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SLICE ALGEBRAS OF BOUNDED ANALYTIC FUNCTIONS

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1.

For function algebras A and B, let $S(A,B)$ denote the function algebra of all continuous functions f on the product $\mathfrak{M}_A \times \mathfrak{M}_B$ of the maximal ideal spaces of A and B, respectively, which satisfy

$$f(\cdot,N) \in \hat{A} \quad \text{for each} \quad N \in \mathfrak{M}_B ,$$
$$f(M,\cdot) \in \hat{B} \quad \text{for each} \quad M \in \mathfrak{M}_A .$$

Here $\hat{A}$ and $\hat{B}$ as usual denote the isomorphic Gelfand representations of A and B as function algebras on $\mathfrak{M}_A$ and $\mathfrak{M}_B$. During the discussion of [1] at the Tulane Symposium on Function Algebras, 1965, H. Rossi suggested comparing the λ-tensor product $H_\infty \otimes_\lambda H_\infty$ of the algebra of bounded analytic functions on the disc with the algebra $S(H_\infty,H_\infty)$ and the algebra $H_\infty(D \times D)$ of bounded analytic functions on the unit polydisc $D \times D$ in C^2. In [2], the relation between $H_\infty \otimes_\lambda H_\infty$ and $H_\infty(D \times D)$ is extensively investigated. The purpose of this note is to relate $S(H_\infty,H_\infty)$ to $H_\infty(D \times D)$ and the results of [2]. Our main result characterizes $S(H_\infty,H_\infty)$ as the subalgebra of $H_\infty(D \times D)$ consisting of those functions having continuous extensions to $\mathfrak{M}_{H_\infty} \times \mathfrak{M}_{H_\infty}$. In two complex dimensions the algebra $H_\infty \otimes_\lambda H_\infty$ will then play a role with respect to $H_\infty(D \times D)$ analogous to the role of the boundary value algebra with respect to H_∞ in one complex dimension.

2.

The algebra $S(H_\infty,H_\infty)$ restricted to $D \times D$ is clearly contained in $H_\infty(D \times D)$, since each function f in $S(H_\infty,H_\infty)$ is bounded on $\mathfrak{M}_{H_\infty} \times \mathfrak{M}_{H_\infty}$, which contains $D \times D$, and is separately, hence jointly, analytic there. Also each f in $S(H_\infty,H_\infty)$ is uniquely determined by its restriction to $D \times D$ and sup norms taken on $D \times D$ and $\mathfrak{M}_{H_\infty} \times \mathfrak{M}_{H_\infty}$ agree. Thus,

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via restriction to $D \times D$, the algebra $S(H_\infty, H_\infty)$ is isometrically isomorphic to a sub-algebra of $H_\infty(D \times D)$.

3.

Let T denote the unit circle. $H_\infty(D \times D)$ corresponds isomorphically and isometrically to the subalgebra $H_\infty(T \times T)$ of $L_\infty(T \times T)$ determined by the boundary values of functions in $H_\infty(D \times D)$. See [5]. By $L_\infty \otimes_\lambda L_\infty$ we mean the subalgebra of $L_\infty(T \times T)$ which is isometrically isomorphic, via the Gelfand representation, to $C(X \times X)$ where X is the maximal ideal space of $L_\infty(T)$. Alternatively, it can be described as the completion in $L_\infty(T \times T)$ of the algebra of elements of $L_\infty(T \times T)$ of the form

$$(s,t) \rightarrow \sum \alpha_i(s)\beta_i(t)$$

with $\sum$ denoting a finite sum. Let A be the closed subalgebra $H_\infty(T \times T) \cap L_\infty \otimes_\lambda L_\infty$ of $L_\infty(T \times T)$. Every function in this algebra can be represented as a continuous function on $X \times X$.

THEOREM 1. *The algebra* $A = H_\infty(T \times T) \cap L_\infty \otimes_\lambda L_\infty$ *is isometrically isomorphic to* $S(H_\infty, H_\infty)$.

