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INEQUALITIES FOR THE GREEN FUNCTION AND BOUNDARY CONTINUITY OF THE GRADIENT OF SOLUTIONS OF ELLIPTIC DIFFERENTIAL EQUATIONS

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1. Introduction.

In the first part of this paper we establish various inequalities for the Green function of Laplace's operator in a Liapunov–Dini region $\Omega \subset R^n$, $n \geqq 3$. In the case of Liapunov regions in R^3 they were previously known, see [6] and [4], but apart from being more general, our derivation seems easier and more natural.

As a consequence of these inequalities we can give an elementary proof of a theorem of Schauder [7], assuring the Hölder continuity of the gradient of a harmonic function in $\bar{\Omega}$, given that u has Hölder continuous tangential derivatives on the Liapunov surface $\partial\Omega$. In fact we shall establish a more general theorem involving Liapunov–Dini surfaces and Dini continuity instead of Hölder continuity, and by an easy example we show that this condition is the right one to ensure boundary continuity. One should also compare the situation in two dimensions.

In the second part the corresponding theorem for solutions of a relatively wide class of semi-linear second order elliptic equations is proved. This class contains e.g. the uniformly elliptic equation

$$a^{ij}u_{ij} + b^i u_i + cu = f,$$

where a^{ij} are boundary Hölder continuous and where b^i, c, and f are measurable with a limitation of their growth near the boundary. For details, see Section 3 and Theorem 3.1. A somewhat wider class of elliptic equations was considered in [9], and we shall use several results and methods from that paper.

The literature dealing with the boundary behavior of solutions of elliptic equations is rather formidable. We refer the reader to the fundamental work of Agmon, Douglis, and Nirenberg in [1] and [2], where also extensive bibliographies can be found. However, to the author's

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knowledge the results in this paper are more general than existing ones for the case of second order operators, since we have considerably weaker regularity assumptions on both the coefficients and the boundary. A comparative discussion will be found at the end of Section 3.

Since we believe that the first part of this paper might be of interest to a wider audience, we have tried to include more details there, in contrast to the second part, where the proofs are more sketchy.

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NOTATIONS. We place ourselves in R^n, $n \geqq 3$, the points of which are denoted by $X, Y, \ldots$, and $X = (x_1, \ldots, x_n)$, $X' = (x_1, \ldots, x_{n-1})$. The Euclidean distance $(x_1^2 + \ldots + x_n^2)^{\frac{1}{2}}$ is denoted $|X|$. Integrals over n-dimensional regions will be denoted by $\int (\cdot)\, dX$, over $(n-1)$-dimensional surfaces by $\int (\cdot)\, dS$, dS being the surface element.

By a Dini function we shall mean a non-negative, monotonic, continuous function $\varepsilon(t)$, $t > 0$, having the properties that $\varepsilon(t)/t$ is also monotonic and

$$\int_0 \frac{\varepsilon(t)}{t}\, dt < \infty .$$

The important property here is of course the integral condition, the other ones being introduced for technical reasons. A common type of Dini functions are $\varepsilon(t) = t^\alpha$, $0 < \alpha < 1$.

A Liapunov–Dini surface is a closed, bounded $(n-1)$-dimensional surface S satisfying the following conditions:

1°. At every point of S there is a uniquely defined tangent (hyper-) plane, and thus also a normal.

2°. There exists a Dini function $\varepsilon(t)$ such that if β is the angle between two normals, and r is the distance between their foot points, then the inequality $\beta \leqq \varepsilon(r)$ holds.

3°. There is a constant $\varrho > 0$ such that if Σ_ϱ is a sphere with radius ϱ and center $X_0 \in S$, then a line parallel to the normal at X_0 meets S at most once inside Σ_ϱ.

A Liapunov–Dini surface is called a Liapunov surface if $\varepsilon(t) = kt^\alpha$, $0 < \alpha < 1$. Liapunov–Dini and Liapunov regions are regions bounded by Liapunov–Dini and Liapunov surfaces respectively. For the properties of Liapunov regions see Günther [5]. Important properties are carried over to Liapunov–Dini surfaces; in particular the Green formula is valid in a Liapunov–Dini region.

The boundary of any set D is denoted by ∂D, and $\bar{D}$ is the closed hull of D. The distance from X to ∂D is denoted by $\delta(X)$.

K denotes a generic constant changing its value from one occurrence to another, and ω_n is the area of the n-dimensional unit sphere.

2.

We start with a lemma on the Green function for the Laplacian in a half space. Since this function is explicitely known and has a very simple form, the following estimates follow without difficulty with elementary methods.

LEMMA 2.1. *Let $G(X,Y)$ be the Green function for $x_n>0$. Then for $i,j,k=1,2,\ldots,n$,*

$$G(X,Y) \leqq \begin{cases} |X-Y|^{2-n}, \\ Ky_n|X-Y|^{1-n}, \\ Kx_ny_n|X-Y|^{-n} \end{cases}$$

$$\left|\frac{\partial}{\partial x_i}G(X,Y)\right| \leqq \begin{cases} K|X-Y|^{1-n}, \\ Ky_n|X-Y|^{-n}, \end{cases}$$

$$\left|\frac{\partial^2}{\partial x_i\partial x_j}G(X,Y)\right| \leqq \begin{cases} K|X-Y|^{-n}, \\ Ky_n|X-Y|^{-1-n}, \end{cases}$$

$$\left|\frac{\partial}{\partial y_k}\frac{\partial^2}{\partial x_i\partial x_j}G(X,Y)\right| \leqq K|X-Y|^{-1-n}.$$

In the next theorem we consider a region D of special form: Let $\varphi(t)=\int_0^t\varepsilon(s)ds$. We note that $\varphi(t)$ is convex and that $\frac{1}{4}t\varepsilon(t)<\varphi(t)<t\varepsilon(t)$, a fact we shall use several times. Now D is defined by

$$D = \{X \mid |X'|<1,\ -\varphi(|X'|)<x_n<2\}.$$

In D we define a harmonic function u by requiring that u be equal to zero on $\partial' D=\partial D\cap\{|X'|<1,x_n<0\}$ and equal to one on $\partial'' D=\partial D\setminus\partial' D$.

THEOREM 2.2. *If u is defined as above, then*

$$u(0,\ldots,0,x_n) \leqq Kx_n,$$

where K depends on ε and n only.

PROOF. We construct the region D_r by taking away from D its intersection with the ball

$$\sum_{i=1}^{n-1} x_i^2+(x_n+r)^2 \leqq r^2$$

and denote the harmonic function that is one on $\partial'' D_r$ and zero on $\partial' D_r$ by u^r.

