f_\gamma,\mathfrak{B})$ is a subset of B^{n_γ}.

DEFINITION 2. The partial algebra $\mathfrak{B}'=\langle B'; F\rangle$ is a *subalgebra* of the partial algebra $\mathfrak{B}=\langle B; F\rangle$ if $B'\subseteq B$, the operation f_γ on $\mathfrak{B}'$ is the restriction of the f_γ on $\mathfrak{B}$ to B', so

$$D(f_\gamma,\mathfrak{B}') = D(f_\gamma,\mathfrak{B}) \wedge B'^{n_\gamma} \quad \text{for all } f_\gamma \in F .$$

DEFINITION 3. The partial algebra $\mathfrak{B}'=\langle B'; F\rangle$ is a *relative subalgebra* of the partial algebra $\mathfrak{B}=\langle B; F\rangle$ if $B'\subseteq B$ and for all $f_\gamma\in F$

$$D(f_\gamma,\mathfrak{B}') = \{(b_0,\ldots,b_{n_\gamma-1});\ (b_0,\ldots,b_{n_\gamma-1}) \in D(f_\gamma,\mathfrak{B}) \wedge B'^{n_\gamma} \text{ and } f_\gamma(b_0,\ldots,b_{n_\gamma-1}) \in B'\} .$$

We also say that $\mathfrak{B}'$ is embedded in $\mathfrak{B}$.

DEFINITION 4. θ is a *congruence relation* on the partial algebra $\mathfrak{B}$ if it is an equivalence relation and

$$x_0 \equiv y_0(\theta),\ldots,x_{n_\gamma-1} \equiv y_{n_\gamma-1}(\theta)$$

implies

$$f_\gamma(x_0,\ldots,x_{n_\gamma-1}) \equiv f_\gamma(y_0,\ldots,y_{n_\gamma-1})(\theta)$$

for all $f_\gamma\in F$ with

$$\{(x_0,\ldots,x_{n_\gamma-1}),(y_0,\ldots,y_{n_\gamma-1})\} \subseteq D(f_\gamma,\mathfrak{B}) .$$

DEFINITION 5. θ is a *strong congruence relation* on the partial algebra $\mathfrak{B}$ if it is a congruence relation and if

$$x_0 \equiv y_0(\theta), \ldots, x_{n_\gamma-1} \equiv y_{n_\gamma-1}(\theta)$$

and

$$(x_0, \ldots, x_{n_\gamma-1}) \in D(f_\gamma, \mathfrak{B})$$

then

$$(y_0, \ldots, y_{n_\gamma-1}) \in D(f_\gamma, \mathfrak{B}) .$$

DEFINITION 6. If $\mathfrak{B}'$ is embedded in $\mathfrak{B}$ and θ is a congruence relation on $\mathfrak{B}$, we denote the restriction of θ to $\mathfrak{B}'$ by $\theta_{B'}$.

Now we are in a position to state and prove our characterization theorems.

3. Characterization theorems.

THEOREM 1. *Let $\mathfrak{B} = \langle B; F\rangle$ be a partial algebra and let θ be a congruence relation on $\mathfrak{B}$. Then there exists an algebra $\mathfrak{A}$ which contains $\mathfrak{B}$ as a relative subalgebra, and there is a congruence relation $\bar{\theta}$ on $\mathfrak{A}$ such that $\bar{\theta}_B = \theta$.*

THEOREM 2. *Let $\mathfrak{B} = \langle B; F\rangle$ be a partial algebra and let θ be a congruence relation on $\mathfrak{B}$. Then θ is strong if and only if $\mathfrak{B}$ can be embedded into an algebra $\mathfrak{A}$ and θ can be extended to a congruence relation $\bar{\theta}$ on $\mathfrak{A}$ such that*

$$[b]\theta = [b]\bar{\theta} \quad \textit{for all } b \in B,$$

where $[b]\theta$ denotes the congruence class of θ containing b.

Before we state Theorem 3, we introduce a new concept needed in that theorem.