PROOF. Let $S_y = \{f_y : X \rightarrow C \mid f_y(x) = f(x,y), f \in A\}$. Since

$$H_\infty \otimes 1 \subset L_\infty \otimes_\lambda L_\infty \cap H_\infty(T \times T),$$

$\hat{H}_\infty \subset S_y$. Let $\bar{S}_y$ denote the completion of S_y in the sup norm on X. Suppose $\bar{\imath}$ denotes the $L_\infty(T)$-function

$$\bar{\imath}(t) = -t \qquad (z \rightarrow \bar{z}).$$

Let $\bar{\imath}$ be represented by $\hat{\bar{\imath}} \in C(X)$; if $\hat{\bar{\imath}} \in \bar{S}_y$, then there exists a sequence $f_y^{(n)} \in S_y$ such that

$$\|f_y^{(n)} - \hat{\bar{\imath}}\|_\infty \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$$

(sup norm computed on X). Now $S_y \otimes 1 \subset A$ and

$$\|f_y^{(n)} \otimes 1 - \hat{\bar{\imath}} \otimes 1\|_\infty \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$$

on $X \times X$. This implies $\hat{\bar{\imath}} \otimes 1 \in A$, which is a contradiction, since $\bar{\imath} \otimes 1 \notin H_\infty(T \times T)$. Then, since $\hat{\bar{\imath}} \notin \bar{S}_y$ by [3, p. 193.], we have $\bar{S}_y = \bar{\hat{H}}_\infty$. So

$$\hat{H}_\infty = \bar{S}_y \supset S_y \supset \hat{H}_\infty$$

implies $f_y \in \hat{H}_\infty$. Similarly, $f_x \in \hat{H}_\infty$. Thus A consists of all continuous functions f such that $f(x, \cdot)$ and $f(\cdot, y)$ are boundary values of H_∞ on T.

Thus every function $f \in A$ can be extended to a function on $\mathfrak{M}_{H_\infty} \times \mathfrak{M}_{H_\infty}$. To see this, choose measures $M(dx)$ and $N(dx)$ on X representing the multiplicative functionals M and N in $\mathfrak{M}_{H_\infty}$. Then define f on $\mathfrak{M}_{H_\infty} \times \mathfrak{M}_{H_\infty}$ by

$$\hat{f}(M,N) = \int_{X \times X} f(x,y)\, M(dx) \times N(dy).$$

We next show that $M(dx) \times N(dy)$ is a multiplicative measure on A. Let $f,g \in A$,

$$\begin{aligned}(fg)\hat{}(M,N) &= \int_{X\times X} (fg)(x,y)\, M(dx) \times N(dy) \\ &= \int_X \left\{ \int_X f(x,y)\, g(x,y)\, M(dx) \right\} N(dy) \\ &= \int_X \left\{ \int_X f(x,y)\, N(dy) \int_X g(x,y)\, N(dy) \right\} M(dx),\end{aligned}$$

since $f(x,\cdot)$ and $g(y,\cdot)$ are in $\hat{H}_\infty$. Now consider

$$x \to \int_X f(x,y)\, N(dy)\,.$$

This function corresponds uniquely to a function

$$t \to \int_X f(t,y)\, N(dy)$$

in $L_\infty(T)$. To show this function is actually in $H_\infty(T)$, we compute its negative Fourier coefficients:

$$\int_T \int_X f(t,y)\, N(dy)\, e^{int} dt = \int_X \left(\int_T f(t,y)\, e^{int} dt \right) N(dy) = 0$$

for $n > 0$. Thus

$$x \to \int_X f(x,y)\, N(dy)$$

is in $\hat{H}_\infty$ and, similarly, so is

$$x \to \int_X g(x,y)\, N(dy).$$

Therefore,

$$\int_X \left\{ \int_X f(x,y)\, N(dy) \int_X g(x,y)\, N(dy) \right\} M(dx)$$

$$= \int_{X \times X} f(x,y)\, N(dy) \times M(dx) \int_{X \times X} g(x,y)\, N(dy) \times M(dx) .$$

As a consequence we have

$$(fg)^{\wedge}(M,N) = \hat{f}(M,N)\, \hat{g}(M,N) .$$

To recapitulate, we now have:

(i) Every multiplicative linear functional on $H_\infty \otimes_\lambda H_\infty$ (that is, every element of $\mathfrak{M}_{H_\infty} \times \mathfrak{M}_{H_\infty}$) extends to a multiplicative linear functional on A.

