MATH. SCAND. 21 (1967), 90–104

ON THE ABSOLUTE CONVERGENCE OF A SERIES ASSOCIATED WITH A FOURIER SERIES

S. M. MAZHAR

1.

Let $f(t)$ be integrable (L) over $(-\pi,\pi)$ and periodic with period 2π and let

$$f(t) \sim \tfrac{1}{2}a_0 + \sum_1^\infty (a_n \cos nt + b_n \sin nt) = \sum A_n(t) . \tag{1.1}$$

Numbers x and s being fixed, we write

$$\varphi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)-2s\} ,$$

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\, \varphi(u)\, du, \qquad \alpha > 0 ,$$

$$\varphi_\alpha(t) = \Gamma(\alpha+1)\, t^{-\alpha}\, \Phi_\alpha(t), \qquad \varphi_0(t) = \varphi(t) ,$$

and

$$s_n = \sum_0^n A_k = \sum_0^n A_k(x) .$$

2.

In this note we are concerned with the series

$$\sum (s_n - s)/n . \tag{2.1}$$

Cesàro summability of this series was first investigated by Hardy and Littlewood [5, p. 238, Theorem 5], whereas a necessary and sufficient condition for its convergence was given by Zygmund [10, p. 61]. Recently Mohanty and Mohapatra [8] have investigated the absolute convergence and summability $|C,\delta|$ of this series. In this connection they proved the following theorems:

Theorem A. *If*

(i) $\varphi_1(t) \log \frac{k}{t} \in \mathrm{BV}(0,\pi)$,

Received August 15, 1964; revised November 21, 1966.

(ii) $\int_0^\pi \frac{|\varphi_1(t)|}{t}\, dt < \infty$,

(iii) $\{n^\delta A_n\} \in \mathrm{BV}$ *for some* $\delta > 0$,

then the series (2.1) *is absolutely convergent.*

THEOREM B. *If the series* (2.1) *is absolutely convergent, then*

$$\int_0^\pi \frac{|\varphi_{1+\delta}(t)|}{t}\, dt < \infty, \qquad \delta > 0 .$$

THEOREM C. *If*

$$\int_0^\pi \frac{|\varphi(t)|}{t}\, dt < \infty ,$$

then the series (2.1) *is summable* $|C,\delta|$, $\delta > 0$.

The object of the present paper is to generalise these theorems of Mohanty and Mohapatra.

3.

We prove the following theorems.

THEOREM 1. *A necessary and sufficient condition that the series* (2.1) *be absolutely convergent, whenever*

$$\varphi_1(t) \log \frac{k}{t} \in \mathrm{BV}(0,\pi) ,$$

is that

$$\begin{cases} \text{(i)} \ \displaystyle\int_0^\pi \frac{|\varphi_2(t)|}{t}\, dt < \infty , \\ \text{(ii)} \ \displaystyle\left\{\frac{1}{e^{n^\alpha}} \sum_1^n e^{m^\alpha} \frac{s_m}{m}\right\} \in \mathrm{BV} , \end{cases}$$

where $0 < \alpha < 1$.

THEOREM 2. *If the series* (2.1) *is summable* $|C,\alpha|$, $\alpha \geqq 0$, *then*

$$\int_0^\pi \frac{|\varphi_\beta(t)|}{t}\, dt < \infty ,$$

where $\beta > \alpha + 1$.

THEOREM 3. *If*

$$\int_0^\pi \frac{|\varphi_\alpha(t)|}{t}\,dt < \infty, \quad \alpha \geqq 0,$$

then the series (2.1) *is summable* $|C,\beta|$, *where* $\beta > \alpha$.

4.

The following lemmas are pertinent for the proof of these theorems:

LEMMA 1 (Bosanquet [2]). *If* $t^{-1}\varphi_\alpha(t) \in L(0,\pi)$, *then* $t^{-1}\varphi_\beta(t) \in L(0,\pi)$, *where* $\beta > \alpha \geqq 0$.

LEMMA 2 (Mohanty [7]). *If* $\varphi(t)\log(t^{-1}k) \in \mathrm{BV}(0,\pi)$, $k > \pi$, *then the Fourier series* (1.1) *at* $t=x$ *is summable* $|R,e^{n^\alpha},1|$, *where* $0<\alpha<1$.

LEMMA 3 (Mohanty and Mohapatra [8]). *If* $\varphi(t)\log(t^{-1}k) \in \mathrm{BV}(0,\pi)$, $k>\pi$, *and* $t^{-1}\varphi(t) \in L(0,\pi)$, *then the series* (2.1) *is absolutely convergent.*

LEMMA 4. *If* $\varphi_\alpha(t)\log(t^{-1}k) \in \mathrm{BV}(0,\pi)$, $k>\pi$, *then* $\varphi_\beta(t)\log(t^{-1}k) \in \mathrm{BV}(0,\pi)$, *where* $\beta > \alpha \geqq 0$.