DEFINITION 7. If the partial algebra $\mathfrak{B}$ with the congruence relation θ is given, we call a subalgebra $\mathfrak{B}'$ *θ-strong* if $\theta_{B'}$ is a strong congruence relation on $\mathfrak{B}'$. Thus, $\mathfrak{B}$ is a θ-strong subalgebra of $\mathfrak{B}$ if and only if θ is strong on $\mathfrak{B}$.

Using this terminology we get

THEOREM 3. *Let $\mathfrak{B}$ be a partial algebra and let θ be a congruence relation on $\mathfrak{B}$. Then $\mathfrak{B}$ can be embedded into a universal algebra $\mathfrak{A}$ so that*

(i) *There is a congruence relation $\bar{\theta}$ on $\mathfrak{A}$ such that $\bar{\theta}_B = \theta$.*
(ii) *For all θ-strong subalgebras $\mathfrak{B}'$ of $\mathfrak{B}$ there is a subalgebra $\mathfrak{A}'$ of $\mathfrak{A}$ extending $\mathfrak{B}'$ such that $[b']\theta_{B'} = [b']\psi$ for all $b' \in B'$ and some congruence relation ψ on $\mathfrak{A}'$.*

Since Theorem 1 is identical with Theorem 3(i) and since the "only if" part of Theorem 2 follows from Theorem 3(ii) and the fact that $\mathfrak{B}$ is a θ-strong subalgebra of itself provided that θ is strong, we can confine our attention to proving the "if" part of Theorem 2 and Theorem 3.

PROOF OF THE „IF"-PART OF THEOREM 2. Let $\mathfrak{B}$ be a relative subalgebra of the algebra $\mathfrak{A}$ and let $\bar{\theta}$ be a congruence relation on $\mathfrak{A}$ extending the congruence relation θ on $\mathfrak{B}$. We assume that $[b]\theta = [b]\bar{\theta}$ for all $b \in B$ and have to show that θ is strong. To do so, assume

$$x_0 \equiv y_0(\theta), \ldots, x_{n_\gamma - 1} \equiv y_{n_\gamma - 1}(\theta)$$

and

$$(x_0, \ldots, x_{n_\gamma - 1}) \in D(f_\gamma, \mathfrak{B}) ,$$

that is

$$f_\gamma(x_0, \ldots, x_{n_\gamma - 1}) \in B .$$

Since $\mathfrak{A}$ is an algebra, $f_\gamma(y_0, \ldots, y_{n_\gamma - 1})$ exists in $\mathfrak{A}$, and we conclude that

$$f_\gamma(x_0, \ldots, x_{n_\gamma - 1}) \equiv f_\gamma(y_0, \ldots, y_{n_\gamma - 1})(\bar{\theta})$$

holds. But we know by assumption that

$$[f_\gamma(x_0, \ldots x_{n_\gamma - 1})]\theta = [f_\gamma(x_0, \ldots x_{n_\gamma - 1})]\bar{\theta}$$

and, hence,

$$f_\gamma(y_0, \ldots, y_{n_\gamma - 1}) \in B .$$

Since $\mathfrak{B}$ is a relative subalgebra of $\mathfrak{A}$, we conclude that

$$(y_0, \ldots, y_{n_\gamma - 1}) \in D(f_\gamma, \mathfrak{B})$$

and therefore θ is strong. This completes the proof.

PROOF OF THEOREM 3(i). Let $\mathfrak{B} = \langle B; F \rangle$ be the partial algebra in question and denote the congruence classes of θ by

$$C_0, C_1, \ldots, C_\gamma, \ldots, \qquad \gamma < \alpha .$$

We define the set

$$A = B \vee \{c_\gamma\}_{\gamma < \alpha} \vee \{m\},$$

where $c_\gamma \neq c_\delta$ if $\gamma \neq \delta$ $(\gamma, \delta < \alpha)$ and $c_\gamma \neq m$, $c_\gamma \notin B$ for all $\gamma < \alpha$ and $m \notin B$. Let

$$\{m\},\ C_0 \vee \{c_0\},\ C_1 \vee \{c_1\}, \ldots, C_\gamma \vee \{c_\gamma\}, \ldots, \qquad \gamma < \alpha ,$$

be the blocks of a partition $\bar{C}$ of A. Set

$$C_\gamma' = C_\gamma \vee \{c_\gamma\}, \qquad \gamma < \alpha .$$

Then we define f_γ $(f_\gamma \in F)$ for $(x_0, \ldots, x_{n_\gamma - 1}) \in A^{n_\gamma}$ as follows:

α) $f_\gamma(x_0, \ldots, x_{n_\gamma - 1})$ is as defined before if

$$(x_0, \ldots, x_{n_\gamma - 1}) \in C_{\nu_0} \times \ldots \times C_{\nu_{n_\gamma - 1}} \wedge D(f_\gamma, \mathfrak{B}) .$$

β) $f_\gamma(x_0,\ldots,x_{n_\gamma-1})=c_\delta$ if

$$(x_0,\ldots,x_{n_\gamma-1}) \in C'_{\nu_0}\times\ldots\times C'_{\nu_{n_\gamma-1}}\setminus D(f_\gamma,\mathfrak{B})$$

and if there is some

$$(x_0',\ldots,x'_{n_\gamma-1}) \in C_{\nu_0}\times\ldots\times C_{\nu_{n_\gamma-1}}\wedge D(f_\gamma,\mathfrak{B})$$

such that $f_\gamma(x_0',\ldots,x'_{n_\gamma-1})\in C_\delta$.

γ) $f_\gamma(x_0,\ldots,x_{n_\gamma-1})=m$ if $x_i=m$ for some $i=0,\ldots,n_\gamma-1$ or if

$$(x_0,\ldots,x_{n_\gamma-1}) \in C'_{\nu_0}\times\ldots\times C'_{\nu_{n_\gamma-1}}$$

and

$$C_{\nu_0}\times\ldots\times C_{\nu_{n_\gamma-1}}\wedge D(f_\gamma,\mathfrak{B}) = \emptyset\,.$$

This definition makes each $f_\gamma\in F$ an operation on A and, thus, $\mathfrak{A}=\langle A\,;F\rangle$ a universal algebra. Part α) of the above definition shows that $\mathfrak{B}$ is a relative subalgebra of $\mathfrak{A}$ since β) and γ) yield elements outside of $\mathfrak{B}$. The partition $\bar{C}$ induces an equivalence relation $\bar\theta$ on $\mathfrak{A}$, and $\bar\theta_B=\theta$ follows from the fact that

(a) $\{m\},C_0'\ldots,C_\gamma',\ldots,\ \gamma<\alpha$, are the equivalence classes of $\bar\theta$,

(b) $\{m\}\wedge B=\emptyset,\ C_0'\wedge B=C_0,\ldots,C_\gamma'\wedge B=C_\gamma,\ldots,\ \gamma<\alpha$,

(c) θ induces the congruence classes $C_0,\ldots,C_\gamma,\ldots,\ \gamma<\alpha$, on B.

Thus, all that is left to be shown in part (i) of Theorem 3 is the fact that $\bar\theta$ is a congruence relation:

Let $x_0\equiv y_0(\bar\theta),\ldots,x_{n_\gamma-1}\equiv y_{n_\gamma-1}(\bar\theta)$:

a) If $x_i=m$ for some i, then $y_i=m$ and, hence,

$$m = f_\gamma(x_0,\ldots,x_{n_\gamma-1}) = f_\gamma(y_0,\ldots,y_{n_\gamma-1})$$

by γ).

b) If $(x_0,\ldots,x_{n_\gamma-1})$ is in $C'_{\nu_0}\times\ldots\times C'_{\nu_{n_\gamma-1}}$, then

$$(y_0,\ldots,y_{n_\gamma-1}) \in C'_{\nu_0}\times\ldots\times C'_{\nu_{n_\gamma-1}}\,.$$

