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ON SOME TRIGONOMETRICAL POLYNOMIALS

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1. Introduction.

Let $n_1, n_2, \ldots, n_k$ be distinct integers in increasing order and let

$$M(k) = \inf_{0 \leqq x \leqq \pi} \left(\sum_{m=1}^{k} \cos n_m x \right).$$

N. C. Ankeny and S. Chowla [1] have made the following conjecture:

For any $K > 0$, there is an N such that $M(k) < -K$ for any $k > N$ and for any sequence (n_m).

This was solved by M. and S. Uchiyama [4]. One of us [3] proved the following

THEOREM I. *Let* $0 < \alpha < \frac{1}{2}$. *Then*

$$M(k) \leqq -k^{\alpha} \qquad \textit{for all sufficiently large } k ,$$

if the number of solutions (i, j, k) *of the equation*

$$n_i + n_j = n_k, \qquad i < j < k \leqq n ,$$

is less than $(\frac{1}{4} - \varepsilon) n^{2-\alpha}$ *for an* $\varepsilon > 0$.

Recently, S. Chowla [2] has proved the following theorem.

THEOREM C. *If* $n_m = O(m^{1+\varepsilon})$, $\varepsilon > 0$, *then*

$$\limsup_{k \to \infty} (-M(k)/k^{\beta}) > 0 \qquad \textit{for} \quad \beta < \alpha_0 , \tag{1}$$

α_0 *being the root in the interval* $(0, 1)$ *of the equation in* α:

$$\int_0^{3\pi/2} \cos x \cdot x^{-\alpha} dx = 0 . \tag{2}$$

The value of α_0 *is* $0.30844\ldots$.

The proof of Theorem C depends on a theorem of A. Selberg which is stated and proved in [2], i.e.

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THEOREM S. *If* (a_m) *is any sequence of non-negative numbers such that*

$$\sum_{m=0}^{k} a_m \cos mx \geqq 0 \qquad \text{for all } x \text{ and all } k \geqq 0\,, \tag{3}$$

then the series

$$\sum_{m=1}^{\infty} a_m/m^{1-\beta}$$

converges for $\beta < \alpha_0$.

We can prove that the theorem does not hold for $\beta = \alpha_0$ (in Section 4).

2. Theorems.

In this paper we shall prove the following theorems.

THEOREM 1. *Let* (a_m) *be a sequence of positive numbers such that*

$$\sum_{m=1}^{\infty} a_m/{n_m}^{1-\beta} = \infty \qquad \text{for some } \beta \;\; 0 \leqq \beta < \alpha_0\,,$$

α_0 *being the root of the equation* (2). *Then, for any* $\varepsilon > 0$,

$$\min_{0\leqq x\leqq \pi} \sum_{m=1}^{N} a_m \cos n_m x \leqq -(A(\beta)-\varepsilon)s_N \qquad \text{for infinitely many } N\,, \tag{4}$$

where

$$s_N = \sum_{m=1}^{N} a_m\,,$$

that is

$$\limsup_{N\to\infty} \left\{ -\min_{0\leqq x\leqq \pi} \left(\frac{1}{s_N} \sum_{m=1}^{N} a_m \cos n_m x \right) \right\} \geqq A(\beta)\,, \tag{5}$$

where

$$A(\beta) = -\int_0^{3\pi/2} \cos x \cdot x^{-\beta}\, dx \cdot (1-\beta) \left(\frac{2}{3\pi}\right)^{1-\beta} > 0\,.$$

In the particular case $\beta = 0$, we get

COROLLARY 1. *If* (a_m) *is a positive sequence such that*

$$\sum_{m=1}^{\infty} a_m/n_m = \infty\,,$$

then

$$\limsup_{N\to\infty}\left\{-\min_{0\leqq x\leqq\pi}\left(\frac{1}{s_N}\sum_{m=1}^{N}a_m\cos n_m x\right)\right\}\geqq\frac{2}{3\pi}. \tag{6}$$

If $a_m=1$ and (n_m) is an arithmetic sequence, then the left side of (6) is $2/(3\pi)$. Therefore, (6), and then (5), is best possible.