PROOF. (The special case: $\alpha = 0$, $\beta = 1$, is due to Mohanty [7].) Since

$$\Phi_\beta(t) = \frac{1}{\Gamma(\beta-\alpha)}\int_0^t (t-u)^{\beta-\alpha-1}\,\Phi_\alpha(u)\,du,$$

we have

$$\varphi_\beta(t)\log\frac{k}{t} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)\Gamma(\alpha+1)}\int_0^t (t-u)^{\beta-\alpha-1}\frac{u^\alpha}{t^\beta}\log\frac{k}{t}\,\varphi_\alpha(u)\,du$$

$$= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)\Gamma(\alpha+1)}\int_0^1 (1-x)^{\beta-\alpha-1}x^\alpha\log\frac{k}{t}\,\varphi_\alpha(xt)\,dx.$$

Let $0<t_0<t_1<t_2<\ldots<t_n=\pi$. Then

$$\sum_0^{n-1}\left|\varphi_\beta(t_{m+1})\log\frac{k}{t_{m+1}} - \varphi_\beta(t_m)\log\frac{k}{t_m}\right|$$

$$\leqq \sum_{m=0}^{n-1}\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)\Gamma(\alpha+1)}\int_0^1 x^\alpha(1-x)^{\beta-\alpha-1}\left|\log\frac{k}{t_{m+1}}\varphi_\alpha(xt_{m+1}) - \log\frac{k}{t_m}\varphi_\alpha(xt_m)\right|dx$$

$$\leqq \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)\Gamma(\alpha+1)}\int_0^1 x^\alpha(1-x)^{\beta-\alpha-1}\sum_{m=0}^{n-1}\left|\log\frac{k}{xt_{m+1}}\varphi_\alpha(xt_{m+1}) - \log\frac{k}{xt_m}\varphi_\alpha(xt_m)\right|dx$$

$$+\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)\Gamma(\alpha+1)}\int_0^1 x^\alpha(1-x)^{\beta-\alpha-1}\log\frac{1}{x}\sum_{m=0}^{n-1}|\varphi_\alpha(xt_{m+1})-\varphi_\alpha(xt_m)|\,dx$$

$$= O\left\{\int_0^1 x^\alpha(1-x)^{\beta-\alpha-1}dx\right\}+O\left\{\int_0^1 x^\alpha(1-x)^{\beta-\alpha-1}\log\frac{1}{x}\,dx\right\} = O(1)\,,$$

by hypothesis.

LEMMA 5. *If*

(i) $\varphi_1(t)\log(t^{-1}k)\in \mathrm{BV}(0,\pi),\ k>\pi$,
(ii) $t^{-1}\varphi_2(t)\in L(0,\pi)$,

then the series (2.1) *is summable* $|R,e^{n^\alpha},1|$, *where* $0<\alpha<1$.

PROOF. We have

$$\begin{aligned}\frac{s_n-s}{n} &= \frac{2}{n\pi}\int_0^\pi \varphi(t)\,\frac{\sin(n+\frac12)t}{2\sin\frac12 t}\,dt\\ &= \frac{1}{\pi}\int_0^\pi \varphi(t)\cot\tfrac12 t\,\frac{\sin nt}{n}\,dt+\frac{1}{\pi}\int_0^\pi\varphi(t)\,\frac{\cos nt}{n}\,dt\\ &= \alpha_n+\beta_n,\quad \text{say}\,.\end{aligned}$$

Now integrating by parts we have

$$\begin{aligned}\alpha_n &= -\frac{1}{\pi}\int_0^\pi \Phi_1(t)\cos nt\cot\tfrac12 t\,dt+\frac{1}{\pi}\int_0^\pi \Phi_1(t)\,\frac{\sin nt}{2n\sin^2\frac12 t}\,dt\\ &= -\gamma_n+\delta_n,\quad \text{say}\,.\end{aligned}$$

From hypothesis (i) we observe that $\Phi_1(t)\cot\frac12 t\log(t^{-1}k)\in \mathrm{BV}(0,\pi)$ and therefore by Lemma 2, $\sum\gamma_n$ is summable $|R,e^{n^\alpha},1|$, $0<\alpha<1$. Again integrating by parts we have

$$\begin{aligned}\delta_n &= -\frac{1}{\pi}\int_0^\pi \Phi_2(t)\,\frac{\cos nt}{2(\sin\frac12 t)^2}\,dt+\frac{1}{\pi}\int_0^\pi \Phi_2(t)\,\frac{\sin nt}{2n}\,\frac{\cos\frac12 t}{(\sin\frac12 t)^3}\,dt\\ &= -p_n+q_n,\quad \text{say}\,.\end{aligned}$$

By Lemma 4

$$\Phi_2(t)\log\frac{k}{t}\cdot\frac{1}{(\sin\frac12 t)^2}\in \mathrm{BV}(0,\pi)\,,$$

hence applying Lemma 2, we find that $\sum p_n$ is summable $|R,e^{n^\alpha},1|$, $0<\alpha<1$.