The regions D_r have the following property: If we shrink D_r by a length factor 16, say, and denote this region by D_r', and place a region $D'_{r,s}$, congruent to D_r', with the node of $\partial' D'_{r,s}$ at $(X', -2\varphi(s))$, $|X'| = s$, and its axis parallel to the x_n-axis, then $\partial' D'_{r,s}$ will not intersect $\partial' D_r$ (see Fig. 1). This follows by using elementary geometric considerations and the fact that $\varepsilon(2t) \leqq 2\varepsilon(t)$.

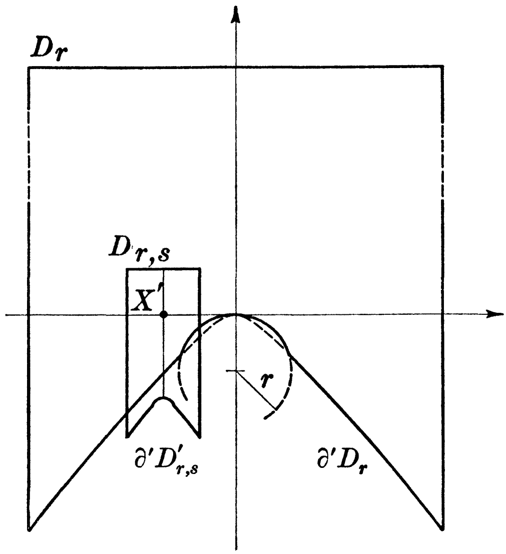

Fig. 1.

Now we note that $-K \leqq \partial u^r/\partial n < 0$ on $\partial'' D_r \cap \{x_n > 0\}$, K independent of r, and represent $u^r(0, \ldots, 0, y_n)$ by using the Green function G of $\{x_n > 0\}$ in $D_r \cap \{x_n > 0\}$. We get, if $Y = (0, \ldots, 0, y_n)$,

$$\omega_n \cdot u^r(Y) = \int_{\partial D \cap \{x_n > 0\}} \left\{ \frac{\partial G}{\partial n} u^r - G \frac{\partial u^r}{\partial n} \right\} dS_X + \int_{|x'| \leqq 1} \frac{\partial}{\partial n_{X'}} G(X', Y)\, u^r(X')\, dX' .$$

Using the estimates of Lemma 2.1 and

$$|u^r| \leqq 1, \qquad |\partial u^r / \partial n| \leqq K ,$$

we see that the first integral is less than $K y_n$, K independent of r. Next by using the maxium principle, we see that for s small enough $u^r(X')$ is less than the corresponding function in $D'_{r,s}$, with $s = |X'|$. Hence

$$u^r(X') \leqq u^r(0,\ldots,32\,\varphi(|X'|))\,.$$

In the second integral we use this estimate to conclude

$$u^r(Y) \leqq Ky_n + Ky_n \int\limits_{|X'|\leqq 1} \frac{u^r(0,\ldots,32\,\varphi(|X'|))}{|X'-Y|^n}\,dX'\,.$$

Now put

$$m_r = \sup\nolimits_{0<y_n<2} u^r(0,\ldots,y_n)/y_n\,.$$

Since u^r is certainly less than y_n times a constant depending on r, m_r is finite for every $r>0$. We get, with $|u^r|\leqq 1$,

$$m_r \leqq K + K \int\limits_{|X|\leqq\varepsilon_1} \frac{m_r\,\varphi(|X'|)}{|X'-Y|^n}\,dX' + K \int\limits_{\varepsilon_1\leqq|X'|\leqq 1} \frac{dX'}{|X'-Y|^n}$$

$$\leqq K + Km_r \int\limits_0^{\varepsilon_1} \frac{\varepsilon(t)}{t}\,dt + \frac{K}{\varepsilon_1}.$$

Obviously we can choose an ε_1 independent of r such that

$$K \int\limits_0^{\varepsilon_1} \frac{\varepsilon(t)}{t}\,dt < \tfrac{1}{2}\,.$$

Then

$$m_r \leqq K + K/\varepsilon_1\,.$$

Since the right hand side is independent of r it follows that

$$u(0,\ldots,y_n) \leqq Ky_n$$

either by letting $r \to 0$ and noting that $u^r \to u$ or by the maximum principle. The theorem is proved.

THEOREM 2.3. *Let Ω be a Liapunov–Dini region. For $X,Y \in \Omega$, $i,j = 1,2,\ldots,n$, the Green function $G(X,Y)$ of Ω satisfies*

(i) $G(X,Y) \leqq K\delta(X)\,|X-Y|^{1-n}$,

(ii) $\left|\dfrac{\partial}{\partial x_i} G(X,Y)\right| \leqq K\,|X-Y|^{1-n}$,

(iii) $\left|\dfrac{\partial}{\partial x_i} G(X,Y)\right| \leqq K\delta(Y)\,|X-Y|^{-n}$,

(iv) $\left|\dfrac{\partial}{\partial x_i}\dfrac{\partial}{\partial y_i} G(X,Y)\right| \leqq K\,|X-Y|^{-n}$,

where K depends on Ω only.

PROOF. Let the diameter of Ω be d. There is some $s_0 < 1$ such that regions congruent to D shrunk by a factor $1/s$ can be placed at every point of $\partial\Omega$ in such a way that the bent part intersects $\partial\Omega$ at this point only, and the symmetry axis is along the normal, for all $s \leqq s_0$.

Now let Y be fixed. If $\delta(X) \geqq s_0$ we have

$$\delta(X) \geqq s_0 \geqq Kd \geqq K|X-Y|$$

and hence

$$G \leqq |X-Y|^{2-n} \leqq K\delta(X)|X-Y|^{1-n}.$$

The same type of argument holds if

$$\delta(X) < s_0 \quad \text{but} \quad |X-Y| < 2\delta(X).$$

Thus it is sufficient to consider the case

$$\delta(X) < s_0, \quad \delta(X) < \tfrac{1}{2}|X-Y|.$$

Let $X^* \in \partial\Omega$ be such that $\delta(X) = |X^* - X|$. At X^* we place a region D' congruent to D shrunk by a factor

$$1/s = \begin{cases} 4/|X-Y| & \text{if} \quad |X-Y| < 4s_0, \\ 1/s_0 & \text{otherwise}. \end{cases}$$

It is easy to see that the distance from Y to $\partial D'$ is greater than $|X-Y|/4$, which means that

$$G(Z,Y) \leqq K|X-Y|^{2-n} \quad \text{for} \quad Z \in \partial D'.$$

By the maximum principle and Theorem 2.2 we find

$$G(X,Y) \leqq K\,\delta(X)\,s^{-1}\,|X-Y|^{2-n} = K\,\delta(X)\,|X-Y|^{1-n}$$

if $|X-Y| < 4s_0$, and

$$\begin{aligned} G(X,Y) &\leqq K\,\delta(X)\,s_0^{-1}|X-Y|^{2-n} \\ &\leqq K\delta(X)\,d^{-1}\,|X-Y|^{2-n} \leqq K\,\delta(X)\,|X-Y|^{1-n} \end{aligned}$$

in the opposite case, and (i) is proved.