(ii) $f \in A$ implies $\hat{f}(\cdot,N) \in \hat{H}_\infty$ and $\hat{f}(M,\cdot) \in \hat{H}_\infty$ for each $M,N \in \mathfrak{M}_{H_\infty}$.

It remains to establish:

(iii) $f \in A$ implies $\hat{f} \in C(\mathfrak{M}_{H_\infty} \times \mathfrak{M}_{H_\infty})$.

(iv) $S(H_\infty, H_\infty)$ can be identified with a closed subalgebra of A.

(iii) is verified by showing that the weakest topology τ on

$$\{M(dx) \times N(dy) : \; M,N \in \mathfrak{M}_{H_\infty}\} \subset M(X \times X)$$

which renders $(M,N) \to \hat{f}(M,N)$ continuous for each $f \in A$ coincides with the weak* topology on $\mathfrak{M}_{H_\infty} \times \mathfrak{M}_{H_\infty}$ determined by $H_\infty \otimes_\lambda H_\infty$. Certainly τ is coarser than the weak* topology. And the sub-basic neighborhood of $M_0(dx) \times N_0(dy)$ given by

$$\left\{ M(dx) \times N(dy) : \left| \int f(x,y)\, M(dx) \times N(dy) - \int f(x,y)\, M_0(dx) \times N_0(dy) \right| < C\varepsilon \right\}$$

contains

$$\left\{ M(dx) : \left| \int f(x,y_i)\, M(dx) - \int f(x,y_i)\, M_0(dx) \right| < \varepsilon \right\} \times$$

$$\times \left\{ M(dy) : \left| \int f(x_i,y) N(dy) - \int f(x_i,y) N_0(dy) \right| < \varepsilon \right\}$$

where

$$\{f(\cdot,y_i) : \; i = 1,2,\ldots,n\} \quad \text{and} \quad \{f(x_i,\cdot) : \; i = 1,2,\ldots,n\}$$

are each ε-dense in $f(X,\cdot)$ and $f(\cdot,X)$. A straightforward process of estimating yields this assertion. Thus each $f \in A$ determines a continuous function $(M,N) \to \hat{f}(M,N)$ on $\mathfrak{M}_{H_\infty} \times \mathfrak{M}_{H_\infty}$.

For (iv), use the comments of Section 2, and the representation of elements in A by continuous functions on $X \times X$.

COROLLARY 1. *Let $\mathscr{L}^K(L_1(T)/H_1{}^0, H_\infty)$ denote the compact linear operators from $L_1(T)/H_1{}^0$ into H_∞ where $L_1(T)/H_1{}^0$ is the quotient space of L^1 whose dual is H_∞. Then $S(H_\infty, H_\infty)$ is isometrically isomorphic to $\mathscr{L}^K(L_1(T)/H_1{}^0, H_\infty)$.*

PROOF. By the above theorem and Theorem 2 of [2] Corollary 1 follows.

Although we defined the slice algebra $S(H_\infty, H_\infty)$ by restricting to slices of $\mathfrak{M}_{H_\infty} \times \mathfrak{M}_{H_\infty}$, for the proof of the theorem it is only necessary to require continuous representations of functions in $H_\infty(D \times D)$ on $X \times X$. Of course, this is because $X \times X$ is the Šilov boundary of $S(H_\infty, H_\infty)$ and $H_\infty \otimes_\lambda H_\infty$.