Also

$$q_n = \frac{1}{\pi}\int_0^\pi t\Phi_2(t)\,\frac{\cos\frac12 t}{2(\sin\frac12 t)^3}\,\frac{\sin nt}{nt}\,dt\,.$$

Now Lemma 3 asserts that if

$$\chi(t)\log\frac{k}{t}\in \mathrm{BV}(0,\pi) \quad\text{and}\quad \frac{\chi(t)}{t}\in L(0,\pi),$$

then

$$\sum |t_n| < \infty ,$$

where

$$t_n = \frac{1}{\pi}\int_0^\pi \chi(t)\frac{\sin nt}{nt}\,dt .$$

Taking

$$\chi(t) = \frac{t\Phi_2(t)\cos\frac{1}{2}t}{2(\sin\frac{1}{2}t)^3},$$

it follows using also hypothesis (ii) that $\sum|q_n|<\infty$ and hence summable $|R,e^{n^\alpha},1|$.

We will now consider β_n.

$$\begin{aligned}\beta_n &= \frac{1}{\pi}\int_0^\pi \varphi(t)\frac{\cos nt}{n}\,dt\\ &= \frac{1}{\pi}\int_0^\pi \frac{\cos nt}{n}\left\{\frac{d}{dt}\left(\frac{t\varphi_1(t)\log(t^{-1}k)}{\log(t^{-1}k)}\right)\right\}dt\\ &= \frac{1}{\pi}\int_0^\pi \frac{\cos nt}{n}\,\frac{t}{\log(t^{-1}k)}\,\frac{d}{dt}\left(\varphi_1(t)\log\frac{k}{t}\right)dt+\frac{1}{\pi}\int_0^\pi\frac{\cos nt}{n}\,\varphi_1(t)\left(1+\frac{1}{\log(t^{-1}k)}\right)dt\\ &= X_n+Y_n, \quad \text{say}.\end{aligned}$$

From Lemma 2, $\sum nY_n$ is summable $|R,e^{n^\alpha},1|$ and hence $\sum Y_n$ is also summable $|R,e^{n^\alpha},1|$. Also by proceeding as in the proof of Theorem 5A (Mohanty [7]) we easily see that $\sum X_n$ is summable $|R,e^{n^\alpha},1|$.

This completes the proof of the lemma.

Lemma 6 (Mohanty [6]). *Let $\mu_m>0$, $\lambda_m=\sum_1^m\mu_n$, and*

$$d_m = \frac{1}{\lambda_m}\sum_1^m \mu_n c_n .$$

Then, if $\{c_m\}$ is a sequence of bounded variation, the sequence $\{d_m\}$ is also of bounded variation.

Lemma 7. *If the sequence*

$$\left\{e^{-n^\alpha}\sum_1^n e^{m^\alpha}a_m\right\}$$

is of bounded variation, then the sequence

$$\left\{e^{-n^\alpha}\sum_1^n e^{m^\alpha} m^{-1}t_m\right\}$$

is also of bounded variation, where $t_m=\sum_1^m a_\nu$, $0<\alpha<1$.

PROOF. We have

$$\frac{1}{e^{n^\alpha}}\sum_1^n\frac{e^{m^\alpha}}{m}\sum_1^m\frac{a_\nu e^{\nu^\alpha}}{e^{\nu^\alpha}} = \frac{1}{e^{n^\alpha}}\sum_1^n\frac{e^{m^\alpha}}{m}\left\{\sum_1^{m-1}\Delta(e^{-\nu^\alpha})\sum_1^\nu a_s e^{s^\alpha}+e^{-m^\alpha}\sum_1^m a_s e^{s^\alpha}\right\}.$$

$$= R_1+R_2, \quad \text{say}.$$

But

$$R_2 = \left(\frac{1}{e^{n^\alpha}}\sum_1^n\frac{e^{m^\alpha}}{m}\right)\frac{\{\sum_1^n e^{m^\alpha}m^{-1}\, e^{-m^\alpha}\sum_1^m a_s e^{s^\alpha}\}}{\sum_1^n e^{m^\alpha}m^{-1}}$$

$$= R_{21}R_{22}, \quad \text{say}.$$

From hypothesis and Lemma 6, R_{22} is of bounded variation and it is therefore sufficient to prove that

$$\left\{\frac{1}{e^{n^\alpha}}\sum_1^n e^{m^\alpha}m^{-1}\right\}\in \mathrm{BV}. \tag{4.1}$$

The proof is just like the proof given below of

$$\left\{\frac{1}{e^{n^\alpha}}\sum_1^n m^{\alpha-1}e^{m^\alpha}\right\}\in \mathrm{BV}. \tag{4.2}$$

