MATH. SCAND. 21 (1967), 281–289

ON THE THEOREM OF SZEGÖ-SOLOMENTSEV

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1.

An important theorem of G. Szegö [4] concerning Hardy classes states that a function f belonging to the Hardy class H_p on the open unit disk, $0 < p \leqq +\infty$, admits a representation of the form

$$f = gb , \tag{1.1}$$

where

$$g(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log|f^*(e^{i\theta})|\,\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta\right\}, \qquad |z|<1 ,$$

f^* being the Fatou radial limit function of f, and b is a bounded analytic function on the open unit disk having Fatou radial limits of modulus 1 p.p.[1] The equality (1.1) taken with the fact that g is also a member of H_p and has a Fatou radial limit function having the same modulus as f^* p.p. yields the maximal principal of Szegö which may be formulated as follows: the function g, which is determined by $\log|f^*|$, is a member of H_p whose modulus is maximal among the moduli of those members of H_p whose Fatou radial limit functions have the same modulus as f^* p.p. The treatment of Szegö makes use of the theory of Toeplitz forms.

A theorem given subsequently by Solomentsev [3], which pertains to a class of subharmonic functions in the open unit ball in n-dimensional euclidean space, implies the theorem of Szegö as an immediate corollary.

The theorem of Solomentsev may be stated as follows: Suppose that u is a subharmonic function ($\not\equiv -\infty$) in the open unit ball in n-dimensional euclidean space and that φ satisfies the condition stated below in Section 4. If $\varphi \circ u$ has a harmonic majorant, then (1) so does u, (2) the least harmonic majorant of u admits a representation as a Poisson

Received December 12, 1966.

Work on this paper was carried out during the tenure of National Science Grant NSF GP 3432.

[1] Szegö proved this result for $p=2$ in the cited paper [4]. The general case was treated by F. Riesz (cf. *Collected Works*, Vol. 1, pp. 616–624, 645–653) with the aid of the theorem of F. and M. Riesz.

integral with respect to a measure whose singular part is non-positive, and (3) the least harmonic majorant of $\varphi \circ u$ is given by the Poisson–Lebesgue integral with boundary function $\varphi \circ u^*$, u^* being the Fatou limit function of u (only finite limits being taken into account).

The result of Solomentsev was rediscovered by Gårding and Hörmander [1]. In their short elegant paper they noted that the theorem of F. and M. Riesz was readily obtained as a consequence of the Solomentsev theorem. It should also be noted that Privalov treated a special case of the Solomentsev theorem $[\varphi(x)=(x^+)^\alpha, 1<\alpha<+\infty]$, and that Privalov and Kuznetsov extended the work of Solomentsev to the case of regions in euclidean space having a reasonably regular character. I am indebted to Professor Lars Gårding for these historical indications.

The object of the present note is to formulate and establish a theorem of Solomentsev type for Riemann surfaces. However we proceed internally without invoking ambient boundary aspects of the problem before we reach the stage of showing that the Solomentsev theorem follows in a straightforward way from the one we shall give. It will be seen that the possibility of an exclusively internal argument is made available by use of the notions of quasi-bounded and singular non-negative harmonic functions given by Parreau in his thesis [2]. The following point is to be emphasized. We may certainly obtain our theorem in the nontrivial cases with the aid of the original Solomentsev theorem as stated by Gårding and Hörmander together with uniformisation methods. On comparison it becomes clear that the present arguments are simpler and appeal to a more primitive aspect of proof. It is also to be noted that our arguments apply, mutatis mutandis, for the case of regions in arbitrary euclidean spaces as well as for the case of Green spaces. While it is not excluded, of course, that the general theorem may be established by boundary considerations reminiscent of those given for the case of the unit ball, it is not to be expected that such access to the question could equal in simplicity an internal argument.

The theorem of Section 5 (infra), which is a special case of one due to Parreau [2, Th. XIV], plays an essential role.

In what follows S will denote a Riemann surface.

