MATH. SCAND. 20 (1967), 290–296

CHARACTERIZATIONS OF A CLASS OF CONVEX SETS

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Introduction.

Suppose the real vector spaces E and F form a dual system with respect to a bilinear form $\langle\cdot,\cdot\rangle$. For each subset X of E and each point y of F the *y-support* of X is defined as the set

$$S(X,y) = \{w\in E : \langle w,y\rangle \leqq \sup_{x\in X}\langle x,y\rangle\} .$$

Note that $S(\emptyset,y)=\emptyset$, while $S(X,y)=E$ when $y=0$ and $X\neq\emptyset$ as well as when $\sup_{x\in X}\langle x,y\rangle=\infty$. If these cases are excluded and $\langle\cdot,\cdot\rangle$ is an inner product then $S(X,y)$ is the smallest closed halfspace which contains X and has y as an outer normal.

For $Y\subset F$, a subset X of E will be called *Y-convex* provided that X is the intersection of its Y-supports; that is, $X=\bigcap_{y\in Y} S(X,y)$. When Y is symmetric ($Y=-Y$) this amounts to saying that each point of $E\sim X$ is strongly separated from X by a hyperplane determined by some member of Y. As is well known, X is F-convex if and only if X is convex and is closed for the weak topology $w(E,F)$.

In connection with a problem from control theory, we became interested in characterizing those proper subsets X of E such that X is Y-convex for every dense subset Y of F (relative to a given admissible topology for F). When E is finite-dimensional they are exactly the closed convex sets which contain no line. When F is a locally convex barrelled space, a proper subset X of F^* is Y-convex for all dense $Y\subset F$ if and only if X is convex, contains no line, and is closed and locally compact for the weak* topology $w(F^*,F)$. These characterizations are corollaries of the more general results obtained below.

Statements of theorems.

A class $\mathscr{A}$ of subsets of E will be called *admissible* provided that it satisfies the following conditions:

(A1) Every member A of $\mathscr{A}$ is $w(E,F)$-bounded; that is, $\sup_{a\in A}\langle a,b\rangle < \infty$ for all $b\in F$.

Received January 2, 1967.
Research supported in part by the National Science Foundation, U.S.A. (NSF-GP-3579).

(A2) $\mathscr{A}$ includes the convex hulls, the $w(E,F)$-closures, and all subsets of its members.
(A3) The union of any two members of $\mathscr{A}$ is a member of $\mathscr{A}$.
(A4) If $A \in \mathscr{A}$, $p \in E$, and λ is a nonzero real number, then $p+\lambda A \in \mathscr{A}$.
(A5) E is covered by $\mathscr{A}$.

With $\mathscr{A}$ as described, $\mathscr{T}_{\mathscr{A}}$ will denote the topology (for F) of uniform convergence on members of $\mathscr{A}$. Thus $(F,\mathscr{T}_{\mathscr{A}})$ is a locally convex space in which a basis for the neighborhoods of the origin 0 is formed by the class of all polars

$$A^\circ = \{b \in F : \sup_{a \in A} \langle a,b \rangle \leqq 1\}$$

of members A of $\mathscr{A}$.

In the proofs below, conditions (A1)–(A5) are used freely without explicit reference. The space E is always equipped with the weak topology $w(E,F)$ and F with the admissible topology $\mathscr{T}_{\mathscr{A}}$.

THEOREM 1. *Suppose that the real vector spaces E and F form a dual system, $w(E,F)$ is the associated weak topology for E, $\mathscr{A}$ is an admissible class of subsets of E, and $\mathscr{T}_{\mathscr{A}}$ is the topology (for F) of uniform convergence on members of $\mathscr{A}$. Then the following five conditions are equivalent for any proper subset X of E:*

(*D*) *X is Y-convex for every $\mathscr{T}_{\mathscr{A}}$-dense subset Y of F.*
(*D'*) *X is F-convex; there is a point $p \in E \sim X$ such that for every $\mathscr{T}_{\mathscr{A}}$-dense symmetric subset Y of F, p is strongly separated from X by a hyperplane determined by some member of Y.*
(*P*) *X is F-convex and the polar X° has nonempty $\mathscr{T}_{\mathscr{A}}$-interior.*
(*L*) *X is F-convex; there is a $w(E,F)$-closed halfspace H such that X's intersection with any translate of H is a member of $\mathscr{A}$.*
(*L'*) *For each point $p \in E \sim X$ there is a $w(E,F)$-closed halfspace H such that p is interior to H, H is disjoint from X, and X's intersection with any translate of H is a member of $\mathscr{A}$.*

THEOREM 2. *If $\mathscr{A}$ is the class of all $w(E,F)$-bounded subsets of E then the five conditions of Theorem 1 are all equivalent to the following:*

(*D''*) *X is Y-convex for every ubiquitous symmetric subset Y of F.*

(A subset Y of F is called *ubiquitous* provided that Y is dense in the very strong sense that each point of F is a point of Y or an endpoint of an open line segment contained in Y.)