If we take $a_m=1/\log m$, then we get

COROLLARY 2. *If*

$$\sum_{m=2}^{\infty}1/(n_m\log m)=\infty,$$

then, for an $\varepsilon>0$,

$$\min_{0\leqq x\leqq\pi}\left(\sum_{m=2}^{N}\frac{\cos n_m x}{\log m}\right)\leqq-\left(\frac{2}{3\pi}-\varepsilon\right)\frac{N}{\log N}\quad \textit{for infinitely many } N.$$

The final corollary of Theorem 1 is

COROLLARY 3. *If*

$$\sum_{m=1}^{\infty}1/{n_m}^{1-\beta}=\infty\qquad \textit{for some } \beta,\quad 0\leqq\beta<\alpha_0,$$

α_0 *being the root of the equation* (2), *then*

$$\limsup_{N\to\infty}\left\{-\min_{0\leqq x\leqq\pi}\left(\frac{1}{N}\sum_{m=1}^{N}\cos n_m x\right)\right\}>0.$$

This contains Theorem C as a particular case.

For the proof of Theorem 1, we don't use Theorem S, but the idea of its proof is used. By our method we can give a slightly different proof of Theorem S (see Section 4). Both of Theorem 1 and Theorem S are easily derived from the identity (8) which is proved in the next section. Furthermore, we can generalize Theorem S in the following form.

THEOREM 2. *Let* $0\leqq\beta<\alpha_0$. *If* (a_m) *is any sequence of non-negative numbers, for which there are* λ *and* A *such that*

$$A>0,\qquad 0\leqq\lambda<1-\beta,\qquad \sum_{m=1}^{N}a_m\cos mx\geqq -AN^{\lambda}\ \textit{for all } N, \tag{7}$$

then the series

$$\sum_{m=1}^{\infty}a_m/m^{1-\beta}$$

converges.

If (7) *holds and*

$$s_m = a_1 + a_2 + \ldots + a_m \geqq 0 \qquad \text{for all } m\,,$$

then the series

$$\sum_{m=1}^{\infty} s_m/m^{2-\beta}$$

converges.

Theorem 2 does not hold for $\beta = \alpha_0$ and Theorem S is the special case $\lambda = 0$ of Theorem 2.

As a corollary of Theorem 2, we get

COROLLARY 4. *For any sequence* (a_m) *satisfying the condition* (3), *the series*

$$\sum_{m=1}^{\infty} s_m/m^{2-\beta}$$

converges, where

$$s_m = a_1 + a_2 + \ldots + a_m\,.$$

3. Proof of Theorem 1.

For $0 \leqq \beta < \alpha_0$, let $B(\beta)$ be defined by

$$B(\beta) = \int_0^{3\pi/2} \cos x \cdot x^{-\beta}\, dx < 0\,.$$

Then

$$B(\beta) \sum_{m=1}^{N} \frac{a_m}{n_m^{1-\beta}} = \sum_{m=1}^{N} \frac{a_m}{n_m^{1-\beta}} \int_0^{3\pi/2} \frac{\cos x}{x^\beta}\, dx \tag{8}$$

$$= \sum_{m=1}^{N} a_m \sum_{j=m}^{\infty} \int_{3\pi/2n_{j+1}}^{3\pi/2n_j} \frac{\cos n_m x}{x^\beta}\, dx$$

$$= \sum_{j=1}^{N} \sum_{m=1}^{j} + \sum_{j=N+1}^{\infty} \sum_{m=1}^{N}$$

$$= \sum_{j=1}^{N-1} \int_{3\pi/2n_{j+1}}^{3\pi/2n_j} \left(\sum_{m=1}^{j} a_m \cos n_m x \right) \frac{dx}{x^\beta} +$$

$$+ \int_0^{3\pi/2n_N} \left(\sum_{m=1}^{N} a_m \cos n_m x \right) \frac{dx}{x^\beta}.$$

If we put $s_j = \sum_{m=1}^{j} a_m$ and suppose that

$$M(j) = \min_{0 \leqq x \leqq \pi} \left(\sum_{m=1}^{j} a_m \cos n_m x \right) \geqq -\delta A(\beta) s_j \tag{9}$$

for all j, $1 \leqq j \leqq N$, and for a δ, $0<\delta<1$, then (8) gives

$$B(\beta)\sum_{m=1}^{N}\frac{a_m}{{n_m}^{1-\beta}} \geqq -\delta A(\beta)\left(\sum_{j=1}^{N-1} s_j \int_{3\pi/2n_{j+1}}^{3\pi/2n_j}\frac{dx}{x^\beta}+s_N\int_0^{3\pi/2n_N}\frac{dx}{x^\beta}\right)$$

