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PAST AND FUTURE

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1.

Professor A. M. Yaglom calls attention in his paper [7] to a natural problem in the theory of stationary stochastic processes. The problem is to characterize the spectrum of a strongly mixing process, or, in other words, to tell from the covariance function whether the distant past and distant future of the process are nearly orthogonal. In the present paper we shall solve this problem, in the sense that other second-order prediction problems have been solved, by giving an analytic condition on the covariance function that is necessary and sufficient for the process to be strongly mixing. Our condition has a rather different character from a necessary condition recently found by I. A. Ibragimov [4], [5]. It is not known whether Ibragimov's condition is sufficient. Although we are able to derive from our condition a number of Ibragimov's secondary results, we do not fully understand the relation between his main result and ours.

In [2], G. Szegö and one of us studied a related prediction problem, that of determining when the past and future of a process are at positive angle. The techniques we develop below enable us to extend the results of [2]. This extension will be discussed at the end of the paper.

2.

Let μ be a finite positive Borel measure on the unit circle in the complex plane. Let χ be the function on the circle defined by $\chi(e^{ix})=e^{ix}$. For each integer n we form in the Hilbert space $L^2(\mu)$ the subspace $\mathscr{F}_n$ spanned by the functions $\chi^n,\chi^{n+1},\chi^{n+2},\ldots$, and the subspace $\mathscr{P}_n$ spanned by the functions $\chi^n,\chi^{n-1},\chi^{n-2},\ldots$. The subspace $\mathscr{F}_1$ is called the *future* and $\mathscr{P}_{-1}$ is called the *past* in $L^2(\mu)$.

For any two subspaces $\mathscr{M}$ and $\mathscr{N}$ of a Hilbert space we can define a number

$$\varrho(\mathscr{M},\mathscr{N}) = \sup|(f,g)|\,,$$

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where f and g range over the unit balls of $\mathscr{M}$ and $\mathscr{N}$, respectively. Evidently ϱ cannot exceed 1; if $\varrho < 1$ we say that $\mathscr{M}$ and $\mathscr{N}$ are *at positive angle.* In the present context we define

$$\varrho_n = \varrho(\mathscr{P}_0, \mathscr{F}_n), \qquad n = 1, 2, \ldots .$$

If p and q are any positive integers with sum equal to n we have also

$$\varrho_n = \varrho(\mathscr{P}_{-p}, \mathscr{F}_q),$$

because multiplication by χ^p is a unitary operator in $L^2(\mu)$ that carries $\mathscr{P}_{-p}$ onto $\mathscr{P}_0$ and $\mathscr{F}_q$ onto $\mathscr{F}_{p+q}$.

The problem we study is this: *which measures μ have the property that ϱ_n tends to 0 as n tends to infinity?*

3.

The search is narrowed by pointing out some necessary conditions that follow from known results. If μ is not absolutely continuous there are nonnull functions in the intersection of every $\mathscr{F}_n$ with $\mathscr{P}_0$, namely the functions carried on the support of the singular part of μ [3, p. 58]. Then $\varrho_n = 1$ for every n. Thus we need only consider measures of the form $d\mu = w\,d\sigma$, where σ stands for normalized Lebesgue measure on the unit circle and w is a nonnegative function in $L^1(\sigma)$. Furthermore, we may assume $\log w$ is summable, because otherwise all the subspaces $\mathscr{F}_n$ and $\mathscr{P}_n$ coincide with $L^2(w)$ [3, p. 114], and once more $\varrho_n = 1$ for all n.

Let W stand for the collection of all nonnegative summable functions w such that ϱ_n tends to 0 in $L^2(w)$. (We exclude the null function.) Each such function w is $|h|^2$ for an outer function h in H^2 [3]. This h is uniquely determined if we require $h(0)$ to be positive. Let $\varphi(z)$ be the argument of $h(z)$, determined so that $\varphi(0) = 0$, with $\varphi(e^{ix})$ defined almost everywhere as the radial limit of $\varphi(z)$. The function $\varphi(e^{ix})$ is the conjugate function of $\frac{1}{2}\log w$. Whenever they are mentioned, h and φ will be related in this way to the weight function w under consideration.

