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THE LATTICE-GRAPH OF THE TOPOLOGY OF A TRANSITIVE DIRECTED GRAPH

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A directed graph D is transitive if whenever the directed lines uv and vw are in D, then the directed line uw is also in D. It was shown by Evans, Harary, and Lynn [1] that there is a one-to-one correspondence between the labeled topologies on n points and the labeled transitive directed graphs with n points; therefore, under this correspondence, there is associated a topology with every labeled transitive directed graph D. We call the directed graph of the lattice of open sets determined by this topology the lattice-graph of the topology of D. It is the object of this note to investigate the lattice-graphs of the topologies of transitive directed graphs. In particular, we characterize those transitive directed graphs whose topologies have isomorphic lattice-graphs.

Graphical definitions not given here may be found in [2].

The associated topology of a transitive directed graph.

We find it convenient to describe here the one-to-one correspondence between the labeled topologies with n points and the labeled transitive directed graphs on n points, as it appeared in [1].

With each topology $\mathcal{O}$ on a set V of n points, a directed graph $D(\mathcal{O})$ can be defined on V by drawing a line directed from $u \in V$ to $v \in V$ if and only if u is in every open set containing v. It is then easy to verify that $D(\mathcal{O})$ is transitive and uniquely determined by $\mathcal{O}$. For example, the topology $\mathcal{O}$ on $V=\{a,b,c,d\}$, defined by

$$\mathcal{O} = \{\emptyset, \{a\}, \{a,b\}, V\},$$

gives rise to the directed graph $D_1 = D_1(\mathcal{O})$ in Figure 1.

Conversely, with each labeled transitive directed graph D having n points, a topology $\mathcal{O}=\mathcal{O}(D)$ on this set V of points is induced by the basis B whose elements consist of the sets:

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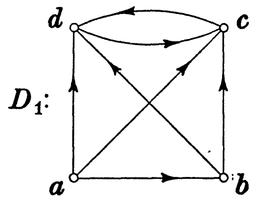

Fig. 1.

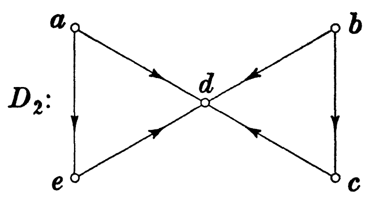

Fig. 2.

$$Q(u) = \{u\} \cup \{v \in V \mid vu \text{ is a directed line in } D\}$$

for all $u \in V$. The transitive directed graph D_2 of Figure 2 generates the basis

$$B = \{Q(a)=\{a\},\ Q(b)=\{b\},\ Q(c)=\{c,b\},\ Q(d)=V,\ Q(e)=\{e,a\}\}$$

which induces the topology

$$\mathcal{O}(D_2) = \{\emptyset,\ \{a\},\ \{b\},\ \{a,b\},\ \{b,c\},\ \{a,e\},\ \{a,b,c\},\ \{a,b,e\},\ \{a,b,c,e\},\ V\}.$$

The lattice-graph associated with a topology.

With each topology $\mathcal{O}$ on a finite set, a lattice, whose directed graph is called the lattice-graph of $\mathcal{O}$, may be associated in a natural way by letting the open sets O_i be represented by points v_i and drawing an arc from v_i to v_j if $O_i \subset O_j$ and there exists no O_h such that $O_i \subset O_h \subset O_j$. The lattice-graph of $\mathcal{O}(D_2)$ is shown in Figure 3.

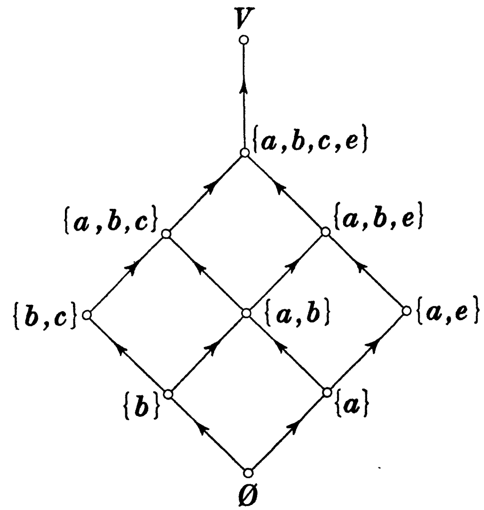

Fig. 3.