2. Harmonic majorants.

We recall that a family Φ of subharmonic functions on S is called a *Perron family* provided that it is closed with respect to the taking of the maximum of two members and to the taking of a Poisson modification of a member. Central in the study of Perron families is the trichotomy theorem which states that the upper envelope of a Perron family on S is one of the following: the constant $-\infty$, the constant $+\infty$, a function harmonic on S.

Suppose now that u is a subharmonic function on S, not the constant $-\infty$, and that h is a harmonic function on S dominating u, that is, satisfying $h \geqq u$. Let Φ denote the smallest Perron family containing u and let Ψ denote the set of subharmonic functions on S dominated by h. Then $\Phi \subset \Psi$. The upper envelope of Φ is harmonic and is dominated by h. We conclude that if u is a subharmonic function on S, not the constant $-\infty$, and u has a harmonic majorant, then u has a least harmonic majorant, in fact the upper envelope of the family Φ just introduced. We denote the least harmonic majorant of u by LHMu. The corresponding notions prevail dually for superharmonic functions and we are led to the existence of the greatest harmonic minorant (GHM) of a superharmonic function on S, not the constant $+\infty$, possessing a harmonic minorant.

3. Quasi-bounded and singular non-negative harmonic functions.

The notions in question are due to Parreau [2]. A non-negative harmonic function p on S is termed *quasi-bounded* provided that there exists a never decreasing sequence (b_n) of non-negative bounded harmonic functions on S such that

$$p = \lim_{n\to\infty} b_n ;$$

p is termed *singular* provided that the only non-negative bounded harmonic function on S dominated by p is the constant 0. A fundamental decomposition theorem states that a non-negative harmonic function p on S admits a unique representation of the form

$$p = q+s , \tag{3.1}$$

where q is a quasi-bounded and s is a singular non-negative harmonic function on S. For the sake of completeness we indicate the argument. Indeed, it is easy to show that with $b_n = \mathrm{GHM}\min\{p,n\}$ we have a decomposition of the desired type when we take

$$q = \lim_{n\to\infty} b_n$$

and $s=p-q$. Indeed q is by its very definition quasi-bounded. Let b denote *henceforth* a non-negative bounded harmonic function on S. If b is dominated by s and the constant taking the value m, a positive integer, then

$$b+b_n \leqq \min\{p, m+n\},$$

so that $b+b_n \leqq b_{m+n}$. On taking the limit, we are led to the conclusion that $b=0$. Hence s is singular. Uniqueness: Suppose that $p=q_1+s_1$ where q_1 is quasi-bounded and s_1 is singular. If b is dominated by q_1, then $b \leqq b_n$, n large. We conclude $q_1 \leqq q$. If b is dominated by q, we have $(b-q_1)^+ \leqq s_1$. Hence $\text{LHM}(b-q_1)^+=0$ and $b \leqq q_1$. We conclude $q \leqq q_1$ and thereupon $q=q_1$. The representation (3.1) is unique.

It is immediate that if q_1 and q_2 are quasi-bounded, then so is q_1+q_2. If s_1 and s_2 are singular and $b \leqq s_1+s_2$, then $(b-s_1)^+ \leqq s_2$. Hence $\text{LHM}(b-s_1)^+=0$, so that $b \leqq s_1$ and consequently $b=0$. We conclude that s_1+s_2 is singular. We are led to the following important lemma.

LEMMA. *If p and p_1 are non-negative harmonic functions on S and $p_1 \leqq p$, then the quasi-bounded (resp. singular) component of p dominates that of p_1.*

The proof follows on considering the decompositions of p_1 and of $p-p_1$.

In order to see the connection between the results which we shall prove and the Solomentsev theorem it will be convenient to have available the well-known characterizations for quasi-bounded and singular non-negative harmonic functions on the open unit disk. The quasi-bounded (resp. singular) are precisely those given by Poisson integrals taken with respect to non-negative absolutely continuous (singular) measures.

4.

We consider a function φ with domain $\{-\infty \leqq x < +\infty\}$ which is assumed to take non-negative real values and to satisfy the following conditions:

(a) φ is continuous,
(b) the restriction of φ to the real line is convex,
(c) $\lim_{x\to+\infty} \varphi(x)/x = +\infty$.