To prove (ii), if $\delta(X) \leqq |X-Y|$ represent G by its Poisson integral over a sphere with radius $\frac{1}{2}\delta(X)$ and use (i) after having differentiated under the integral sign; if $|X-Y| < \delta(X)$, take a sphere of radius $\frac{1}{2}|X-Y|$ and use the inequality $G(X,Y) < |X-Y|^{2-n}$.

To prove (iii) we first show that for fixed $X \in \Omega$,

$$\frac{\partial}{\partial x_i} G(X,Y) \to 0 \quad \text{as } Y \to \partial\Omega.$$

When Y is on a sphere around X with radius ϱ, we have $G(X,Y) \geqq$

$\frac{1}{2}|X-Y|^{2-n}$ if ϱ is small enough. Now let h be a vector with length $<\frac{1}{2}\varrho$. The difference quotient satisfies

$$\left|\frac{G(X+h,Y)-G(X,Y)}{|h|}\right| \leqq |X-Y|^{1-n}K \leqq K\varrho^{-1}G(X,Y)$$

for $|X-Y|=\varrho$. Since the difference quotient is zero on $\partial\Omega$ the stated inequality between the quotient and G holds in Ω minus the sphere. After fixing Y we let $h \to 0$ and see that

$$\frac{\partial}{\partial x_i} G(X,Y) \to 0 \quad \text{as } \delta(Y) \to 0 .$$

Now, using (ii), (iii) follows as in the proof of (i), and then (iv) follows from (iii) as (ii) followed from (i).

The theorem is proved.

Remark. (i) was proved in [6] with $n=3$ for Liapunov regions, and in [8] for regions with stronger regularity but for general n. (ii) was proved by Eĭdus in [4] for Liapunov regions in R^3, with the help of integral equations.

Theorem 2.4. *Let u be a harmonic function in a Liapunov–Dini region Ω, continuous in $\bar{\Omega}$, and with the property that to every $X_0 \in \partial\Omega$ there is a linear polynomial $L_{X_0}(X)$ such that*

$$|u(X)-L_{X_0}(X)| \leqq \varepsilon(|X-X_0|)\,|X-X_0|, \qquad X \in \partial\Omega ,$$

where the Dini function $\varepsilon(t)$ satisfies the additional condition that $\varepsilon(t)/t^\gamma$ is monotonic for some γ, $0<\gamma<1$.

Then $\partial u/\partial x_i$ are continuous in $\bar{\Omega}$.

In particular, if Ω is a Liapunov region and $\varepsilon(t)=kt^\alpha$, then the functions $\partial u/\partial x_i$ are α-Hölder continuous in $\bar{\Omega}$.

Remark. It is clear that Dini or Hölder continuity of the tangential derivatives of u on $\partial\Omega$ implies the respective conditions in the theorem. It is also clear that the additional requirement on the Dini function is a fairly mild one.

Finally, we remark that the α appearing is the same as in the definition of the Liapunov surface; if the Hölder exponent of the function is smaller than that of the surface, we can always diminish the latter.

Proof of Theorem 2.4. It is easy to see that the coefficients of $L_{X_0}(X)$ can be chosen so as to be uniformly bounded on $\partial\Omega$.

Our first step will be to prove that $|\operatorname{grad} u|$ is bounded in Ω.

Let Y be an arbitrary point with $\delta(Y)$ small, say $< \frac{1}{4}\varrho$, where ϱ is the number appearing in the axiom 3° of Liapunov–Dini surfaces. If

$$Y^* \in \partial\Omega, \qquad |Y - Y^*| = \delta(Y),$$

let Σ be the intersection of Ω with a ball of radius $\frac{1}{2}\varrho$ and center Y^*. To simplify notations we assume that

$$Y^* = O, \qquad Y = (0,\ldots,y_n), \quad y_n > 0 .$$

Since the coefficients of the linear polynomials are uniformly bounded we may subtract L_0 from u and assume that u satisfies

$$|u(X)| \leqq K|X|\varepsilon(|X|), \qquad X \in \partial\Omega .$$

Put

$$u^t = u(x_1,\ldots,x_{n-1},x_n+t), \qquad t>0 .$$

Then $\operatorname{grad} u^t$ is continuous in $\bar{\Sigma}$ and if $\partial\Omega$ is given by $x_n = \Psi(X')$ we represent Du^t in

$$\Sigma^\tau = \Sigma \cap \{x_n > \tau + \Psi(X')\}, \qquad \tau > 0$$

as follows:

$$\omega_n Du^t(Y) = \int\limits_{\partial\Sigma^\tau \cap \Sigma} \left\{u^t(X) D\frac{\partial}{\partial n} G(X,Y) - \frac{\partial}{\partial n} u^t(X) DG(X,Y)\right\} dS_X + \\ + \int\limits_{\partial\Sigma^\tau \cap \partial\Sigma} (\cdot)\, dS_X ,$$

where D denotes differentiation with respect to any Y-variable and G is Green's function for Ω. Now for fixed Y and t we let $\tau \to 0$. $D\partial G/\partial n$ tends in weak (L^2) sense, say, to some function on $\partial\Sigma \cap \partial\Omega$ satisfying the same inequalities as $D\partial G/\partial n$. Since u^t and $\operatorname{grad} u^t$ are continuous in $\bar{\Sigma}$ no other convergence problems arise here. We observe that the second term in the first integral disappears after this operation.