THEOREM 3. *If all members of $\mathscr{A}$ are $w(E,F)$-relatively compact then the five conditions of Theorem 1 are all equivalent to the following:*

(L'') *X is F-convex and contains no line; each point of X admits a $w(E,F)$-neighborhood (relative to X) which is a member of $\mathscr{A}$.*

Proof of Theorem 1.

$(D) \Rightarrow (D')$. This is obvious.

$(D') \Rightarrow (P)$. Let X and p be as described in (D'), and suppose that the interior of the polar X° is empty. Then for each $\eta > 0$ the set

$$\begin{aligned} Y_\eta &= F \sim \eta(X^\circ \cup -X^\circ) \\ &= \{b \in F : \sup_{x\in X}\langle x, -b\rangle > \eta < \sup_{x \in X}\langle x, b\rangle\} \end{aligned}$$

is a dense symmetric subset of F. Using this fact, we shall produce a dense symmetric subset Y of F such that $p \in \bigcap_{y\in Y} S(X,y)$, thus contradicting (D') and showing that (D') implies (P). For each point q of F let

$$G(q) = \{b \in F : \langle p,b\rangle < \langle p,q\rangle + 1 \text{ and } \langle p,-b\rangle < \langle p,-q\rangle + 1\},$$

an open neighborhood of q in F, and let

$$\eta(q) = \max(\langle p,q\rangle + 1,\ \langle p,-q\rangle + 1) > 0 .$$

Then $G(q) \cap Y_{\eta(q)}$ is a dense subset of $G(q)$ and from the relevant definitions it follows that $p \in S(X,y)$ whenever y or $-y$ is a member of $G(q) \cap Y_{\eta(q)}$. Thus the desired end is achieved by defining

$$Y = \bigcup_{q\in F}(G(q) \cap Y_{\eta(q)}) \cup -(G(q) \cap Y_{\eta(q)}) .$$

$(P) \Rightarrow (L)$. Suppose (P) holds and let q be an interior point of X°. Then for each $\lambda > 0$ the origin is interior to the convex hull $\operatorname{con}(X^\circ \cup \{-\lambda q\})$, which therefore contains a set of the form $A_\lambda{}^\circ$ for some $A_\lambda \in \mathscr{A}$. We may assume without loss of generality that A_λ is convex, closed, and includes the origin, whence $A_\lambda{}^{\circ\circ} = A_\lambda$. It then follows that

$$\begin{aligned} A_\lambda = A_\lambda{}^{\circ\circ} &\supset (\operatorname{con}(X^\circ \cup \{-\lambda q\}))^\circ \\ &= X^{\circ\circ} \cap \{-\lambda q\}^\circ \supset X \cap \{w \in E : \langle w, -q\rangle \leqq 1/\lambda\}, \end{aligned}$$

whence the final intersection is a member of $\mathscr{A}$ and the desired conclusion follows.

$(L) \Rightarrow (L')$. Suppose (L) holds and consider an arbitrary point $p \in E \sim X$. Since X is F-convex there exists $y \in F$ such that $\sup_{x\in X}\langle x,y\rangle < \langle p,y\rangle$. And by the second part of (L) there exists $z \in F$ such that for each real λ the set $\{x \in X : \langle x,z\rangle \geqq \lambda\}$ is a member of $\mathscr{A}$. This implies $\sup_{x\in X}\langle x,z\rangle < \infty$ and hence for a sufficiently small $\mu > 0$ it is true that

$$\sup_{x\in X}\langle x, y+\mu z\rangle < \langle p, y+\mu z\rangle .$$

Let β be a number strictly between those on the two sides of this inequality and let

$$H = \{w \in E : \langle w, y+\mu z\rangle \geqq \beta\}.$$

Then H plainly satisfies the first two parts of condition (L). For the last part, note that if $x \in X$ and $\langle x, y+\mu z\rangle \geqq \gamma$ then

$$\langle x,z\rangle \geqq \frac{\gamma - \langle x,y\rangle}{\mu} > \frac{\gamma - \langle p,y\rangle}{\mu}.$$

Thus each set of the form

$$\{x \in X : \langle x, y+\mu z\rangle \geqq \gamma\}$$

is contained in a set of the form

$$\{x \in X : \langle x,z\rangle \geqq \lambda\},$$

and since the latter is a member of $\mathscr{A}$, so is the former. This shows that X's intersection with any translate of H is a member of $\mathscr{A}$.