Let m denote the maximum of the x satisfying $\varphi(x)=\varphi(-\infty)$. The restriction of φ to $\{m \leqq x < +\infty\}$ is increasing. We denote its inverse by ψ. When $m > -\infty$, ψ is concave on its domain. When $m = -\infty$, the restriction of ψ to $\{\varphi(-\infty) < x < +\infty\}$ is concave. The following equalities hold:

$$\varphi[\psi(x)] = x, \qquad \varphi(-\infty) \leqq x < +\infty;$$
$$\psi[\varphi(x)] = \max\{x, m\}, \qquad -\infty \leqq x < +\infty.$$

5.

We consider first a non-negative harmonic function p on S and show

THEOREM. *If $\varphi \circ p$ has a harmonic majorant, then p and* LHM$\varphi \circ p$ *are quasi-bounded* [Parreau].

PROOF. Let s denote the singular component of p. Let A denote a positive number. There exists a positive number B such that

$$Ax \leqq \varphi(x) + B \tag{5.1}$$

for all real x. Hence

$$As \leqq Ap \leqq \text{LHM}\, \varphi \circ p + B\,.$$

Applying the lemma of Section 3 we obtain

$$As \leqq \text{LHM}\, \varphi \circ p\,,$$

and hence conclude, given the arbitrariness of A, that s is the constant zero. Consequently, p is quasi-bounded.

We let (b_n) denote a monotone never decreasing sequence of bounded non-negative harmonic functions on S having limit p. We have

$$\varphi \circ b_n \leqq \varphi \circ b_{n+1} \leqq \text{LHM}\, \varphi \circ p\,,$$

$n=0,1,\ldots$; whence

$$\text{LHM}\, \varphi \circ b_n \leqq \text{LHM}\, \varphi \circ b_{n+1} \leqq \text{LHM}\, \varphi \circ p\,,$$

$n=0,1,\ldots$. We observe that

$$\lim_{n\to\infty} \text{LHM}\, \varphi \circ b_n\,, \tag{5.2}$$

which is dominated by LHM $\varphi \circ p$, is harmonic. Since (5.2) dominates $\varphi \circ b_m$, $m=0,1,\ldots$, it dominates $\varphi \circ p$. We conclude that

$$\text{LHM}\, \varphi \circ p = \lim_{n\to\infty} \text{LHM}\, \varphi \circ b_n\,.$$

Hence LHM$\varphi \circ p$ is quasi-bounded. This conclusion can be inferred, of course, once it is known that LHM$\varphi \circ p$ is dominated by the quasi-bounded (5.2).

6.

We suppose now that u is subharmonic on S but is not the constant $-\infty$ and that $\varphi \circ u$ has a harmonic majorant. Using

$$\varphi(x^+) \leqq \varphi(x) + \varphi(0)$$

we see that $\varphi \circ u^+$ has a harmonic majorant and from (5.1) that u^+ has a harmonic majorant. We denote LHM u by v and LHM u^+ by w.

In the theorem that follows the concavity property of ψ plays an essential role.

THEOREM. (a) *$\varphi \circ w$ has a harmonic majorant.* (b) *The equalities*

$$\text{LHM}\,\varphi \circ u = \text{LHM}\,\varphi \circ v \tag{6.1}$$

and

$$\text{LHM}\,\varphi \circ u^+ = \text{LHM}\,\varphi \circ w \tag{6.2}$$

hold.

PROOF. Starting from

$$\text{LHM}\,\varphi \circ u^+ \geqq \varphi \circ u^+$$

we obtain

$$\psi \circ (\text{LHM}\,\varphi \circ u^+) \geqq u^+ . \tag{6.3}$$

Since the left hand side of (6.3) is superharmonic, we have

$$\psi \circ (\text{LHM}\,\varphi \circ u^+) \geqq w ,$$

and hence

$$\text{LHM}\,\varphi \circ u^+ \geqq \varphi \circ w .$$

Consequently $\varphi \circ w$ has a harmonic majorant and

$$\text{LHM}\,\varphi \circ u^+ \geqq \text{LHM}\,\varphi \circ w .$$

The equality (6.2) follows. It is to be noted that the fact that u^+ has a harmonic majorant may be inferred directly from (6.3). The proof of (6.1) follows that given for (6.2) with u replacing u^+.