In view of Theorem 2.3 and the elementary inequality

$$|\operatorname{grad} u^t(X)| \leqq KM\delta^{-1}(X), \qquad M = \max|u| ,$$

where K is independent of t, there is no difficulty involved in letting $t \to 0$. We get

$$\omega_n Du(Y) = \int\limits_{\partial\Sigma \cap \partial\Omega} u(X)\, D\frac{\partial}{\partial n} G(X,Y)\, dS_X + \\ + \int\limits_{\partial\Sigma \cap \Omega} \left\{u(X)\, D\frac{\partial}{\partial n} G(X,Y) - \frac{\partial}{\partial n} u(X) DG(X,Y)\right\} dS_X .$$

Now

$$\left| \int\limits_{\partial\Sigma \cap \partial\Omega} (\cdot)\, dS_X \right| \leqq K \int\limits_{|X'|<\frac{1}{2}\varrho} \frac{\varepsilon(|X'|)\,|X'|}{|X'|^n}\, dX' \leqq K \int\limits_0^{\frac{1}{2}\varrho} \frac{\varepsilon(t)}{t}\, dt \leqq K < \infty ,$$

$$\int\limits_{\partial\Sigma \cap \Omega} (\cdot)\, dS_X \leqq KM \int\limits_{|X|=\frac{1}{2}\varrho} \frac{dS_X}{|X-Y|^n} \leqq KM\varrho^{-1} = K < \infty .$$

The next step will be to prove that

$$|D^2u(X)| \leqq K\varepsilon(\delta(X))\, \delta^{-1}(X) . \tag{2.4.1}$$

To do that construct an egg-shaped region E the boundary of which consists of the part of the surface $x_n = k\varphi(|X'|)$ that lies below $x_n = s$, plus a cap put on top so as to make E convex, of diameter $2s$, say, and symmetric with respect to the x_n-axis. If k is large and s small enough, a region congruent to E can be placed inside Ω at every point of $\partial\Omega$ in such a way that the symmetry axis lies along the normal, and the node of E is the only point common with $\partial\Omega$.

Keeping the notation, we take $Y=(0,\ldots,y_n)$, $Y^* = O$, and represent $u(Y)$ by its values on ∂E, using the Green function G of $x_n > 0$. After having differentiated twice we get

$$\omega_n D^2u(Y) = \int\limits_{\partial E} \left\{ u(X)\, D^2 \frac{\partial}{\partial n} G(X,Y) - \frac{\partial u}{\partial n} D^2 G(X,Y) \right\} dS_X ,$$

D^2 denoting any second derivative with respect to the Y-variables. In order to estimate the first term in the integral we note that on ∂E

$$\begin{aligned} |u(X)| &\leqq |u(X) - u(X^*)| + |u(X^*)| \\ &\leqq K|X - X^*| + K\varepsilon(|X^*|)|X^*| \leqq K\varepsilon(|X|)\,|X| \end{aligned}$$

which follows since $|\mathrm{grad}\, u|$ is bounded and we have assumed that the linear polynomial is already subtracted from u. Now we divide the domain of integration into three parts:

$$\begin{aligned} \partial E_1 &= \partial E \cap \}x_n < s\} \cap \{|X'| < y_n\} \\ \partial E_2 &= \partial E \cap \{x_n < s\} \cap \{|X'| > y_n\} \\ \partial E_3 &= \partial E \cap \{x_n > s\} . \end{aligned}$$

$$\left| \int_{\partial E_1} u(X)\, D^2 \frac{\partial}{\partial n} G(X, Y)\, dS_X \right| \leqq K \int_{\partial E_1} \frac{\varepsilon(|X|)\,|X|}{|X-Y|^{n+1}}\, dS_X$$

$$\leqq K \frac{\varepsilon(y_n)}{y_n{}^n} \int_{|X'|<y_n} dS_X = K\,\varepsilon(y_n) y_n{}^{-1} ,$$

$$\left| \int_{\partial E_2} (\cdot)\, dS_X \right| \leqq K \int_{\partial E_2} \frac{\varepsilon(|X|)\,|X|}{|X-Y|^{n+1}}\, dS_X$$

$$\leqq K \int_{y_n}^{d} \frac{\varepsilon(r)}{r}\, dr$$

$$\leqq K y_n{}^{-1} \int_{1}^{d/y_n} \frac{\varepsilon(y_n t)}{t^2}\, dt$$

$$\leqq K y_n{}^{-1} \varepsilon(y_n) \int_{1}^{\infty} t^{\gamma-2}\, dt \leqq K y_n{}^{-1} \varepsilon(y_n) ,$$

$$\left| \int_{\partial E_3} (\cdot)\, dS_X \right| \leqq K(s) \max |u| .$$

The second term is estimated similarly, using the boundedness of $\partial u/\partial n$.

The continuity of $\operatorname{grad} u$ in $\bar{\Omega}$ is a trivial consequence of (2.4.1) since then $\operatorname{grad} u$ is continuous with the same modulus of continuity along every normal onto $\partial\Omega$, and since the direction of the normals is a continuous function on $\partial\Omega$.

In the case of Liapunov surfaces and $\varepsilon(t)=kt^{\alpha}$, the Hölder continuity follows with a similar argument.

The theorem is proved.

REMARK 1. That the Dini condition is actually necessary to ensure even boundedness of the gradient is seen from the following example.

Let $\varepsilon(t)$ be a highly regular function on $R^1 \setminus \{0\}$, satisfying the requirements of a Dini function, except that

$$\int_0 \frac{\varepsilon(t)}{t}\, dt = \infty .$$

Put

$$\varepsilon(-t) = \varepsilon(t) \quad \text{and} \quad f(t) = \int_0^t \varepsilon(s)\, ds .$$

If $u(x,y)$ is the harmonic function in $y>0$ having f as its boundary function, $\operatorname{grad} u$ is not bounded in a neighborhood of the origin, because

$$\left|\frac{\partial}{\partial y} u(0,y)\right| \geqq K\int_y^1 \frac{f(t)}{y^2+t^2}\,dt - K \geqq K\int_y^1 \frac{\varepsilon(t)}{t}\,dt - K \to \infty, \qquad y \to 0 .$$

Remark 2. It is possible to avoid the limiting process involving τ in the proof above by first proving that the derivatives of the Green function are continuous in $\bar{\Omega}$, since in this case we know that $|\operatorname{grad} G|$ is bounded by Theorem 2.3, and then we can apply the argument with the region E immediately.

Theorem 2.5. *Let $G(X,Y)$ be the Green function of a Liapunov–Dini region Ω. Then for fixed $Y \in \Omega$ there is a constant $C>0$ such that*

$$\frac{\partial}{\partial n_X} G(X,Y) \geqq C .$$

Proof. Consider the egg-shaped region E of the proof of Theorem 2.4, and let

$$X = (0,\ldots,x_n), \qquad Y = (0,\ldots,y_n), \qquad y_n > 2x_n .$$

If $G^*(X,Y)$ is the Green function of $\{x_n>0\}$ and G that of E, then

$$G(X,Y) = G^*(X,Y) - \frac{1}{\omega_n}\int_{\partial E} G^*(T,X)\frac{\partial}{\partial n_T} G(T,Y)\,dS_T .$$

It is easy to check that there is a constant $C>0$ such that

$$G^*(X,Y) \geqq Cx_n|X-Y|^{1-n} \quad \text{if} \quad |X-Y| > y_n/2 .$$

Moreover from Theorem 2.3

$$\left|\frac{\partial G}{\partial n_T}(T,Y)\right| \leqq K|T-Y|^{1-n}, \quad \leqq Ky_n|T-Y|^{-n} .$$