$(L') \Rightarrow (D)$. Consider an arbitrary dense subset Y of F and point p of $E \sim X$. Let H be as described in (L'), whence there exist $q \in F \sim \{0\}$ and real numbers σ and δ such that

(1) $$H = \{w \in E : \langle w,q\rangle \geqq \sigma+\delta\}$$

and $\sup_{x \in X}\langle x,q\rangle = \sigma < \sigma+\delta < \langle p,q\rangle$.

Choose

(2) $$x_0 \in X \quad \text{with} \quad \langle x_0,q\rangle > \sigma - 1$$

and define

(3) $$X_0 = \{x \in X : \langle x,q\rangle = \sigma-1\}, \qquad H' = \{w \in E : \langle w,q\rangle \geqq \sigma - 1\},$$

so that both X_0 and $X \cap H'$ are members of $\mathscr{A}$. For notational convenience assume $x_0 = 0$, as can be done without loss of generality. Then

$$\begin{aligned}\sup_{x \in X_0}\langle x,q\rangle < \sigma - 1 < \langle x_0,q\rangle = 0 \leqq \sup_{x \in X \cap H'}\langle x,q\rangle \\ = \sigma < \sigma+\eta < \langle p,q\rangle.\end{aligned}$$

Let G denote the set of all $g \in F$ such that

(4) $$\sup_{x \in X_0}\langle x,g\rangle < \tfrac{1}{2}(\sigma-1) < 0 < \sup_{x \in X \cap H'}\langle x,g\rangle < \sigma+\eta < \langle p,g\rangle.$$

Since $q \in G$, and since the sets X_0, $X \cap H'$, and $\{p\}$ are all members of $\mathscr{A}$, G is a nonempty open subset of F and hence intersects Y. For $g \in Y \cap G$ we have $p \notin S(X,g)$, as follows from (4) in conjunction with the fact that

(5) $$X \subset (X \cap H') \cup [0,\infty[X_0.$$

Since p was an arbitrary point of $E \sim X$ it follows that $X = \bigcap_{y \in Y} S(X,y)$. We have now proved that (L') implies (D) and have thus completed the proof of Theorem 1.

Proof of Theorem 2.

Plainly (D) implies (D''). Now suppose that $\mathscr{A}$ is the class of all bounded subsets of E and that (L') fails for some $p \in E \sim X$. We shall produce a ubiquitous symmetric subset Y of F such that $p \in \bigcap_{y \in Y} S(X,y)$, whence (D'') is contradicted and it will follow that (D'') implies (L'). For an arbitrary point q of F, consider the following three possibilities:

(i) $p \in S(X,q) \cap S(X,-q)$;
(ii) $p \notin S(X,q)$;
(iii) $p \notin S(X,-q)$.

When (i) holds let $Y(q) = \{-q,q\}$.

When (ii) holds there exist σ, δ, H, x_0, X_0 and H' such that conditions (1)–(3) above are satisfied. If the set $X \cap H'$ is bounded, then so is X_0, and with the aid of (5) above it can be seen that X's intersection with any translate of H is bounded. As this contradicts the assumption about P, we conclude that $X \cap H'$ is unbounded and hence there exists $z \in F$ such that $\sup_{x \in X \cap H'} \langle x,z \rangle = \infty$. Let $\varrho = \inf_{x \in X \cap H'} \langle x,z \rangle$ and

$$\varepsilon = \begin{cases} \dfrac{\langle p,q \rangle - \sigma}{\varrho - \langle p,z \rangle} > 0 & \text{when } \varrho > \langle p,z \rangle, \\ \infty & \text{when } \varrho \leqq \langle p,z \rangle. \end{cases} \tag{6}$$

Let

$$Y(q) = \{-q - \mu z : \ 0 < \mu < \varepsilon\} \cup \{q + \mu z : \ 0 < \mu < \varepsilon\},$$

a symmetric union of open segments or open rays havings $-q$ and q among their endpoints. To see that $p \in S(X, q + \mu z)$, note that the function $\langle \cdot, q \rangle$ is bounded below on the set $X \cap H'$, while $\langle \cdot, z \rangle$ is unbounded above there, and consequently

$$\sup_{x \in X} \langle x, q + \mu z \rangle = \infty .$$

To see that $p \in S(X, -q - \mu z)$ for $0 < \mu < \varepsilon$, note that

$$\langle p, -q - \mu z \rangle = -\langle p,q \rangle - \mu \langle p,z \rangle < -\sigma - \mu\varrho$$

by (6), while

$$\begin{aligned} -\sigma - \mu\varrho &= -\sup_{x \in X \cap H'} \langle x,q \rangle - \mu \inf_{x \in X \cap H'} \langle x,z \rangle \\ &= \inf_{x \in X \cap H'} \langle x, -q \rangle + \mu \sup_{x \in X \cap H'} \langle x, -z \rangle \\ &\leqq \sup_{x \in X \cap H'} (\langle x, -q \rangle + \mu \langle x, -z \rangle) = \sup_{x \in X} \langle x, -q - \mu z \rangle . \end{aligned}$$

The procedure for (iii) is essentially the same as that for (ii), and finally, having defined $Y(q)$ for every $q \in F$, we set $Y = \bigcup_{q \in F} Y(q)$. The set Y will then have the desired properties and the proof of Theorem 2 is complete.