Using the theorem of Section 5 we see that w and $\text{LHM}\,\varphi \circ w$ are quasi-bounded. Hence $\text{LHM}\,\varphi \circ u^+$ and $\text{LHM}\,\varphi \circ u$ are quasi-bounded.

7.

If a harmonic function h on S is dominated by a non-negative harmonic function on S or — what is equivalent — it is representable as the difference $p_1 - p_2$ of non-negative harmonic functions p_k on S, then h admits a representation as such a difference for which the component terms are least. Indeed, the representation in question is obtained by taking $p_1 = \text{LHM}\,h^+$ and $p_2 = p_1 - h$.

We represent v of Section 6 in this way and we note that $\text{LHM}\,v^+ \leqq w$ so that $\text{LHM}\,v^+$ is quasi-bounded. We conclude that

$$v = Q - s , \tag{7.1}$$

where Q is the difference of quasi-bounded harmonic functions and s is singular. A representation of the form (7.1) is unique.

Keeping in mind the interpretation of the notions "quasi-bounded" and "singular" in terms of Poisson integrals for the case of the open unit disk (ball), we see that (7.1) implies that the singular component of the measure yielding v as a Poisson integral is non-positive. Further the results of Sections 5, 6 imply that the measure $d\sigma$ yielding LHM $\varphi \circ u$ as a Poisson integral is absolutely continuous. These two statements comprise part of the theorem of Solomentsev as stated by Gårding and Hörmander [1]. In addition, there is also established there that $d\sigma$ is simply $\varphi[Q^*(e^{i\theta})]d\theta$ (mutatis mutandis for the higher dimensional case). Here Q^* denotes the Fatou boundary function of Q; only finite radial limits are taken into account.

We shall now prove in the setting of the theorem of Section 6 that

$$\text{LHM}\,\varphi \circ u = \text{LHM}\,\varphi \circ Q\,. \tag{7.2}$$

This result together with the Riesz representation theorem for subharmonic functions applied to $\varphi \circ Q$ yields the above equality for $d\sigma$. One can also prove this formula for $d\sigma$ by establishing the mean convergence (order 1) of $\varphi[Q(re^{i\theta})]$ as r tends to 1, using the device of Gårding and Hörmander. Cf. (2) of [1].

We turn to the proof of (7.2). To that end let ε be a positive number. Let c be a positive number satisfying

$$c + \psi[\varphi(-\infty) + \varepsilon] > 0\,.$$

We proceed from the observation that $x \leqq \psi[\varphi(x)]$ and obtain

$$Q - s \leqq \psi \circ [\text{LHM}\,\varphi \circ v + \varepsilon]\,.$$

We conclude

$$c + q_1 \leqq s + q_2 + [c + \text{GHM}\,\psi \circ (\text{LHM}\,\varphi \circ v + \varepsilon)]\,,$$

where q_1 and q_2 are quasi-bounded non-negative harmonic functions on S satisfying $Q = q_1 - q_2$. Applying the lemma of Section 3 we conclude the inequality

$$Q \leqq \psi \circ (\text{LHM}\,\varphi \circ v + \varepsilon)\,,$$

whence

$$\varphi \circ Q \leqq \text{LHM}\,\varphi \circ v + \varepsilon\,.$$

On taking the least harmonic majorant of $\varphi \circ Q$ we are led at once to (7.2), given the arbitrariness of ε.