This gives us for $a>0$

$$\left|\int_{|T'|\leqq a} G^*(T,X)\frac{\partial}{\partial n_T} G(T,Y)\,dS_T\right| \leqq \frac{Kx_n}{|X-Y|^{n-1}}\int_{|T'|<a}\frac{t_n}{|X-T|^n}\,dS_T$$

$$\leqq \frac{K_1x_n}{|X-Y|^{n-1}}\int_0^a \frac{\varepsilon(r)}{r}\,dr ,$$

$$\left|\int_{|T'|>a} G^*(T,X)\frac{\partial}{\partial n_T} G(T,Y)\,dS_T\right| \leqq K(a)\,x_ny_n .$$

Now choose a so small that

$$K_1 \int_0^a \frac{\varepsilon(r)}{r} dr < \tfrac{1}{4}C ,$$

and then y_n so small that

$$y_n^n K(a) < \tfrac{1}{8}C .$$

Then

$$K(a)y_n \leqq K(a)2(y_n - x_n) \leqq \tfrac{1}{4}C|Y-X|^{n-1}$$

and hence

$$G(X,Y) \geqq \frac{Cx_n}{|X-Y|^{n-1}} - \frac{K_1 x_n \int_0^a \varepsilon(r)/r\, dr}{|X-Y|^{n-1}} - K(a)\, x_n y_n$$

$$\geqq \frac{Cx_n}{2|X-Y|^{n-1}} .$$

With $x_n \to 0$ we get

$$\frac{\partial}{\partial n_O} G(O,Y) \geqq \tfrac{1}{2}C/|Y|^{n-1} .$$

The theorem then follows with the maximum principle and Harnack's inequality.

Remark 1. The proof of Theorem 2.5 is actually independent of the rest of the paper; in fact, the only use of the earlier parts made were the inequalities for $\partial G/\partial n_T$. But since E is convex, these follow immediately with the maximum principle.

Remark 2. The integrability condition on $\varepsilon(t)$ is necessary. To see that assume that for fixed Y

$$\partial G/\partial n_X \geqq C > 0, \qquad X \in \partial E .$$

Here E is defined using $\varepsilon(t)$ as above. Put

$$u_t(X) = \frac{x_n + t}{[|X'|^2 + (x_n+t)^2]^{\frac{1}{2}n}} .$$

Then

$$u_t(Y) = \frac{1}{\pi} \int_{\partial E} \frac{\partial G}{\partial n_X}(X,Y)\, u_t(X)\, dS_X$$

$$\geqq \frac{C}{\pi} \int_{\partial E} \frac{x_n + t}{[|X'|^2 + (x_n+t)^2]^{\frac{1}{2}n}} dS_X$$

$$\geqq K \int\limits_{\substack{\partial E \cap \{s_1 \geqq |X'| \geqq t\} \\ x_n < s_2}} \frac{|X'|\, \varepsilon(|X'|)}{[|X'|^2 + (k\varepsilon(|X'|)|X'| + |X'|)^2]^{\frac{1}{2}n}} dS_X$$

$$\geqq K \int\limits_t^{s_1} \frac{\varepsilon(r)}{r} dr \to \infty, \qquad t \to 0\,,$$

if

$$\int\limits_0^1 \frac{\varepsilon(r)}{r} dr = \infty\,,$$

whilst on the other hand

$$u_t(Y) \to y_n/|Y|^n\,.$$

REMARK 3. The inequalities for the Green function obtained in Theorems 2.3 and 2.5 are of course valid for the Green function of any second order homogeneous elliptic operator with constant coefficients. If we consider a class of such operators the ellipticity constants of which are all bounded below by $\lambda > 0$, the constants in the inequalities can be chosen uniformly with respect to the class, and depend only on λ and n. To see that we check that the only difference in the proofs will be the constants in the inequalities for the Green function of a half space, and these constants can be chosen uniformly with respect to the class, since the dilatation of distance is bounded above and below by λ^{-1} and λ when applying a linear transformation which takes a given operator into the Laplacian.

3.

In this section we study solutions of the equation

$$(*) \qquad a^{ij}(X)u_{ij} = F(X, u, u_i, u_{ij})$$

in a Liapunov region Ω. Apart from measurability, we shall make the following assumptions on a^{ij} and F:

$$|a^{ij}(X) - a^{ij}(Y)| \leqq K|X-Y|^\alpha, \qquad X \in \partial\Omega,\ Y \in \bar{\Omega},\ 0 < \alpha < 1\,,$$

$$\lambda|\xi|^2 \leqq a^{ij}(X)\,\xi_i\xi_j \leqq \lambda^{-1}|\xi|^2, \qquad X \in \bar{\Omega},\ \xi = (\xi_1, \ldots, \xi_n) \neq 0, \lambda > 0\,,$$

$$a^{ij} = a^{ji}\,,$$

$$|F(X, u, u_i, u_{ij})| \leqq K\delta^\alpha(X)|u_{ij}| + K\delta^{\alpha-1}(X)|u_i| + $$
$$+ K\varepsilon(\delta(X))\delta^{-1}(X)|u| + K\varepsilon(\delta(X))\delta^{-1}(X)\,,$$

where $\varepsilon(t)$ is a Dini function satisfying

$$\varepsilon(t)/t^{\gamma} \text{ decreasing for some } \gamma < 1 .$$

We shall think of a solution u as a function with two continuous derivatives in the interior of Ω, although it is possible to relax this condition (see [9, p. 487]). We construct a regularization of (*) by taking functions $\bar{a}^{ij}$ belonging to $C^{\infty}(\Omega)$ and $C^{\alpha}(\bar{\Omega})$, and satisfying $\bar{a}^{ij} = a^{ij}$ on $\partial\Omega$,

$$|\operatorname{grad} \bar{a}^{ij}| \leqq K\delta^{\alpha-1}(X) .$$

A solution $u(X)$ of (*) will then also be a solution of

$$\bar{a}^{ij}(X)u_{ij} = F(X,u,u_i,u_{ij}) + [\bar{a}^{ij}(X) - a^{ij}(X)]u_{ij} .$$

For the construction of the functions $\bar{a}^{ij}$, see Lemma 3.9 of [9].

In [9, p. 523] a mapping was constructed which maps a neighborhood Σ_{ϱ} of fixed size around any point $X_0 \in \partial\Omega$ onto a neighborhood of the origin in such a way that the image of $\Sigma_{\varrho} \cap \partial\Omega$ lies in the plane $y_n = 0$ and the image of Σ_{ϱ} contains a hemisphere $\{|X| < \sigma,\ 0 < y_n\}$, and such that to (*) corresponds an equation of the same type.