Also

$$R_1 = \frac{1}{e^{n^\alpha}}\sum_1^n\frac{e^{m^\alpha}}{m^\alpha m^{1-\alpha}}\sum_1^{m-1}e^{\nu^\alpha}\Delta(e^{-\nu^\alpha})\frac{\sum_1^{m-1}\Delta(e^{-\nu^\alpha})\,e^{\nu^\alpha}\,(e^{-\nu^\alpha}\sum_1^\nu a_s e^{s^\alpha})}{\sum_1^{m-1}e^{\nu^\alpha}\Delta(e^{-\nu^\alpha})}$$

$$= \frac{1}{e^{n^\alpha}}\sum_1^n e^{m^\alpha}m^{\alpha-1}\frac{1}{m^\alpha}\sum_1^{m-1}e^{\nu^\alpha}\Delta(e^{-\nu^\alpha})\psi(m)$$

$$= \frac{1}{e^{n^\alpha}}\sum_1^n e^{m^\alpha}m^{\alpha-1}\frac{\sum_1^n e^{m^\alpha}\, m^{\alpha-1}\,\psi(m)\, m^{-\alpha}\sum_1^{m-1}e^{\nu^\alpha}\Delta(e^{-\nu^\alpha})}{\sum_1^n e^{m^\alpha}m^{\alpha-1}},$$

where $\psi(m)\in \mathrm{BV}$, and thus R_1 will be of bounded variation provided (4.2) holds and

$$\left\{\frac{1}{m^\alpha}\sum_1^{m-1}e^{\nu^\alpha}\Delta(e^{-\nu^\alpha})\right\}\in \mathrm{BV}. \tag{4.3}$$

PROOF OF (4.2). Since

$$\frac{1}{e^{n^\alpha}}\sum_1^n e^{m^\alpha} m^{\alpha-1} = \frac{e^{n^\alpha}-1}{e^{n^\alpha}}\,\frac{1}{e^{n^\alpha}-1}\sum_1^n \frac{(e^{m^\alpha}-e^{(m-1)^\alpha})e^{m^\alpha} m^{\alpha-1}}{(e^{m^\alpha}-e^{(m-1)^\alpha})}$$

and

$$e^{m^\alpha} m^{\alpha-1}/(e^{m^\alpha}-e^{(m-1)^\alpha}) \in \mathrm{BV}\,,$$

the result follows from Lemma 6.

PROOF OF (4.3). In this case we have

$$\frac{1}{m^\alpha}\sum_1^{m-1} e^{\nu^\alpha}\Delta(e^{-\nu^\alpha}) = \frac{1}{m^\alpha}\sum_1^{m-1}\frac{e^{(\nu+1)^\alpha}-e^{\nu^\alpha}}{e^{(\nu+1)^\alpha}}\cdot\frac{(\nu+1)^{\alpha-1}}{(\nu+1)^{\alpha-1}}$$

$$= \frac{1}{m^\alpha}\sum_1^{m-1}(\nu+1)^{\alpha-1}\frac{\sum_1^{m-1}(e^{(\nu+1)^\alpha}-e^{\nu^\alpha})(\nu+1)^{\alpha-1}/(\nu+1)^{\alpha-1}e^{(\nu+1)^\alpha}}{\sum_1^{m-1}(\nu+1)^{\alpha-1}}\,.$$

Since

$$e^{\nu^\alpha}\nu^{\alpha-1}/(e^{\nu^\alpha}-e^{(\nu-1)^\alpha}) \in \mathrm{BV}$$

and is greater than some fixed positive number, it follows that

$$(e^{\nu^\alpha}-e^{(\nu-1)^\alpha})/e^{\nu^\alpha}\nu^{\alpha-1} \in \mathrm{BV}\,.$$

Also it can easily be seen that

$$\frac{1}{m^\alpha}\sum_1^m \nu^{\alpha-1} \in \mathrm{BV}\,.$$

Applying Lemma 6, the result is obvious.

LEMMA 8 (Bhatt [1]). *If*

(i) $\sum a_n$ *is summable* $|R,\lambda,k|$, $k>0$,

(ii) $\left\{\frac{1}{\lambda_m}\sum_1^m a_n\lambda_n\right\} \in \mathrm{BV}$,

and

(iii) $\left\{\frac{\lambda_n}{\lambda_{n+1}}\right\} \in \mathrm{BV}$,

then $\sum a_n$ *is absolutely convergent.*

LEMMA 9 (Pati [9]; Zygmund [10, p. 258]). *Let* $k_n^\beta(t)$ *denote the n-th Cesàro mean of order* β *of the series*

$$\tfrac{1}{2}+\sum_1^\infty \cos nt\,,$$

then

$$\frac{d^\varrho}{dt^\varrho} k_n{}^\beta(t) = \begin{cases} O(n^{\varrho+1}), & 0 < t \leqq n^{-1}, \\ O(t^{-1-\beta} n^{\varrho-\beta}) + O(n^{-1} t^{-\varrho-2}), & n^{-1} < t \leqq \pi, \end{cases}$$

where ϱ is zero or a positive integer and $\beta \geqq 0$.