$$= -\frac{\delta A(\beta)}{1-\beta}\left(\frac{3\pi}{2}\right)^{1-\beta}\left\{\sum_{j=1}^{N-1} s_j\left(\frac{1}{n_j^{1-\beta}}-\frac{1}{n_{j+1}^{1-\beta}}\right)+s_N\frac{1}{n_N^{1-\beta}}\right\}$$

$$= \delta B(\beta)\sum_{j=1}^{N}\frac{s_j-s_{j-1}}{{n_j}^{1-\beta}}$$

$$\geqq \delta B(\beta)\sum_{j=1}^{N}\frac{a_j}{{n_j}^{1-\beta}},$$

since $B(\beta)<0$. This is a contradiction, by $0<\delta<1$. Hence, for any δ, $0<\delta<1$, there is a_j, $1\leqq j\leqq N$, such that the relation (9) does not hold. By the divergence of the series

$$\sum_{m=1}^{\infty} a_m/{n_m}^{1-\beta},$$

we have

$$M(j) < -\delta A(\beta)s_j$$

for infinitely many j and some δ, $0<\delta<1$, that is

$$\limsup_{N\to\infty}\{-M(N)/s_N\} \geqq \delta A(\beta)\,.$$

Since δ is any positive number <1, we get the required relation (5).

4. Proof of Theorems S and 2.

In the relation (8), we take $n_m=m$ and suppose (7), then

$$B(\beta)\sum_{m=1}^{N}\frac{a_m}{m^{1-\beta}} \geqq \sum_{j=1}^{N}\int_{3\pi/2(j+1)}^{3\pi/2j}\left(\sum_{m=1}^{j} a_m\cos mx\right)\frac{dx}{x^\beta}+$$

$$+\int_0^{3\pi/2N}\left(\sum_{m=1}^{N} a_m\cos mx\right)\frac{dx}{x^\beta}$$

$$\geqq -A\left(\sum_{j=1}^{N} j^\lambda\int_{3\pi/2(j+1)}^{3\pi/2j}\frac{dx}{x^\beta}+N^\lambda\int_0^{3\pi/2N}\frac{dx}{x^\beta}\right)$$

$$\geqq -A\int_0^{3\pi/2}\frac{dx}{x^{\beta+\lambda}}$$

$$= -\frac{A}{1-\beta-\lambda}\left(\frac{3\pi}{2}\right)^{1-\beta-\lambda},$$

since $1-\beta-\lambda>0$, where A denotes a positive constant different in different occurrences. Hence we have

$$\sum_{m=1}^{N} \frac{a_m}{m^{1-\beta}} \leqq \frac{A}{1-\beta-\lambda}\left(\frac{3\pi}{2}\right)^{1-\beta-\lambda} \Big/ \left(-\int_0^{3\pi/2} \frac{\cos x}{x^\beta}\,dx\right) < \infty .$$

Thus, if $a_m \geqq 0$ for all m, then

$$\sum_{m=1}^{\infty} a_m/m^{1-\beta}$$

converges.

If $s_m \geqq 0$ for all m, then we have the identity

$$\sum_{m=1}^{N} \frac{a_m}{m^{1-\beta}} = \sum_{m=1}^{N-1} s_m \left(\frac{1}{m^{1-\beta}} - \frac{1}{(m+1)^{1-\beta}}\right) + \frac{s_N}{N^{1-\beta}},$$

where all terms on the right side are non-negative, and thus the series

$$\sum_{m=1}^{\infty} s_m/m^{2-\beta}$$

converges.

We shall prove that Theorem S (and hence Theorem 2) does not hold for $\beta=\alpha_0$. We put

$$a_0 = -\inf_{k,x} \sum_{m=1}^{k} m^{-\alpha_0} \cos mx ,$$

$$a_m = m^{-\alpha_0} \qquad \text{for } m \geqq 1 .$$

Then

$$\sum_{m=0}^{k} a_m \cos mx \geqq 0 \qquad \text{for all } k \text{ and all } x$$

(see [5, p. 191]), but the series

$$\sum_{m=1}^{\infty} a_m/m^{1-\alpha_0} = \sum_{m=1}^{\infty} 1/m = \infty .$$

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