4.

We begin by following an idea of [2]. In $L^2(w)$ we have

$$\varrho_n = \sup \left| \int f g \chi^n w \, d\sigma \right|, \tag{1}$$

where f and g are polynomials in χ subject to the conditions

$$\int |f|^2 w\, d\sigma \leqq 1, \qquad \int |g|^2 w\, d\sigma \leqq 1 .$$

Now (1) can be written

$$\varrho_n = \sup\left|\int (fh)(gh)\chi^n e^{-2i\varphi}\, d\sigma\right|. \tag{2}$$

As f and g vary in their prescribed way, fh and gh range over a dense subset of the unit ball of H^2, and their product ranges over a dense subset of the unit ball of H^1. Thus (2) expresses ϱ_n as the norm of $\chi^n e^{-2i\varphi}$ as a linear functional on H^1.

By the Hahn–Banach theorem this quantity is equal to

$$\inf\|\chi^n e^{-2i\varphi} - A\|_\infty = \inf\|e^{-2i\varphi} - \chi^{-n}A\|_\infty, \tag{3}$$

where A ranges over the functions in H^∞ with mean value zero. The functions $\chi^{-n}A$ in the second expression are arbitrary sums $P+A$, where P is any trigonometric polynomial with frequencies lying above $-n$, and A is in H^∞. The limit of (3) as n tends to infinity is

$$\inf\|e^{-2i\varphi} - (P+A)\|_\infty,$$

where P ranges over all trigonometric polynomials and A over H^∞. Thus we have this characterization of the elements of W:

THEOREM 1. *A weight function w is in W if and only if its logarithm is summable and $e^{-2i\varphi}$ can be approximated uniformly by functions $P+A$, where P is a trigonometric polynomial and A is in H^∞.*

5.

Let R be the uniform closure of the set of functions $P+A$. Evidently R is a closed subspace of L^∞, containing all continuous functions as well as H^∞. But the product of two functions $P+A$ has the same form again, so that R is a Banach algebra. From this observation we can deduce a result of [6]: *if w is continuous and strictly positive, then w is in W.*

For the proof write

$$e^{-2i\varphi} = e^{\log w}\, e^{-\log w - 2i\varphi}.$$

The first factor on the right is continuous. The second factor is h^{-2} and belongs to H^∞. Since each factor is in R, the product is too, so w is in W by Theorem 1.

The fact that R is an algebra shows furthermore that the product of two functions from W is in W, provided only this product is summable. We shall prove later that the product is indeed always summable.

The algebra R has presented itself in other connections [1], and it would be useful to have a simple criterion by which to identify functions in R. In this direction we can only offer the following remark. Let C be the space of continuous complex valued functions on the circle.

THEOREM 2. *R is equal to $C+H^\infty$, the set of all sums $f+g$ where f is in C and g is in H^∞.*

By definition $C+H^\infty$ is uniformly dense in R; the point is that this set is actually closed. Denote by Z the intersection of C with H^∞. By the F. and M. Riesz theorem, the dual of C/Z is H_0^1 (the subspace of H^1 consisting of the functions with mean value zero). The space H_0^1 has L^∞/H^∞ as its dual, and so the latter is the second dual of C/Z. The canonical map of C/Z into L^∞/H^∞ carries $f+Z$ to $f+H^\infty$, and the range of this map is closed. Hence the inverse image of this range in L^∞, under the canonical projection of L^∞ onto L^∞/H^∞, is closed, and this inverse image is exactly $C+H^\infty$.

6.

We cannot develop the criterion of Theorem 1 further, on account of lack of information about R. Instead we go back to (3) to get this ungainly proposition.

THEOREM 3. *In order for w to belong to W it is necessary and sufficient that for each positive number ε there exist a function A in H^∞ and a positive integer n such that*

$$(4) \qquad -\varepsilon < \arg(Ah^2\chi^{-n}) < \varepsilon \pmod{2\pi}, \qquad \textit{and} \qquad -\varepsilon < \log|A| < \varepsilon$$

almost everywhere on the circle.