As we have seen, given a labeled transitive directed graph D, there corresponds a topology $\mathcal{O}(D)$ on the point set of D and thus a lattice-graph $\mathcal{L}$ associated with the open sets of $\mathcal{O}$. We refer to $\mathcal{L}=\mathcal{L}(D)$ as the *lattice-graph of the topology* of D. We now investigate the relationship between transitive directed graphs and the lattice-graphs of their topologies. In order to describe this relationship, a few intermediate concepts are useful.

A *clique* in a directed graph is a maximal complete symmetric subgraph. Although the cliques of a directed graph D do not, in general, partition the point set of D, such is the case if D is transitive.

LEMMA 1. *The cliques of a transitive directed graph partition its point set.*

PROOF. Clearly, every point of a transitive directed graph D lies in at least one clique. Suppose v is a point of D which lies in two distinct cliques C_1 and C_2. Then if $u \in C_1$ and $w \in C_2$, both the lines uw and wu are in D since D is transitive. However, this implies that C_1 and C_2 are properly contained in a complete symmetric subgraph contradicting the fact that C_1 and C_2 are maximal. Hence v belongs to a single clique and the cliques partition the point set of D.

In view of this result, we define the *clique-condensation graph* D' of a transitive directed graph D to be that directed graph whose points are in one-to-one correspondence with the cliques $\{C_1, C_2, \ldots\}$ of D and in which there is a line directed from the point corresponding to C_i to the point corresponding to C_j in D' if there is a line directed from a point of C_i to a point of C_j in D. A transitive directed graph D_3 and its clique-condensation graph are shown in Figure 4.

THEOREM 1. *The clique-condensation graph D' of a transitive directed graph D is transitive and asymmetric.*

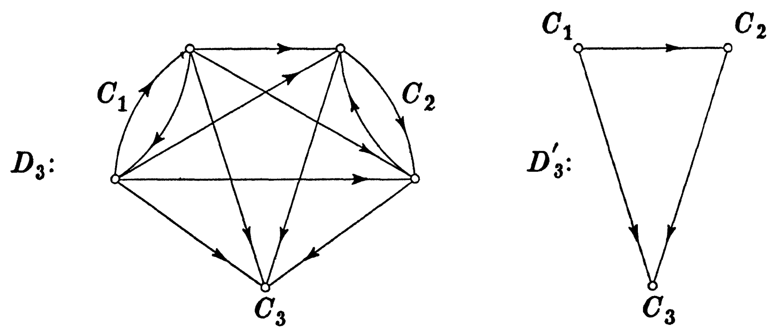

Fig. 4.

PROOF. By the transitivity of D, if there is a line directed from a point of C_i to a point of C_j, then there must be a line from every point of C_i to every point of C_j. Furthermore, there can be no lines directed from C_j to C_i since cliques are maximal.

COROLLARY 1a. *A transitive directed graph D is isomorphic to its clique-condensation graph if and only if D is asymmetric.*

It is not difficult to find two different transitive directed graphs whose topologies have isomorphic lattice-graphs, but, as we shall see, such directed graphs must enjoy a common property.

LEMMA 2. *The topologies of a transitive directed graph D and its clique-condensation graph D' have isomorphic lattice-graphs.*

PROOF. If a point v of D lies in an open set O of the topology $\mathcal{O}(D)$, then every point in the clique to which v belongs also lies in O. Since the transitivity relation among the cliques in D is the same as that among the corresponding points of D', $\mathcal{L}(D)$ and $\mathcal{L}(D')$ are isomorphic, where the point

$$\{C_{i_1}, C_{i_2}, \ldots, C_{i_k}\}$$

of $\mathcal{L}(D')$ corresponds to the point

$$C_{i_1} \cup C_{i_2} \cup \ldots \cup C_{i_k}$$

of $\mathcal{L}(D)$.