Lemma 10 (Bosanquet [3]). *Let*

$$\gamma_\alpha(x) = \int_0^1 (1-u)^{\alpha-1} \cos xu \, du ,$$

then for $\sigma > 0$, $t > 0$, $n > 0$, we have

$$\left| \Delta^p \left(\frac{d}{dt}\right)^\lambda \gamma_\sigma(nt) \right| \leqq \begin{cases} A n^\lambda t^p & \lambda \geqq 0,\ p \geqq 0, \\ A n^{-p-2} t^{-\lambda-2}, & p+\lambda \leqq \sigma - 2, \\ A n^{\lambda-\sigma} t^{p-\sigma}, & p+\lambda > \sigma - 2. \end{cases}$$

Lemma 11 (Hardy [4, p. 101]). *If $\alpha > -1$ and $\sum a_n$ is summable (C,α) then $S_n{}^{\alpha'} = o(n^\alpha)$, where $\alpha' < \alpha$ and $S_n{}^\alpha$ is the n-th Cesàro sum of order α of the series $\sum a_n$.*

Lemma 12. *Let $A_n{}^\alpha = \binom{n+\alpha}{\alpha}$ and*

$$J_m(t) = \sum_{n=m}^{\infty} A_{n-m}^{h-\alpha} \Delta^{h+1} \{(n+1)\Delta\gamma_\beta(nt)\} ,$$

then

$$J_m(t) = \begin{cases} O(t^\alpha) , \\ O(t^{1+\alpha-\beta} m^{1-\beta}) , \end{cases}$$

where $0 < \alpha < \beta - 1 < h+1$ and $h = [\alpha]$.

Proof. We split J_m as follows,

$$J_m(t) = \sum_{m}^{m+p} + \sum_{m+p+1}^{\infty} = \Sigma_1 + \Sigma_2, \quad \text{say} ,$$

where $p = [1/t]$. Now

$$|\Sigma_1| = O\left[\sum_{m}^{m+p} (n-m+1)^{h-\alpha} t^{h+1} \min\big(1, (mt)^{1-\beta}\big)\right]$$
$$= O[t^\alpha \min\big(1, (mt)^{1-\beta}\big)] ,$$

by Lemma 10. Further

$$|\Sigma_2| \leqq A_{p+1}^{h-\alpha} \max_{N > m+p+1} \left| \sum_{m+p+1}^{N} \Delta^{h+1}\{(n+1)\Delta\gamma_\beta(nt)\} \right|$$
$$= O\Big[t^{\alpha-h} t^h \min\Big(1, \big((m+p)t\big)^{1-\beta}\Big)\Big]$$
$$= O\Big[t^\alpha \min\big(1, (mt)^{1-\beta}\big)\Big] .$$

This completes the proof of the lemma.

5.

REMARKS. (1) By virtue of Lemma 1, Condition (ii) of Theorem A implies Condition (i) of Theorem 1 but the converse is not true.

(2) Condition (iii) of Theorem A implies Condition (ii) of Theorem 1. For if $n^{1-\alpha}A_n \in \mathrm{BV}$, then

$$e^{n^\alpha}A_n/(e^{n^\alpha}-e^{(n-1)^\alpha}) \in \mathrm{BV}$$

and applying Lemma 6, we find that

$$\left\{\frac{1}{e^{n^\alpha}}\sum_1^n e^{m^\alpha} A_m\right\} \in \mathrm{BV}\,.$$

Finally from Lemma 7, we observe that Condition (ii) of Theorem 1 is satisfied.

(3) Theorems B and C are particular cases of Theorem 2 and Theorem 3 of the present paper, which correspond to Theorem 2 and Theorem 1 of Bosanquet [3] for Fourier series $\sum A_n(x)$.

6.

PROOF OF THEOREM 1. *Necessity*: By Theorem B and Lemma 1, it is evident that condition (i) is necessary for the absolute convergence of (2.1). Applying Abel's transformation to the expression

$$\frac{1}{e^{n^\alpha}}\sum_1^n e^{m^\alpha}\frac{s_m-s}{m},$$

it follows from Lemma 6 and the result (4.1) of Lemma 7, that the condition (ii) of Theorem 1 is also necessary.

Sufficiency: In the proof of Lemma 7 we have shown that

$$\left\{\frac{1}{e^{n^\alpha}}\sum_1^n \frac{e^{m^\alpha}}{m}\right\} \in \mathrm{BV}\,,$$

therefore, from condition (ii) it follows that

$$\left\{\frac{1}{e^{n^\alpha}}\sum_1^n \frac{s_m-s}{m}e^{m^\alpha}\right\} \in \mathrm{BV}\,.$$

Also, from Lemma 5 we observe that the series (2.1) is summable $|R,e^{n^\alpha},1|$, $0<\alpha<1$. Hence from Lemma 8 the result follows.

7.