Before we continue with the theorem, we introduce some new notation. The integral $\int(\cdot)\, dX_{(i)}$ over a surface can be interpreted as $\int(\cdot) \cos\gamma_i\, dS$ where $\cos\gamma_i$ is the scalar product of the i-th unit vector and the normalized outer normal of the surface. By $\partial/\partial\nu$ we denote the conormal derivative $a^{ij} \cos\gamma_j\, \partial/\partial x_i$ where $a^{ij}\, \partial^2/\partial x_i \partial x_j$ is an operator with *constant* coefficients with respect to which we use Green's formula.

The summation convention is used freely.

THEOREM 3.1. *Let u be a solution of (*) in a Liapunov region Ω, continuous in $\bar{\Omega}$, and with the property that to every $X_0 \in \partial\Omega$ there is a linear polynomial $L_{X_0}(X)$ such that*

$$|u(X) - L_{X_0}(X)| \leqq \varepsilon(|X - X_0|)\,|X - X_0|, \qquad X \in \partial\Omega ,$$

where $\varepsilon(t)$ is a Dini function as above. Then the derivatives $\partial u/\partial x_i$ are continuous in $\bar{\Omega}$.

In particular, if $\varepsilon(t) = kt^{\alpha}$, then $\partial u/\partial x_i$ are α-Hölder continuous in $\bar{\Omega}$.

REMARK. We have used the same function $\varepsilon(t)$ in the assumptions on u and on the equation. This is of course not necessary, but can be achieved by taking at each point the maximum of $\varepsilon_1(t)$ and $\varepsilon_2(t)$ and taking the largest of the two γ's involved.

For the proof of the theorem we need some lemmata.

LEMMA 3.2. *Suppose u is a solution of (*) in a Liapunov region Ω, continuous in $\bar{\Omega}$. Suppose also that*

$$|u(X)-u(X^*)| \leqq K|X-X^*|^{\beta}, \qquad 0 \leqq \beta \leqq 1 .$$

Then for all $p \geqq 1$ *and all* $\gamma > 0$ *we have*

$$\iint_{\Omega} \delta^{p(1-\beta)-1+\gamma}(X)\, [|u_i|^p + \delta^p(X)\, |u_{ij}|^p]\, dX \;<\; \infty .$$

Proof. It is sufficient to assume $p > 1$ since the finiteness for $p = 1$ then follows by Hölder's inequality.

For a fixed $X_0 \in \Omega$ we put $v(X) = u(X) - u(X_0^*)$ and apply Lemma 3.7 (i) and (ii) of [9]. We get with $l \leqq \frac{1}{4}\delta(X_0)$

$$\left.\begin{array}{l} l^p \displaystyle\iint_{|X-X_0| \leqq \ell} |u_i|^p dX \;\leqq \\ l^{2p} \displaystyle\iint_{|X-X_0| \leqq \ell} |u_{ij}|^p dX \;\leqq \end{array}\right\} K \iint_{|X-X_0| \leqq 3l} \{|u(X)-u(X_0^*)|^p + \ell^{2p}(|F|^p + |[a^{ij}(X)-a^{ij}(X_0^*)]\, u_{ij}|^p)\}\, dX .$$

Now the proof follows the pattern of the proof of Theorem 4.1 in [9], and since the modifications are obvious we omit the rest.

Lemma 3.3. *Assume* u *is a solution of* (*), *with* $\varepsilon(t) = kt^{\alpha}$, *in a Liapunov region* Ω. *Suppose* $|\mathrm{grad}\, u|$ *is bounded, and that to every point* $X_0 \in \partial\Omega$ *there is a linear polynomial* $L_{X_0}(X)$ *such that*

$$|u(X) - L_{X_0}(X)| \;\leqq\; K\,|X - X_0|^{1+\beta}, \qquad 0 \leqq \beta < \alpha,\; X \in \Omega .$$

Then for every $p \geqq 1$ *and every* $\gamma > 0$

$$\iint_{\Omega} \delta^{p(1-\beta)-1+\gamma}\, |u_{ij}|^p\, dX \;<\; \infty .$$

The proof is identical to that of Lemma 3.2, with the exception that for a fixed X_0 we consider

$$v(X) \;=\; u(X) - L_{X_0^*}(X) .$$

Lemma 3.4. *Let* u *be as in Theorem* 3.1. *Then*

$$|\mathrm{grad}\, u| \;\leqq\; K\,\delta^{\gamma-1}(X)$$

for some $\gamma > 0$.

Proof. Choose $Y \in \Omega$ with $\delta(Y)$ small, say $< \frac{1}{2}s$. By the discussion in the beginning of this section we can assume that $Y^* = O$, that the intersection of $\partial\Omega$ with a neighborhood $\{|X| < s\}$ is plane, say, lies in $x_n = 0$, and that the set

$$D = \{X \mid -s < x_i < s, \ i = 1, \ldots, n-1, \ 0 < x_n < s\}$$

is a subset of Ω. We put

$$\partial' D = \partial D \cap \{x_n = 0\} \quad \text{and} \quad \partial'' D = \partial D \setminus \partial' D .$$

Now translate D along the positive x_n-axis by t to D_t, take the Green function of $x_n > t$ for the operator $a^{ij}(O)\, \partial^2/\partial x_i \partial x_j$ and apply Green's formula to it and u in D_t, use the fact that u is a solution of

$$a^{ij}(O) u_{ij} = F + (\bar{a}^{ij} - a^{ij}) u_{ij} + [a^{ij}(O) - \bar{a}^{ij}(X)] u_{ij} ,$$

integrate partially in

$$D_t \setminus \{X \mid |X - Y - (0, \ldots, t)| < \tfrac{1}{2}\delta(Y)\} ,$$

and let $t \to 0$ (cf. formulas 7.1.5–11 in [9]). After differentiating we get, modulo signs,