PROOF. Modifying (3) a little we have

$$\varrho_n = \inf \|1 - Ae^{2i\varphi}\chi^{-n}\|_\infty ,$$

where A ranges over the functions in H^∞ with mean value zero (a qualification that is without importance). For the norm on the right to be small it is necessary and sufficient that $|A|$ be uniformly close to 1, and that $Ah^2\chi^{-n}$ have argument close to 0. The assertion of the theorem is now obvious.

Let W_0 be the set of weight functions w with the following property: for every positive ε we can find real functions r, s, t such that

$$(5) \qquad \log w = r + \tilde{s} + t$$

with $\|r\|_\infty < \varepsilon$, $\tilde{s}$ conjugate to s and $\|s\|_\infty < \varepsilon$, and t continuous.

Actually in (5) t could be a trigonometric polynomial; for we can approximate a continuous function by a trigonometric polynomial and add the remainder to r without increasing its bound beyond ε. By taking conjugates of each element of (5) we see that 2φ (and thus φ itself) admits such representations if $\log w$ does, and conversely. Let us also mention that w^{-1} belongs to W_0 with w; and both are summable to every power, because $\exp|\tilde{s}/\varepsilon|$ is summable as soon as $\|s\|_\infty < \varepsilon\pi/2$.

THEOREM 4. *W_0 is contained in W.*

This is part of our main result, and is only stated independently for convenience. The proof is simple if we start from the following

LEMMA. *Every continuous real function on the circle can be approximated uniformly by functions $-\arg B\chi^{-n}$, where B is a Blaschke product with exactly n zeros and n is variable.*

Taking the lemma for granted, we prove that an arbitrary element w of W_0 belongs to W. As mentioned above we can find representations

$$2\varphi = \tilde{r} + s + t \tag{6}$$

where r, s are bounded real functions with bounds less than ε and t is a real trigonometric polynomial. If we set

$$A = \exp(-r - i\tilde{r}),$$

an analytic function that satisfies the second condition of (4), then the meaning of (6) is

$$\arg Ah^2 = s + t .$$

Now let $B\chi^{-n}$ be a function given by the lemma whose argument approximates $-t$:

$$\arg(ABh^2\chi^{-n}) = s + t + \arg B\chi^{-n} .$$

The right side is bounded by ε if the approximation is good enough, and so (4) is established with AB in place of A. That proves Theorem 4.

The lemma is elementary. The functions $B^{-1}\chi^n$ whose arguments are involved are, explicitly,

$$k\chi^n \prod_1^n \frac{1-\bar{\alpha}_j\chi}{\chi-\alpha_j} = k\prod_1^n \frac{1-\bar{\alpha}_j\chi}{1-\alpha_j\bar{\chi}}$$

where $\alpha_1,\ldots,\alpha_n$ are complex numbers in the open unit disk (not necessarily distinct) and k has modulus 1. So

$$-\arg B\chi^{-n} = \arg k + 2 \arg \prod_{1}^{n} (1-\bar{\alpha}_j\chi)\,, \tag{7}$$

in other words, the functions with which we are trying to approximate are precisely those of the form $2 \arg P(e^{ix})$, where $P(z)$ is a polynomial having no roots in the closed unit disk and having modulus 1 at the origin.

Let Q be any polynomial whose real part vanishes at the origin. The partial sums P_n of the Taylor series for $\exp Q$ are polynomials having modulus 1 at the origin; they converge uniformly to $\exp Q$ on the closed unit disk and therefore have no zeros in the closed disk for n sufficiently large. Hence for n sufficiently large the functions $2 \arg P_n(e^{ix})$ are of the form (7). These functions evidently converge uniformly on the unit circle to twice the imaginary part of $Q(e^{ix})$, which is an arbitrary real trigonometric polynomial. As every real continuous function on the circle can be uniformly approximated by real trigonometric polynomials, the lemma is proved.

7.

Here is our main result.

THEOREM 5. *W is exactly the collection of all functions $|P|^2 w_0$, where P is a polynomial and w_0 belongs to W_0.*

If P is a polynomial then the weight function $|P|^2$ is in W, because for this weight function we have $\varrho_n = 0$ as soon as n exceeds the degree of P. Thus, if w is in W then $|P|^2 w$ is the product of two functions in W and so is itself in W. This proves one half of the theorem.