Thus far we have been unable to prove $S(H_\infty, H_\infty) = \hat{H}_\infty \otimes_\lambda \hat{H}_\infty$, but if this were not the case — as indicated in [2], the Banach basis problem would have a negative solution.

4.

Let $\mathfrak{M}$ be the maximal ideal space of $H_\infty(D \times D)$. $\mathfrak{M}$ is not homeomorphic to $\mathfrak{M}_{H_\infty} \times \mathfrak{M}_{H_\infty}$, the maximal ideal space of $H_\infty \otimes_\lambda H_\infty$. For, if it were, every function in $H_\infty(D \times D)$ would induce a compact operator from $L_1(T)/H_1{}^0$ into H_∞ by the Corollary, since $H_\infty(D \times D)$ would then coincide with $S(H_\infty, H_\infty)$. In [2], a large class of functions in $H_\infty(D \times D)$ which do not determine compact operators is exhibited. Also, L. Stout communicated a more direct proof of the fact $\mathfrak{M} \neq \mathfrak{M}_{H_\infty} \times \mathfrak{M}_{H_\infty}$ based on an observation about products of interpolating sets of $H_\infty(D)$ being interpolating sets for $H_\infty(D \times D)$.

Let $\pi: \mathfrak{M} \to \mathfrak{M}_{H_\infty} \times \mathfrak{M}_{H_\infty}$ be defined by $\pi(\varphi) = \varphi \,|\, H_\infty \otimes_\lambda H_\infty$. π is certainly continuous, $\pi(\mathfrak{M})$ is compact and $\pi(\mathfrak{M}) \supset D \times D$. Thus by Carleson's corona theorem

$$\mathfrak{M}_{H_\infty} \times \mathfrak{M}_{H_\infty} \supset \pi(\mathfrak{M}) \supset \overline{D \times D} \supset \mathfrak{M}_{H_\infty} \times \mathfrak{M}_{H_\infty} .$$

THEOREM 2. *The mapping π defined above is a continuous map of $\mathfrak{M}$ onto $\mathfrak{M}_{H_\infty} \times \mathfrak{M}_{H_\infty}$. Over $D \times D$, the mapping π is one-one and π^{-1} maps $D \times D$ homeomorphically onto an open subset of $\mathfrak{M}$.*

PROOF. The first statement follows from the discussion which preceded the Theorem, so we turn to a proof of the remaining claims. Suppose $\varphi \in \mathfrak{M}$ and $\pi(\varphi) = \varphi(0,0)$. The argument when $\pi(\varphi)$ is point evaluation other than at the origin can be reduced to this case or treated in a similar fashion. Let f be a bounded analytic function on $D \times D$ which vanishes at the origin and set

$$F = f - f(0,\cdot) - f(\cdot,0) + f(0,0)\,.$$

It will suffice to show $\varphi(F)=0$, for then $\varphi=\varphi(0,0)$. On $D \times D$,

$$F(z,w) = \sum_{i,j \geqq 1}^{\infty} a_{ij}\, z^i w^j\,,$$

where $\{a_{ij}\}$ are the coefficients of the power series expansion of f. Convergence is uniform on compact subsets of $D \times D$. Thus

$$F(z,w) = z\, w\, g(z,w)$$

with $g \in \mathrm{Hol}(D \times D)$. Trivially, g is bounded on the complement in $D \times D$ of a neighborhood of

$$\{(z,w):\ |w|=1,\ z=0\} \cup \{(z,w):\ |z|=1,\ w=0\}\,,$$

and, in fact, g is bounded on the trace of this neighborhood on $D \times D$. To see this we examine the growth of g near points where $z=0$ and $0<r \leqq |w|<1$, the other case being similar:

$$g(0,w) = \lim_{z\to 0} \frac{F(z,w)}{z\,w} = \frac{1}{w}\frac{\partial F}{\partial z}(0,w) = \frac{1}{2\pi i w}\int_{|\xi|=1} \frac{F(\xi,w)}{\xi^2}\, d\xi$$