THEOREM 2. *The lattice-graphs $\mathcal{L}(D_1)$ and $\mathcal{L}(D_2)$ of the topologies of two transitive directed graphs D_1 and D_2 are isomorphic if and only if the clique-condensation graphs D_1' and D_2' are isomorphic.*

PROOF. Suppose D_1' and D_2' are isomorphic. Then clearly $\mathcal{L}(D_1')$ and $\mathcal{L}(D_2')$ are isomorphic. However, by Lemma 2, $\mathcal{L}(D_1)$ and $\mathcal{L}(D_1')$ are isomorphic as are $\mathcal{L}(D_2)$ and $\mathcal{L}(D_2')$. Hence $\mathcal{L}(D_1)$ is isomorphic to $\mathcal{L}(D_2)$.

Conversely, to show $\mathcal{L}(D_1)$ and $\mathcal{L}(D_2)$ isomorphic implies D_1' and D_2' isomorphic, it is equivalent to show that the lattice-graph $\mathcal{L}(D)$ uniquely determines the topology $\mathcal{O}(D')$, or that $\mathcal{L}(D')$ uniquely determines the topology of D', which in turn uniquely determines D', since by Lemma 2, $\mathcal{L}(D)$ and $\mathcal{L}(D')$ are isomorphic. Now the points of $\mathcal{L}(D')$ correspond to open sets of a topology on some finite set V. If uv is a directed line of $\mathcal{L}(D')$, then u and v correspond to open sets O_1 and O_2, respectively, for which $O_1 \subset O_2$ but no open set O_3 exists having $O_1 \subset O_3 \subset O_2$. The set $O_2 \setminus O_1$ necessarily consists of a single element of V, for if $\{a,b\} \subseteq O_2 \setminus O_1$, then whenever one of a and b is contained in an open set, then so must the other be contained in the same open set. This implies that both directed lines ab and ba are in D' contradicting the asymmetry condition (see Theorem 1). Also, for every point $a \in V$, the smallest open set containing a corresponds to a point w_1 of $\mathscr{L}(D')$ having indegree one, i.e., having one line directed towards it, for if w_2w_1 and w_3w_1 were directed lines of $\mathscr{L}(D')$, then the union of the open sets corresponding to w_2 and w_3 would be a larger open set not containing a, implying that the lines w_2w_1 and w_3w_1 do not exist. Hence, up to a permutation of V, the one-to-one correspondence between the points of $\mathscr{L}(D')$ and the open sets defined on V is given as follows. First, identify with each point u of $\mathscr{L}(D')$ having indegree one a distinct point a of V (which is possible by the preceding discussion). Then for any point v of $\mathscr{L}(D')$ associate the subset of V consisting of the union of any point identified with v and all points a identified with points u for which there is a directed path from u to v in $\mathscr{L}(D')$. The correspondence is completed by associating the empty set $\emptyset$ with the remaining unlabeled point of $\mathscr{L}(D')$. This uniquely determines D' (up to labeling).

COROLLARY 2a. *For any lattice-graph $\mathscr{L}$ of the topology of a transitive directed graph, there exists a unique asymmetric transitive directed graph D the lattice-graph of whose topology is isomorphic to $\mathscr{L}$.*

It therefore follows that every transitive directed graph whose topology has lattice-graph $\mathscr{L}$ can be obtained from the asymmetric directed graph D whose topology has lattice-graph $\mathscr{L}$ by a suitable replacement of the points of D by complete symmetric directed graphs. Thus, all transitive directed graphs whose topologies have lattice-graphs $\mathscr{L}$ are completely determined and constructible.

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