To prove the other half of the theorem we need a simple lemma about analytic continuation.

LEMMA. *Let the function S be analytic in the unit disk except for a pole of order n (perhaps zero) at the origin. Assume $z^n S$ is in the space $H^{\frac{1}{2}}$ of the unit disk, and that S is real valued and nonnegative almost everywhere on the unit circle. Then S can be continued analytically across the circle.*

We shall derive this from another continuation principle, which is well-known and which we therefore do not prove: *If a function in the space H^1 of an annulus $0 < R < |z| < 1$ is real valued almost everywhere on the unit circle, then it can be continued analytically across the circle.*

Let S satisfy the hypotheses of the lemma. Then there is a factoring $S = S_1 S_2$, where S_1 and $z^n S_2$ are in H^1 of the unit disk and $|S_1| = |S_2|$

almost everywhere on the unit circle. This means that $S_2 = \bar{S}_1$ on the circle. Thus $S_1 + S_2$ and $i(S_1 - S_2)$ are real valued on the circle, and so, by the H^1 continuation principle stated above, they can be continued analytically across the circle. Hence S_1 and S_2 can themselves be continued across the circle, and therefore so also can S.

We remark that the conclusion of the lemma no longer holds if, instead of assuming $z^n S$ is in $H^{\frac{1}{2}}$, one assumes only that it is in H^p for $0 < p < \frac{1}{2}$. The function $-(1+z)^2/(1-z)^2$ furnishes a counter example (with $n = 0$).

We now complete the proof of Theorem 5. Let w be a function in W and ε a positive number less than $\frac{1}{2}\pi$. By Theorem 3, there is a function A in H^∞, a nonnegative integer n, and a real function s in L^∞ such that almost everywhere on the unit circle,

$$-\varepsilon < s < \varepsilon, \qquad -\varepsilon < \log|A| < \varepsilon,$$

$$s + \arg(Ah^2\chi^{-n}) \equiv 0 \pmod{2\pi}.$$

The last condition means that the function

$$S = Ah^2\chi^{-n}e^{-\tilde{s}+is} \tag{8}$$

is nonnegative almost everywhere on the unit circle. The function $\exp(-\tilde{s}+is)$ is in H^1 (and in fact in H^p for $p < \pi/2\varepsilon$). Also h^2 is in H^1, and thus $z^n S$ is in $H^{\frac{1}{2}}$. Applying the continuation principle proved above, we may conclude that S can be continued analytically across the unit circle. The reflection principle tells us that S is actually analytic in the entire plane except for poles at 0 and ∞. Thus S is a polynomial in z and z^{-1}. Restricted to the unit circle, S is a nonnegative trigonometric polynomial, so it has there a representation $S = |P_0|^2$ with P_0 a polynomial in z.

Let P_0 be factored into a product PQ, where P is a polynomial with roots only on the unit circle and Q is a polynomial with no roots on the unit circle. Define the functions r and t on the circle by

$$r = -\log|A| \quad \text{and} \quad t = 2\log|Q|.$$

Then, taking absolute values in (8) and rearranging, we obtain

$$w = |h|^2 = |P|^2 e^{r+\tilde{s}+t}. \tag{9}$$

What we have proved is this: for every ε between 0 and $\frac{1}{2}\pi$ the function w has a representation (9), where P is a polynomial with roots only on the unit circle, and r, s, t are real functions with $\|r\|_\infty \leqq \varepsilon$, $\|s\|_\infty \leqq \varepsilon$, and t continuous.

To complete the proof it remains to show that the polynomial P one obtains in this manner is independent of ε. For this we use an argument of Ibragimov's. Notice that the functions $w/|P|^2$ and $|P|^2/w$ are both summable. Thus, if P_1 and P_2 are the polynomials that result from two different choices of ε, then

$$\int |P_1/P_2|\, d\sigma = \int (|P_1|/w^{\frac{1}{2}})(w^{\frac{1}{2}}/|P_2|)\, d\sigma$$
$$\leqq \left(\int |P_1|^2/w\, d\sigma\right)^{\frac{1}{2}} \left(\int w/|P_2|^2\, d\sigma\right)^{\frac{1}{2}} < \infty ,$$

and so P_2 divides P_1. The same reasoning shows that P_1 divides P_2, and thus P_1 and P_2 are identical except for a multiplicative constant of no importance.