PROOF OF THEOREM 2. Without loss of any generality we can assume that $0<\alpha<\beta-1<h+1$, $h=[\alpha]$. Let $S_n{}^h$ denote the n-th Cesàro sum of

order h of the series $\sum(s_m - s)/(m+1)$. Then applying Lemmas 10 and 11 we have

$$\begin{aligned}
\beta^{-1}\varphi_\beta(t) &= -\frac{s}{\beta} + \sum_{n=0}^{\infty} A_n(x)\,\gamma_\beta(nt) \\
&= \sum_{n=0}^{\infty} \frac{s_n - s}{n+1}(n+1)\Delta\gamma_\beta(nt) \\
&= \sum_{n=0}^{\infty} S_n^h\, \Delta^{h+1}\{(n+1)\Delta\gamma_\beta(nt)\} \\
&= \sum_{n=0}^{\infty} \Delta^{h+1}\{(n+1)\Delta\gamma_\beta(nt)\} \sum_{m=0}^{n} A_{n-m}^{h-\alpha} S_m^{\alpha-1} \\
&= \sum_{m=0}^{\infty} S_m^{\alpha-1} \sum_{n=m}^{\infty} A_{n-m}^{h-\alpha} \Delta^{h+1}\{(n+1)\Delta\gamma_\beta(nt)\} \\
&= \sum_{m=0}^{\infty} S_m^{\alpha-1} J_m(t) \\
&= \sum_{m=0}^{\infty} S_m^{\alpha} \Delta J_m(t), \quad \text{by Lemma 12},
\end{aligned}$$

(7.1) $$= \sum_{m=0}^{\infty} \sigma_m^\alpha A_m^\alpha \Delta J_m(t), \quad \sigma_n^\alpha = S_n^\alpha / A_n^\alpha .$$

Since

$$\begin{aligned}
S_m^{\alpha-1} = S_m^\alpha - S_{m-1}^\alpha &= A_m^\alpha \left[\sigma_m^\alpha - \frac{m}{\alpha+m}\sigma_{m-1}^\alpha\right] \\
&= O(m^\alpha |\sigma_m^\alpha - \sigma_{m-1}^\alpha|) + O(m^{\alpha-1})
\end{aligned}$$

and by Lemma 10,

$$\Delta^h\{(n+1)\Delta\gamma_\beta(nt)\} = O(n^{1-\beta}),$$

we have

$$\begin{aligned}
&\left|\sum_{m=0}^{p} S_m^{\alpha-1} \sum_{n=p+1}^{\infty} A_{n-m}^{h-\alpha}\Delta^{h+1}\{(n+1)\Delta\gamma_\beta(nt)\}\right| \\
&\leq \sum_{m=0}^{p} |S_m^{\alpha-1}| A_{p+1-m}^{h-\alpha} \operatorname*{Max.}_{p'>p+1} \left|\sum_{n=p+1}^{p'} \Delta^{h+1}\{(n+1)\Delta\gamma_\beta(nt)\}\right| \\
&= O\left\{p^{1-\beta} \sum_{m=0}^{p} |S_m^{\alpha-1}|\, A_{p-m}^{h-\alpha}\right\} \\
&= O\left\{p^{1-\beta} \sum_{m=0}^{p} m^\alpha |\sigma_m^\alpha - \sigma_{m-1}^\alpha|\, A_{p-m}^{h-\alpha}\right\} + O\left\{p^{1-\beta} \sum_{m=0}^{p} A_m^{\alpha-1} A_{p-m}^{h-\alpha}\right\} \\
&= O\left\{p^{1+\alpha-\beta} \sum_{m=0}^{\infty} |\sigma_m^\alpha - \sigma_{m-1}^\alpha|\right\} + O(p^{1+h-\beta}) \\
&= O(p^{1+\alpha-\beta}) = o(1), \quad p \to \infty,
\end{aligned}$$

by virtue of the fact that $\beta > \alpha + 1$ and $\sum |\sigma_n^\alpha - \sigma_{n-1}^\alpha| < \infty$. Thus we get

$$\lim_{p\to\infty} \sum_{m=0}^{p} \sum_{n=p+1}^{\infty} S_m^{\alpha-1} A_{n-m}^{h-\alpha} \Delta^{h+1}\{(n+1)\Delta\gamma_\beta(nt)\} = 0 ,$$

which justifies the change of order of summation.

Let

$$V_m(t) = \sum_{\nu=m}^{\infty} A_\nu^\alpha \Delta J_\nu(t) .$$

We shall now show that

$$V_m(t) = \begin{cases} O(m^\alpha t^\alpha) + O(t) , \\ O(m^{1+\alpha-\beta} t^{1+\alpha-\beta}) . \end{cases} \tag{7.2}$$

We have

$$\begin{aligned} V_m(t) &= \sum_{\nu=m}^{\infty} A_\nu^\alpha \Delta J_\nu(t) \\ &= \sum_{\nu=m}^{\infty} A_\nu^{\alpha-1} J_\nu(t) + A_{m-1}^\alpha J_m(t) \\ &= O\left\{\sum_{\nu=m}^{\infty} \nu^{\alpha-1} \nu^{1-\beta} t^{1+\alpha-\beta}\right\} + O\{m^\alpha m^{1-\beta} t^{1+\alpha-\beta}\} \\ &= O\{m^{1+\alpha-\beta} t^{1+\alpha-\beta}\} . \end{aligned}$$

This proves the second part of (7.2). In the expression

$$s_n - s = \frac{2}{\pi} \int_0^\pi \frac{\varphi(t) \sin(n+\frac{1}{2})t}{2 \sin\frac{1}{2}t} dt$$

let $\varphi(t) = \sin^2 \frac{1}{2}t$ for all t, then

$$s_0 - s = \tfrac{1}{2}, \quad \text{and} \quad \frac{s_n - s}{n+1} = 0 \quad \text{for } n > 0 .$$