If $C_{\nu_0}\times\ldots\times C_{\nu_{n_\gamma-1}}\wedge D(f_\gamma,B)=\emptyset$, then

$$m = f_\gamma(x_0,\ldots,x_{n_\gamma-1}) = f_\gamma(y_0,\ldots,y_{n_\gamma-1})$$

by γ). If $C_{\nu_0}\times\ldots\times C_{\nu_{n_\gamma-1}}\wedge D(f_\gamma,B)\neq\emptyset$, then either b_1) both

$$f_\gamma(x_0,\ldots,x_{n_\gamma-1}) \quad\text{and}\quad f_\gamma(y_0,\ldots,y_{n_\gamma-1})$$

equal c_δ for some $\delta<\alpha$, or b_2) one of them, say $f_\gamma(x_0,\ldots,x_{n_\gamma-1})$, is in C_δ for some $\delta<\alpha$ and the other equals c_δ (by β)), or b_3) both $(x_0,\ldots,x_{n_\gamma-1})$ and $(y_0,\ldots,y_{n_\gamma-1})$ are in $D(f_\gamma,\mathfrak{B})$ which by α) and the fact that θ is a congruence relation on $\mathfrak{B}$ implies that

$$f_\gamma(x_0,\ldots,x_{n_\gamma-1}) \equiv f_\gamma(y_0,\ldots,y_{n_\gamma-1})(\theta)$$

and, hence, that

$$f_\gamma(x_0,\ldots,x_{n_\gamma-1}) \equiv f_\gamma(y_0,\ldots,y_{n_\gamma-1})(\bar{\theta}) .$$

Thus,

$$f_\gamma(x_0,\ldots,x_{n_\gamma-1}) \equiv f_\gamma(y_0,\ldots,y_{n_\gamma-1})(\bar{\theta})$$

holds in all cases, and the proof of part (i) of Theorem 3 is completed.

PROOF OF THEOREM 3(ii). Let $\mathfrak{B}'=\langle B'; F\rangle$ be a θ-strong subalgebra of $\mathfrak{B}$. Then $\theta_{B'}$ is a strong congruence relation on $\mathfrak{B}'$ and, hence, either

$$\tilde{C}_{\nu_0}\times\ldots\times\tilde{C}_{\nu_{n_\gamma-1}} \wedge D(f_\gamma,\mathfrak{B}) = \emptyset$$

or

$$\tilde{C}_{\nu_0}\times\ldots\times\tilde{C}_{\nu_{n_\gamma-1}} \wedge D(f_\gamma,\mathfrak{B}) = \tilde{C}_{\nu_0}\times\ldots\times\tilde{C}_{\nu_{n_\gamma-1}}$$

for

$$\tilde{C}_{\nu_i} = C_{\nu_i}\wedge B' \quad \text{and} \quad f_\gamma\in F .$$

This and the fact that $A'\wedge C_\gamma=\tilde{C}_\gamma$ for $A'=B'\vee\{m\}\vee\{c_\gamma\}_{\gamma<\alpha}$ shows that $\mathfrak{A}'=\langle A'; F\rangle$ is a subalgebra of the algebra $\mathfrak{A}$ since part β) or γ) in the definition of $f_\gamma(x_0,\ldots,x_{n_\gamma-1})$ is applied if

$$(x_0,\ldots,x_{n_\gamma-1})\in A'^{n_\gamma}\setminus D(f_\gamma,\mathfrak{B}) .$$

$\mathfrak{B}'$ is obviously a relative subalgebra of $\mathfrak{A}'$. The congruence classes of $\theta_{B'}$ are

$$\tilde{C}_0,\ldots,\tilde{C}_\gamma,\ldots, \qquad \gamma<\alpha .$$

On the other hand, a congruence ψ on $\mathfrak{A}'$ is defined by the classes

$$\{c_0\},\ldots,\{c_\gamma\},\ldots,\{m\},\tilde{C}_0,\ldots,\tilde{C}_\gamma,\ldots, \qquad \gamma<\alpha .$$

Thus $[b']\theta_{B'}=[b']\psi$ is true for all $b'\in B'$. This completes the proof of Theorem 3(ii).

REFERENCES

1. G. Grätzer, *Universal algebra,* Mimeographed lecture notes, Penn. State U., 1966.
2. G. Grätzer, *Congruence relations on partial algebras,* Notices Amer. Math. Soc. 87 (1966), 146 (# 66T-66).
3. G. Grätzer and E. T. Schmidt, *Characterization of congruence lattices of abstract algebras,* Acta Sci. Math. (Szeged) 24 (1963), 34–59.
4. B. H. Neumann, *Universal algebra,* Lecture notes, Courant Inst. of Math. Sci., New York University, 1962.

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