The proof of Theorem 5 is complete.

8.

We list informally some consequences of our representation theorem.

If w is in W_0, then w^n is summable for every positive and negative n. If w is in W, then w^n is summable for every positive n. The first statement has already been justified, and the second one follows by Theorem 5. (Ibragimov's Sledstvie 3 [5, p. 115; Corollary 3 in Amer. Transl., p. 106] is a consequence of the second statement.)

W is closed under the formation of products. For now we know that the product of functions in W is always summable, and that is all that was needed.

Incidentally, *W is closed under the formation of sums.* This fact is easy to prove directly from (1).

A function in W cannot have a simple discontinuity. (This is due to Ibragimov [5, p. 114].) Suppose on the contrary that w in W has a simple discontinuity at a point e^{ix_0}. Then $1+w$ is a function in W_0 with the same discontinuity, and $\log(1+w)$ possesses representations (5). On the other hand we can write $\log(1+w)$ as the sum of a function f having a jump matching that of $\log(1+w)$ at e^{ix_0}, but smooth everywhere else, and a function g continuous at e^{ix_0} and vanishing there. Thus we can write

$$f = r + \tilde{s} + t - g . \tag{10}$$

Let t be a trigonometric polynomial; take the conjugate of (10):

$$\tilde{f} = \tilde{r} + s + \tilde{t} - \tilde{g} + c , \tag{11}$$

where c is some real constant. Now $\tilde{f}$ is asymptotically equal to $k \log|x-x_0|$ as x tends to x_0, where k is a nonzero constant depending only on the jump of f at e^{ix_0}. Hence there is an integer n, depending only on the jump of f, such that $\exp n\tilde{f}$ is not summable on any interval containing e^{ix_0}. But by taking ε small enough we can make every term on the right of (11) exponentially summable to as high a power as we please, at least in some neighborhood of e^{ix_0}. Indeed s is bounded, $\tilde{r}$ is the conjugate of a function with bound as small as we please, $\tilde{t}$ is continuous, and $\tilde{g}$ is the conjugate of a function that is as small as we please in a sufficiently small neighborhood of e^{ix_0}. So the Schwarz inequality implies $\exp n\tilde{f}$ is summable over an interval containing e^{ix_0} (depending on n), for arbitrarily large $|n|$. This contradiction proves the assertion.

If w is in W_0 then the indefinite integral of $\log w$ *is a uniformly smooth function, in other words,*

$$\int_x^{x+\delta} \log w \, d\sigma - \int_{x-\delta}^{x} \log w \, d\sigma = o(\delta) \quad \textit{as } \delta \to 0\,,$$

uniformly in x. Take $\varepsilon > 0$ and write $\log w = r + \tilde{s} + t$ with $\|r\|_\infty < \varepsilon$, $\|s\|_\infty < \varepsilon$, and t continuous. The indefinite integral of t is obviously uniformly smooth, and the indefinite integral of r is Lipschitzian with Lipschitz constant ε. The indefinite integral of $\tilde{s}$ is the conjugate of a Lipschitz function with Lipschitz constant ε, and this implies that

$$\limsup_{\delta \to 0} \delta^{-1} \left| \int_x^{x+\delta} \tilde{s} \, d\sigma - \int_{x-\delta}^{x} \tilde{s} \, d\sigma \right| \leqq C\varepsilon$$

uniformly in x, where C is an absolute constant [8, Chap. VII, Sec. 5]. The assertion about $\log w$ follows.

If w is a bounded function in W, then the indefinite integral of w is uniformly smooth. This will follow from the preceding result if we can show that e^w is in W. But e^w is the uniform limit of the series $\sum_0^\infty w^n/n!$, and the partial sums of this series are in W by our earlier results. From the definition of W it is easily proved that a weight function which is bounded from zero and uniformly approximable by functions in W is itself in W; hence e^w is in W.