$$\begin{aligned}
\omega_n \frac{\partial}{\partial y_k} u(Y) = & \int_{\partial' D} u(X) \frac{\partial}{\partial y_k} \frac{\partial}{\partial \nu_X} G(X, Y)\, dS_X \, - \\
& - \int_{\partial' D} [\bar{a}^{nj}(X) - \bar{a}^{nj}(O)]\, u(X') \frac{\partial}{\partial y_k} \frac{\partial}{\partial x_j} G(X, Y)\, dX_{(i)} \, + \\
& + \int_{\partial'' D} \bar{a}^{ij}(X)\, u \frac{\partial}{\partial y_k} \frac{\partial}{\partial x_j} G(X, Y)\, dX_{(i)} - \bar{a}^{ij}(X)\, u_i \frac{\partial}{\partial y_k} G(X, Y)\, dX_{(j)} \, + \\
& + \iint_D \frac{\partial}{\partial y_k} G(X, Y)\, [F + (\bar{a}^{ij} - a^{ij}) u_{ij}]\, dX \, + \\
& + \int_{|X-Y| = \frac{1}{2}\delta(Y)} [\bar{a}^{ij}(X) - a^{ij}(O)]\, u \frac{\partial}{\partial y_k} \frac{\partial}{\partial x_j} G(X, Y)\, dX_{(i)} \, - \\
& \qquad - [\bar{a}^{ij}(X) - a^{ij}(O)]\, u_i \frac{\partial}{\partial y_k} G(X, Y)\, dX_{(j)} \, + \\
& + \int_{|X-Y| \leq \frac{1}{2}\delta(Y)} [\bar{a}^{ij}(X) - \bar{a}^{ij}(O)]\, u_{ij} \frac{\partial}{\partial y_k} G(X, Y)\, dX \, + \\
& + \iint_{D \cap \{|X-Y| > \frac{1}{2}\delta(Y)\}} \Big\{ [\bar{a}^{ij}(X) - a^{ij}(O)]\, u \frac{\partial}{\partial y_k} \frac{\partial^2}{\partial x_i \partial x_j} G(X, Y) \, + \\
& \qquad + \bar{a}_i^{ij}\, u \frac{\partial}{\partial y_k} \frac{\partial}{\partial x_j} G(X, Y) - \bar{a}_j^{ij}\, u_i \frac{\partial}{\partial y_k} G(X, Y) \Big\}\, dX .
\end{aligned}$$

By Section 2 of this paper we know that the first integral over $\partial' D$ is bounded, and since u is bounded it is easy to see that the second one is $\leqq K y_n^{\alpha-1}$. By Theorem 4.5 of [9] we know that $|\mathrm{grad}\, u| \leqq K\delta^{-1}(X)$ which implies that the integral over $\partial'' D$ is $\leqq K s^{-1}$. Using the boundedness of u and the Hölder continuity of $\bar{a}^{ij}$ we see that the integral over $\{|X-Y| = \delta(Y)/2\}$ is $\leqq K y_n^{\alpha-1}$.

The integral over D we split into two parts:

$$\left| \iint_{D \cap \{|X-Y| \leqq y_n/2\}} \ldots \, dX \right| \leqq K \iint_{|X-Y| \leqq y_n/2} |X-Y|^{1-n} \{x_n^{\alpha-2} + x_n^{\alpha} |u_{ij}|\} \, dX$$
$$\leqq K y_n^{\alpha-1} + y_n^{\alpha-1+(1-n)/p-\gamma_1/p} \left[\iint_D x_n^{2p-1+\gamma_1} |u_{ij}|^p \, dX \right]^{1/p}$$
$$\leqq K y_n^{\alpha-1-\gamma}$$

by Hölder's inequality and Lemma 3.2, where γ is made arbitrarily small by making p large enough.

$$\left| \iint_{D \cap \{|X-Y| \geqq y_n/2\}} dX \right| \leqq K \iint \frac{x_n}{|X-Y|^n} \{x_n^{\alpha-2} + x_n^{\alpha} |u_{ij}|\} \, dX$$
$$\leqq K \iint \frac{x_n^{\alpha-1}}{|X-Y|^n} dX + K \left[\iint x_n^{2p-1+\alpha} |u_{ij}|^p \, dX \right]^{1/p} \left[\iint \frac{x_n^{\alpha-1}}{|X-Y|^{nq}} \right]^{1/q}$$
$$\leqq K y_n^{\alpha-1-\gamma} .$$

The remaining two integrals are estimated in exactly the same way as the first and second part respectively of the one just treated, and the lemma is proved.

Proof of Theorem 3.1. We need a more easily handled formula than the one above, and we get one by omitting the extra partial integrations:

$$\omega_n \frac{\partial}{\partial y_k} u(Y)$$
$$= \int_{\partial' D} u(X) \frac{\partial}{\partial y_k} \frac{\partial}{\partial \nu_X} G(X,Y) \, dS_X + \int_{\partial'' D} u(X) \frac{\partial}{\partial y_k} \frac{\partial}{\partial \nu_X} G(X,Y) \, dS_X +$$
$$+ \iint_D \frac{\partial}{\partial y_k} G(X,Y) \, \{F + [a^{ij}(O) - a^{ij}(X)] u_{ij}\} \, dX .$$

Due to Lemma 3.3 and 3.2 where we now can take $\beta = \alpha - \gamma > 0$, the double integral converges, for any $\gamma > 0$.

We first prove that $|\mathrm{grad}\, u|$ is bounded by first using Lemma 3.4 and then make repeated use of Lemma 3.3 and 3.2 and estimations of the double integral, successively getting

$$|\operatorname{grad} u| \leqq K\,\delta^{m\alpha-\gamma-1}(X)\,,$$

$m=1,2,\ldots,N$, where $(N+1)\alpha>1$. In the next step the boundedness follows. The estimations are similar to those of Lemma 3.4 and will not be repeated.

In view of the boundedness of $|\operatorname{grad} u|$ the continuity and Hölder continuity of $\partial u/\partial x_i$ is a purely local affair, i.e. we need consider $|\partial u(Y)/\partial x_i - \partial u(Z)/\partial x_i|$ only when Z and Y are close to each other and the boundary. Our first step will be to show that

$$\left|\frac{\partial u}{\partial x_i}(Y)-\frac{\partial u}{\partial x_i}(Y^*)\right| \leqq K\int_0^{|Y-Y^*|}\frac{\varepsilon(t)}{t}\,dt + K|Y-Y^*|^{\alpha-\gamma} \tag{3.1.1}$$

for all $\gamma>0$, and as usual we shall assume $Y^*=O$, $Y=(0,\ldots,y_n)$ so that the formula above is valid. We define $\partial u/\partial x_k(O)$ by putting $Y=O$ in the formula; it is easy to check that the double integral so defined is convergent.