and the integral implies

$$\left|\frac{\partial F}{\partial z}(0,w)\right| \leqq \|F\|_\infty \quad \text{for all} \quad w \in D\,,$$

which shows that $z^{-1}w^{-1}F(z,w)$ is uniformly bounded near these points. Thus $g \in H_\infty(D \times D)$. Therefore,

$$\varphi(F) = \varphi(z)\,\varphi(w)\,\varphi(g) = 0\cdot\varphi(g) = 0\,.$$

That π^{-1} is a homeomorphism is a direct consequence of the fact that

$$H_\infty(D \times D)|K \cong H_\infty \otimes_\lambda H_\infty|K$$

for every compact subset K of $D \times D$.

THEOREM 3. *Let $f \in H_\infty(D \times D)$ and let $(M,N) \in \mathfrak{M}_{H_\infty} \times \mathfrak{M}_{H_\infty}$. Let (z_δ, w_δ) be a net in $D \times D$ converging to (M,N), and suppose that $\vartheta = \lim f(z_\delta, w_\delta)$ exists. Then there is a multiplicative functional $\varphi \in \pi^{-1}(M,N)$ such that $\varphi(f)=\vartheta$.*

PROOF. Let J be the set of functions g in $H_\infty(D\times D)$ for which $\lim g(z_\delta, w_\delta)=0$. Then J is an ideal in $H_\infty(D\times D)$ and therefore is contained in the kernel of a multiplicative functional $\varphi \in \mathfrak{M}$, that is, $\varphi(g)=0$ for each $g\in J$. Observe that

$$\{\hat{f}-\hat{f}(M,N) : f\in H_\infty \otimes_\lambda H_\infty\} \subset J \quad \text{and} \quad f-\vartheta \in J\,.$$

Thus $\pi(\varphi)=\varphi(M,N)$ and $\varphi(f)=\vartheta$.

If the closure of $D\times D$ in $\mathfrak{M}$ exhausts $\mathfrak{M}$ (i.e., the Corona conjecture is true for $H_\infty(D\times D)$), then we would be able to conclude immediately that $S(H_\infty, H_\infty)$ "is" precisely those functions in $H_\infty(D\times D)$ which are constant on all fibers $\pi^{-1}(M,N)$ with $(M,N)\in \mathfrak{M}_{H_\infty}\times \mathfrak{M}_{H_\infty}$. (For example, see [4].) Without the benefit of a two complex dimensional Corona theorem, however, we do have:

THEOREM 4. *Every function $\hat{f}$ in $H_\infty(D\times D)\hat{}$ which is a constant on all fibers $\pi^{-1}(M,N)$ for $(M,N)\in \mathfrak{M}_{H_\infty}\times \mathfrak{M}_{H_\infty}$ is an extension of a function f in $S(H_\infty,H_\infty)\,|\,D\times D$.*

PROOF. If f is constant on $\pi^{-1}(M,N)$ for every $(M,N)\in \mathfrak{M}_{H_\infty}\times \mathfrak{M}_{H_\infty}$, then the last theorem shows that $\hat{f}\,|\,D\times D$ has a continuous extension to $D\times D\cup\{(M,N)\}$. If $\hat{f}\,|\,D\times D$ extends continuously to $D\cup X\times X$, the theorem of Section 3 suffices for f to correspond in a natural fashion to an element of $S(H_\infty,H_\infty)$.

REMARK. Certainly, every $f\in H_\infty\otimes_\lambda H_\infty$ is constant on each fiber $\pi^{-1}(M,N)$ when regarded as an element $\hat{f}$ of $H_\infty(D\times D)\hat{}$. Therefore, either of the two results

(1) $S(H_\infty,H_\infty)=H_\infty\otimes_\lambda H_\infty$

or

(2) the Corona theorem for $H_\infty(D\times D)$

would provide the converse of the last theorem.

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