8. Some remarks.

It was observed by Gårding and Hörmander (op. cit.) that the theorem of F. and M. Riesz concerning the Hardy class H_1 is readily derived with the aid of the theorem of Solomentsev. Their argument uses implicitly the Lebesgue decomposition of a measure. The fact that the boundary measure of a quasi-bounded harmonic function on the open unit disk is absolutely continuous may be proved without reference to the Lebesgue decomposition (thanks to the Fatou theorem for bounded harmonic functions). We may establish the theorem of the Riesz brothers starting with an internal argument as follows. If f belongs to the class H_1 on the open unit disk, then $q = \mathrm{LHM}\,|f|$ is quasi-bounded as follows on considering $u = \log|f|$, $\varphi(x) = \exp x$ and applying the last sentence of Section 6 in the non-trivial case where f is not the constant zero. Since $0 \leqq \mathrm{Re} f + q \leqq 2q$, we see that $\mathrm{Re} f$ is the difference of quasi-bounded harmonic functions and hence admits a Poisson–Lebesgue integral representation. The same remark applies to $\mathrm{Im} f$. We conclude that f admits a Poisson–Lebesgue integral representation. The remaining statements of the theorem of the Riesz brothers follow from this fact.

Szegö's theorem. We return to the Szegö theorem cited at the outset of this paper and show that it is a theorem of Solomentsev type in a function-theoretic setting and thereupon that there is a general maximal theorem of Szegö type in the subharmonic theory. Thus to obtain the representation (1.1) we note that we may put aside the trivial case where f is the constant zero and that we may assume that $p \neq +\infty$. We take

$$u = \log|f| \quad \text{and} \quad \varphi(x) = \exp(px)\,.$$

Applying (7.1) and the fact that $v - \log|f|$ is a sum of Green's functions ($\not\equiv +\infty$), we are led to the representation (1.1) when we note that a sum of Green's functions ($\not\equiv +\infty$) on the open unit disk has Fatou radial limit equal to 0 p.p.

In the general setting when S is hyperbolic for non-trivial u the Riesz representation theorem and (7.1) yield

$$u = Q - (s + G)\,, \tag{8.1}$$

where Q and s are as in (7.1) and G is a Green's potential on S generated by a non-negative measure (the total mass of which may be $+\infty$). The equality (8.1) replaces (1.1). In it the terms Q, s, G are uniquely determined. When S is parabolic, u is necessarily constant.

On specializing S to the case of the open unit disk we see that the Fatou radial limits of u and Q agree p.p. on the unit circumference. Since $\varphi \circ Q$ has a harmonic majorant, we conclude that all the subharmonic functions on the open unit disk, not the constant $-\infty$, satisfying the Solomentsev condition for a given φ and having equivalent

Fatou radial limit functions, have a common Q and that this Q is the maximal such function.

It is now possible given φ to characterize a function U on the unit circumference which agrees p.p. with the Fatou radial limit function of a subharmonic function u on the open unit disk, not the constant $-\infty$, where u satisfies the Solomentsev condition relative to the given φ. We obtain a generalization of the corresponding theorem for Hardy classes — a result closely connected with the Szegö theorem. It is clear that a necessary condition for such a U is that both U and $\varphi \circ U$ be integrable, (φ being extended in definition by $\varphi(+\infty) = +\infty$). This condition is also sufficient. Indeed, if we take u as given by the Poisson–Lebesgue integral with boundary function U, by the Jensen inequality $\varphi \circ u$ is dominated by the harmonic function h given by the Poisson–Lebesgue integral with boundary function $\varphi \circ U$. Since u is a harmonic function on the open unit disk satisfying the Solomentsev condition for the given φ and having Fatou radial limit function agreeing p.p. with U, it follows that the stated condition is sufficient.

For given U satisfying the stated condition, $h = \mathrm{LHM}\, \varphi \circ u$ for all subharmonic u on the open unit disk satisfying the Solomentsev condition for the given φ and having Fatou radial limit function agreeing p.p. with U. This fact follows from the part of the Solomentsev theorem treated in Section 7. It also follows on noting that

$$\varphi \circ u \leqq \mathrm{LHM}\, \varphi \circ u \leqq h ,$$

so that $\mathrm{LHM}\, \varphi \circ u$ is quasi-bounded and has the same Fatou radial limits p.p. as h.

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