The last italicized statement is an easy consequence of Ibragimov's condition, and it is the closest we have been able to come to his condition with our methods.

We seem confronted with three classes of functions: the class S_0 of functions $u+\tilde{v}$ where u and v are real continuous functions on the circle; the class $\log W_0$ of functions $\log w$ with w in W_0; the class S of real L^1 functions whose indefinite integrals are uniformly smooth. The inclusion $\log W_0 \subset S$ is proved above, and the inclusion $S_0 \subset \log W_0$ is trivial. It is natural to ask whether both inclusions are proper. We do not know the answer.

9.

In [2] an analytic condition on w was found for $\mathscr{P}_0$ and $\mathscr{F}_1$ to be at positive angle in $L^2(w)$. The question of finding a similar condition for $\mathscr{P}_0$ and $\mathscr{F}_n$ to be at positive angle, when n is larger than 1, was raised but not completely answered. The method of analytic continuation used in this paper leads to such a condition. We have indeed the following result.

THEOREM 6. *In order for $\mathscr{P}_0$ and $\mathscr{F}_n$ to be at positive angle in $L^2(w)$ it is necessary and sufficient that w have the form*

$$w = |P|^2 e^{r+\tilde{s}}, \tag{12}$$

where P is a polynomial of degree less than n, r is a real bounded function, and $\tilde{s}$ is the conjugate of a real function with bound strictly smaller than $\frac{1}{2}\pi$.

From (3) we have

$$\varrho_n = \inf \|1 - Ae^{2i\varphi}\chi^{1-n}\|_\infty ,$$

where A ranges over all the functions in H^∞ (and not merely those with mean value zero). This quantity is less than 1 if and only if for some positive ε and A in H^∞ we have almost everywhere

$$|\arg(Ah^2\chi^{1-n})| < \tfrac{1}{2}\pi - \varepsilon \pmod{2\pi}, \quad \text{and} \quad |A| > \varepsilon . \tag{13}$$

This condition from [2] is analogous to (4).

Suppose (13) holds, and let s be the function bounded by $\frac{1}{2}\pi - \varepsilon$ such that

$$s + \arg(Ah^2\chi^{1-n}) \equiv 0 \pmod{2\pi} .$$

Then the function

$$S = Ah^2\chi^{1-n} e^{-\tilde{s}+is} \tag{14}$$

is analytic in the circle except at the origin (and even there if $n=1$), and of class $H^{\frac{1}{2}}$ when the pole is removed. The boundary values of S are positive almost everywhere. By the continuation principle, S is analytic everywhere in the plane except for poles at the origin and infinity of order at most $n-1$. Therefore on the circle S has the form

$S = |P|^2$ for a polynomial P of degree less than n. Finally, taking the modulus of (14) and setting $r = -\log|A|$ gives

$$w = |h|^2 = |P|^2 e^{r+\tilde{s}},$$

as desired.

In the other direction, let w be a weight function (12). If P has zeros inside the unit circle we remove them from P and incorporate the corresponding trigonometric polynomial (which is bounded from zero) in the factor e^r. This will reduce the degree of P, leading to a stronger result about w. With this done, we assume that P has degree exactly $n-1$ and has no zeros inside the circle.

Now given (12) we define S to be $|P|^2$ and A to be the outer function with modulus e^{-r}. Then we have

$$\chi^{n-1} S e^{\tilde{s}-is} = A h^2, \tag{15}$$

at least if we incorporate in A the proper multiplicative constant of modulus 1. Indeed $\chi^{n-1}S$ is an analytic trigonometric polynomial whose extension inside the unit circle has no zeros; such a function is outer. Therefore the left side of (15) is an outer function in H^1, and the same is obviously true of the right side. The two sides have the same modulus, and an outer function is determined up to a constant factor by its modulus, so (15) follows.

Thus the argument of

$$S = A h^2 \chi^{1-n} e^{-\tilde{s}+is}$$

is zero almost everywhere on the circle, and (13) holds.

This completes the proof of Theorem 6.

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