Therefore $\sigma_n{}^\alpha = \frac{1}{2}$ for every n. Also

$$\beta^{-1}\varphi_\beta(t) = \frac{1}{2\beta t^\beta} \int_0^t (t-u)^\beta \sin u \, du .$$

Thus from (7.1) we get

$$\sum_0^\infty A_m{}^\alpha \Delta J_m(t) = \frac{1}{\beta t^\beta} \int_0^t (t-u)^\beta \sin u \, du$$

so that

$$\begin{aligned} V_m(t) &= \sum_{n=0}^{\infty} A_n^{\alpha} \Delta J_n(t) - \sum_{0}^{m-1} A_n^{\alpha} \Delta J_n(t) \\ &= \frac{1}{\beta t^{\beta}} \int_0^t (t-u)^{\beta} \sin u \, du - \sum_0^m A_n^{\alpha-1} J_n(t) + A_m^{\alpha} J_m(t) \\ &= O(t) + O\left(\sum_0^m n^{\alpha-1} t^{\alpha}\right) + O(m^{\alpha} t^{\alpha}) \\ &= O(t) + O(m^{\alpha} t^{\alpha}) , \end{aligned}$$

by Lemma 12. This completes the proof of (7.2).

From (7.1) we observe that

$$\beta^{-1} \varphi_{\beta}(t) = \sum_{m=0}^{\infty} \sigma_m^{\alpha} \Delta V_m(t) = \sum_{m=0}^{\infty} (\sigma_m^{\alpha} - \sigma_{m-1}^{\alpha}) V_m(t) .$$

Now

$$\begin{aligned} \int_0^{\pi} \frac{|\varphi_{\beta}(t)|}{t} dt &\leqq \beta \int_0^{\pi} \sum_0^{\infty} |\sigma_m^{\alpha} - \sigma_{m-1}^{\alpha}| \frac{|V_m(t)|}{t} dt \\ &= \beta \sum_{m=0}^{\infty} |\sigma_m^{\alpha} - \sigma_{m-1}^{\alpha}| \int_0^{\pi} \frac{|V_m(t)|}{t} dt . \end{aligned}$$

Since

$$\sum_{m=0}^{\infty} |\sigma_m^{\alpha} - \sigma_{m-1}^{\alpha}| < \infty ,$$

by hypothesis, it is sufficient to show that

$$\int_0^{\pi} \frac{|V_m(t)|}{t} dt < \infty, \quad \text{uniformly in } m .$$

Now

$$\begin{aligned} \int_0^{\pi} \frac{|V_m(t)|}{t} dt &= \int_0^{1/m} + \int_{1/m}^{\pi} \\ &= O\left\{\int_0^{1/m} (m^{\alpha} t^{\alpha-1} + 1) \, dt\right\} + O\left\{\int_{1/m}^{\pi} t^{\alpha-\beta} m^{1+\alpha-\beta} dt\right\} \\ &= O(1) . \end{aligned}$$

This completes the proof of Theorem 2.

8.

Proof of Theorem 3. Without loss of any generality we can assume that in this case $0 < \alpha < \beta < h + 1$, where h is the greatest integer not greater than α. Then

$$n\frac{s_n-s}{n} = \frac{2}{\pi}\int_0^\pi \frac{\varphi(t)\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt\,.$$

Writing T_n^β for the n-th Cesàro mean of order β of the sequence $\{n((s_n-s)/n)\}$ and $K_n^\beta(t)$ for that of the sequence $\{D_n(t)\}$, where

$$D_n(t) = \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t},$$

we have

$$\begin{aligned} T_n^\beta &= \frac{2}{\pi}\int_0^\pi \varphi(t)\,K_n^\beta(t)\,dt \\ &= \left[\sum_{\varrho=1}^{h+1}(-1)^{\varrho-1}\,\Phi_\varrho(t)\left(\frac{d}{dt}\right)^{\varrho-1}K_n^\beta(t)\right]_0^\pi + \\ &\qquad + (-1)^{h+1}\int_0^\pi \Phi_{h+1}(t)\left(\frac{d}{dt}\right)^{h+1}K_n^\beta(t)\,dt \\ &= E_n+F_n, \quad \text{say}\,. \end{aligned}$$

Now

$$E_n = \sum_{\varrho=1}^{h+1}(-1)^{\varrho-1}\,\Phi_\varrho(\pi)\left\{\left(\frac{d}{dt}\right)^{\varrho-1}K_n^\beta(t)\right\}_{t=\pi} = O(n^{\alpha-\beta})\,.$$

Hence

$$\sum\frac{|E_n|}{n} = O\left\{\sum n^{\alpha-\beta-1}\right\} = O(1), \quad \text{since } \beta>\alpha\,.$$