Proof of Theorem 3.

Before proving Theorem 3 we shall describe an example to show that condition (L'') is not always equivalent to those of Theorem 1. Let E' be an infinite-dimensional normed linear space, L a line through the origin in E', and X' the set of all points of E' at distance $\leqq 1$ from L. Let P' be a closed linear subspace supplementary to L in E', so that the cylinder X' has bounded intersection with any strip consisting of all points between two translates of the hyperplane P'. Let F' be the conjugate space of E', and let $\mathscr{A}'$ denote the set of all $w(E',F')$-bounded (equivalently, norm-bounded) subsets of E'. Then X' is F'-convex and each point of X' admits a $w(E',F')$-neighborhood (relative to X') which is a member of $\mathscr{A}'$. Now let E be a norm-dense linear subspace of E' such that $E \cap L = \{0\}$, and let $X = X' \cap E$. Let F be the conjugate space of E and $\mathscr{A}$ the set of all $w(E,F)$-bounded subsets of E. Then X contains no line and hence satisfies condition (L''). However, X does not satisfy condition (L). For, consider an arbitrary closed halfspace H in E whose interior includes the origin. The closure of H in E' is a closed halfspace in E', and the closure of $X \cap H$ is $X' \cap H'$. But $X' \cap H'$ is unbounded, for it contains at least a ray from the line L, and hence the set $X \cap H$ is also unbounded.

Theorem 3 is based on the following result, which does not require any additional assumption about the class $\mathscr{A}$.

Proposition. *If X is a convex subset of E and some point x_0 of X admits a $w(E,F)$-neighborhood (relative to X) which is a member of $\mathscr{A}$, then every $w(E,F)$-bounded subset of X is a member of $\mathscr{A}$.*

Proof. Without loss of generality we may assume $x_0 = 0$, whence by hypothesis there are points $y_1, \ldots, y_n$ of F and positive numbers $\varepsilon_1, \ldots, \varepsilon_n$ such that the set

$$N = \{x \in X : \langle x, y_i \rangle \leqq \varepsilon_i \text{ for } i = 1, \ldots, n\}$$

is a member of $\mathscr{A}$. Now consider an arbitrary $w(E,F)$-bounded subset W of X, and for $1 \leqq i \leqq n$ let

$$\sigma_i = \sup_{w \in W} \langle w, y_i \rangle < \infty .$$

Let

$$\sigma = \max(\sigma_1/\varepsilon_1, \ldots, \sigma_n/\varepsilon_n).$$

Then N is a member of $\mathscr{A}$, and since X is convex it can be verified that $W \subset \sigma N$. This implies $W \in \mathscr{A}$.

To prove Theorem 3, note first that (L) implies (L'') without any additional assumption about $\mathscr{A}$. For the reverse implication, assume that (L'') holds and the members of $\mathscr{A}$ are all $w(E,F)$-relatively compact. Then X is $w(E,F)$-locally compact. Since X is F-convex and contains no line, a theorem of Klee (3.2 of [2]) guarantees the existence of a $w(E,F)$-closed halfspace H such that X's intersection with any translate of H is $w(E,F)$-compact and hence of course $w(E,F)$-bounded. It then follows from the Proposition that each such intersection is a member of $\mathscr{A}$, whence condition (L) is satisfied.

For a special case of the relationship obtained here between conditions (P) and (L''), see Fan (Theorem 1 of [1]).

Corollaries.

For the first corollary below, let $\mathscr{A}$ be the class of all $w(E,F)$-bounded sets contained in finite-dimensional subspaces of E. For the second, let E be the conjugate space F^* of F, $\mathscr{A}$ the class of all $w(E,F)$-compact subsets of E, and note that (when F is barrelled) $\mathscr{T}_{\mathscr{A}}$ is identical with the original topology of F.

COROLLARY. *Suppose that the real vector spaces E and F form a dual system, and X is a proper subset of E. Then X is Y-convex for every $w(F,E)$-dense subset Y of F if and only if X is a finite-dimensional closed convex set which contains no line.*

COROLLARY. *For a locally convex barrelled space F, a proper subset X of F^* is Y-convex for every dense subset Y of F if and only if X is convex' contains no line, and is closed and locally compact for the weak* topology $w(F^*,F)$.*

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