Since by Section 2 the surface integrals present no problems we concentrate on the double integral and split the domain of integration by taking away from D the set $B=\{|X-Y|\leqq 2y_n\}$. In $D\setminus B$ we have

$$\left|\frac{\partial}{\partial y_k}G(X,Y)-\frac{\partial}{\partial y_k}G(X,O)\right| \leqq \begin{cases} K y_n x_n |X|^{-1-n}\,,\\ K y_n |X|^{-n}\,,\end{cases}$$

whence

$$\left|\iint_{D\setminus B}\{F+[a^{ij}(O)-a^{ij}(X)]\,u_{ij}\}\left\{\frac{\partial}{\partial y_k}G(X,Y)-\frac{\partial}{\partial y_k}G(X,O)\right\}dX\right|$$

$$\leqq K\iint_{D\setminus B}\frac{y_n x_n}{|X-Y|^{n+1}}\{x_n^{-1}\varepsilon(x_n)+x_n^{\alpha-1}+|X|^{\alpha}|u_{ij}|\}\,dX$$

$$\leqq K y_n\iint_{y_n\leqq|X-Y|\leqq d}\frac{\varepsilon(|X-Y|)}{|X-Y|^{n+1}}\,dX + K y_n\iint_{|X-Y|\geqq 2y_n}|X-Y|^{\alpha-n-1}\,dX +$$

$$+ K y_n\left[\iint x_n^{p-1+\alpha}\,|u_{ij}|^p\,dX\right]^{1/p}\left[\iint_{|X-Y|\geqq 2y_n}\frac{x_n^{(1-\alpha)/(p-1)}|X|^{\alpha q}}{|X|^{q(n+1)}}\,dX\right]^{1/q}$$

$$\leqq K\varepsilon(y_n) + K y_n^{\alpha} + K y_n^{\alpha-\gamma}$$

$$\leqq K\int_0^{y_n}\frac{\varepsilon(t)}{t}\,dt + K y_n^{\alpha-\gamma},$$

$$\left| \iint_B \frac{\partial}{\partial y_k} G(X,O) \{F + [a^{ij}(O) + a^{ij}(X)]u_{ij}\} \, dX \right|$$

$$\leqq \iint_B \frac{x_n}{|X|^n} \{x_n^{-1} \varepsilon(x_n) + x_n^{\alpha-1} + |X|^\alpha \, |u_{ij}|\} \, dX$$

$$\leqq K \int_0^{3y_n} \frac{\varepsilon(t)}{t} dt + K y_n^\alpha + K y_n^{\alpha-\gamma} .$$

The other integral over B is treated similarly, and (3.1.1) is proved. We proceed to prove that if $|Z-Y| < \frac{1}{4}\delta(Y)$, then

$$\text{(3.1.2)} \qquad \left| \frac{\partial}{\partial y_k} u(Y) - \frac{\partial}{\partial y_k} (Z) \right| \leqq K \, |Z-Y|^{\alpha-\gamma} + K \int_0^{|Z-Y|} \frac{\varepsilon(t)}{t} dt, \qquad \gamma > 0 .$$

Since we still need consider the double integral only, we divide the domain of integration and use the following estimates for the difference $\partial/\partial y_k G(X,Y) - \partial/\partial z_k G(X,Z)$ in the respective parts: In

$\{|X-Y| \leqq \frac{1}{2}|Z-Y|\}$: $\quad K\,|X-Z|^{1-n} + K\,|Z-Y|^{1-n}$,

$\{|X-Z| \leqq \frac{1}{2}|Z-Y|\}$: $\quad K\,|X-Y|^{1-n} + K\,|Z-Y|^{1-n}$,

$\{|X-Z|, |X-Y| > \frac{1}{2}|Z-Y|, |X-Y| < \frac{1}{2}\delta(Y)\}$: $\quad K\,|Y-Z|^{1-n}$,

$\{|X-Y| > \frac{1}{2}\delta(Y)\}$: $\quad K\,|Z-Y|\,|X-Y|^{-n}$.

Now the inequalities (3.1.1) and (3.1.2) obviously imply continuity in the general case and Hölder continuity with exponent $\alpha-\gamma$ for all $\gamma > 0$ in the case $\varepsilon(t) = kt^\alpha$ (see also the proof of Theorem 2.4). To get α-Hölder continuity we apply Lemma 3.3 with $\beta = \alpha - \gamma$ and use the resulting inequality for the integrals containing u_{ij} above, and the theorem is proved.

Remark. By considering the function

$$u(X) = \int_0^{x_n} dy \int_y^1 \frac{\varepsilon(t)}{t} dt + K(2 - x_n) ,$$

which is a solution of

$$|\Delta u| \leqq x_n^{-1} \, \varepsilon(x_n) ,$$

$$|\Delta u| \leqq K \, x_n^{-1} \, \varepsilon(x_n) \, |u| ,$$

and

$$|\Delta u| \leqq K \, x_n^{-1} \, \varepsilon(x_n) \, |u_i|$$

in $0 < x_n < 1$ we see that we cannot relax the hypothesis on $\varepsilon(t)$, since $u_{x_n} \to \infty$ as $x_n \to 0$ if

$$\int_0^1 \frac{\varepsilon(t)}{t}\, dt = \infty ,$$

despite the fact that u is constant on $x_n = 0$.

It is probable though that the Hölder continuity of the a^{ij} can be weakened to a Dini condition, and that a corresponding relaxation may be made on the coefficient in front of $|u_i|$ in the assumptions on the equation.

The following theorem is proved with exactly the same method as above. In fact, the proof is a subset of the proofs given and so it will be omitted.

THEOREM 3.5. *Let u be a solution in a Liapunov region Ω of the equation*

$$a^{ij} u_{ij} = F(X, u, u_i, u_{ij})$$

where a^{ij} satisfy the same hypothesis as above, and F satisfies

$$|F| \leqq K\, [\delta^{\eta-2}(X) + \delta^{\eta-2}(X)\, |u| + \delta^{\alpha-1}(X)\, |u_i| + \delta^{\alpha}(X)\, |u_{ij}|]$$

with $0 < \eta < 1$. Suppose that u is continuous in $\bar{\Omega}$ and that the restriction to $\partial\Omega$ of u is η-Hölder continuous. Then u is η-Hölder continuous in $\bar{\Omega}$.

As mentioned in the introduction, results in this direction have been obtained for higher order equations by Agmon, Douglis, and Nirenberg in [1]. If Theorem 12.10 of [1] is specialized to the case of a second order operator and compared to Theorem 3.1 of this paper, one finds that they have assumed all the coefficients to belong to $C^{1+\alpha}$ while we have second order coefficients belonging to C^{α}, roughly, and lower order coefficients not even continuous, but with a limitation for their growth at the boundary. Our regularity assumption on the boundary is also weaker than theirs: $C^{1+\alpha}$ versus $C^{2+\alpha}$.

We would also like to mention a note by Browder [3] where he states a theorem assuring the Hölder continuity of the gradient assuming the coefficients to be continuous in the closed domain and the solution to be zero on the boundary, but since he also requires the right hand side of the linear equation to belong to L^p, $p > n$, one has to assume higher regularity than necessary of solutions which do not vanish identically on the boundary. Browder also requires the boundary to be of C^2 regularity.

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