Also

$$\begin{aligned} &\int_0^\pi \Phi_{h+1}(t)\left(\frac{d}{dt}\right)^{h+1}K_n^\beta(t)\,dt \\ &= \frac{1}{\Gamma(1+h-\alpha)}\int_0^\pi \Phi_\alpha(u)\,du\int_u^\pi (t-u)^{h-\alpha}\left(\frac{d}{dt}\right)^{h+1}K_n^\beta(t)\,dt \\ &= \frac{1}{\Gamma(1+h-\alpha)}\int_0^\pi \Phi_\alpha(u)\,P(n,u)\,du\,, \end{aligned}$$

where

$$\begin{aligned} P(n,u) &= \int_u^\pi (t-u)^{h-\alpha}\left(\frac{d}{dt}\right)^{h+1}K_n^\beta(t)\,dt \\ &= \int_u^{u+n^{-1}} + \int_{u+n^{-1}}^\pi = F_{n1}+F_{n2}, \quad \text{say}\,. \end{aligned}$$

Now

$$F_{n1} = \int_{u}^{u+n^{-1}} (t-u)^{h-\alpha} O\left[n^{h+2}\{\min(1,(nu)^{-\beta-1})\}\right] dt$$

$$(8.1) \qquad = O(n^{\alpha+1}) \min\{1,(nu)^{-\beta-1}\}.$$

Applying the second mean value theorem and Lemma 9, we have

$$F_{n2} = n^{\alpha-h} \int_{u+n^{-1}}^{\xi} \left(\frac{d}{dt}\right)^{h+1} K_n^{\beta}(t)\, dt, \qquad u+n^{-1} < \xi < \pi,$$

$$(8.2) \qquad = O(n^{\alpha+1}) \min\{1,(nu)^{-\beta-1}\}.$$

Therefore from (8.1) and (8.2) we obtain

$$P(n,u) = \begin{cases} O(n^{\alpha+1}), \\ O(n^{\alpha-\beta}u^{-\beta-1}). \end{cases}$$

Hence

$$\sum \frac{|F_n|}{n} \leqq \frac{1}{\Gamma(1+h-\alpha)\Gamma(\alpha+1)} \int_0^{\pi} \frac{|\varphi_\alpha(u)|}{u} u^{1+\alpha} \sum \frac{|P(n,u)|}{n} du.$$

Now

$$\sum \frac{|P(n,u)|}{n} = \sum_{n \leqq u^{-1}} + \sum_{n > u^{-1}}$$

$$= \sum_{n \leqq u^{-1}} O(n^{\alpha}) + \sum_{n > u^{-1}} O(n^{\alpha-\beta-1}u^{-\beta-1})$$

$$= O(u^{-\alpha-1}) + O(u^{-\alpha-1}) = O(u^{-\alpha-1}).$$

Hence

$$\sum \frac{|F_n|}{n} = O\left\{\int_0^{\pi} \frac{|\varphi_\alpha(u)|}{u} du\right\} = O(1),$$

by hypothesis. Thus

$$\sum \frac{|T_n^{\beta}|}{n} < \infty.$$

If $\tilde{\sigma}_n^{\beta}$ is the n-th Cesàro mean of order β of the series (2.1), then $T_n^{\beta} = n(\tilde{\sigma}_n^{\beta} - \tilde{\sigma}_{n-1}^{\beta})$. It, therefore, follows that the series (2.1) is summable $|C,\beta|$, $\beta > \alpha$.

This completes the proof of Theorem 3.

I am highly grateful to Prof. B. N. Prasad, F.N.I., for his constant encouragement and kind advice during the preparation of this paper and to Mr. T. Gutmann Madsen for giving several useful suggestions.

REFERENCES

1. S. N. Bhatt, *A Tauberian theorem for absolute Riesz summability*, Indian J. Math. 1 (1958), 29–32.
2. L. S. Bosanquet, *Absolute summability* (A) *of Fourier series*, Proc. Edinburgh Math. Soc. (2) 4 (1934–36), 12–17.
3. L. S. Bosanquet, *Absolute Cesàro summability of a Fourier series*, Proc. London Math. Soc. 41 (1936), 517–528.
4. G. H. Hardy, *Divergent Series*, Oxford, 1956.
5. G. H. Hardy and J. E. Littlewood, *The allied series of a Fourier series*, Proc. London Math. Soc. 24 (1926), 211–246.
6. R. Mohanty, *A criterion for the absolute convergence of a Fourier series*, Proc. London Math. Soc. (2) 51 (1950), 186–196.
7. R. Mohanty, *On the absolute Riesz summability of a Fourier series and allied series*, Proc. London Math. Soc. (2) 52 (1951), 295–320.
8. R. Mohanty and S. Mohapatra, *On the absolute convergence of a series associated with a Fourier series*, Proc. Amer. Math. Soc. 7 (1956), 1049–1053.
9. T. Pati, *On the absolute Riesz summability of Fourier series and its conjugate series*, Trans. Amer. Math. Soc. 76 (1954), 351–374.
10. A. Zygmund, *Trigonometric Series*, Warszawa, 1935.

DEPARTMENT OF MATHEMATICS AND STATISTICS
ALIGARH MUSLIM UNIVERSITY
ALIGARH
INDIA