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ON THE ANGULAR DERIVATIVE OF REGULAR FUNCTIONS

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1. Introduction.

Let $z(\zeta)$ be a regular function, defined for $\operatorname{Re}\zeta > 0$, and suppose ζ_0 lies on the imaginary axis. If

$$\lim_{\zeta\to\zeta_0} z(\zeta) = z_0, \quad \lim_{\zeta\to\zeta_0} \frac{z(\zeta)-z_0}{\zeta-\zeta_0} = \tau ,$$

both limits being attained uniformly as $\zeta \to \zeta_0$ in each angle

$$-\tfrac{1}{2}\pi+\alpha \leqq \arg(\zeta-\zeta_0) \leqq \tfrac{1}{2}\pi-\alpha, \qquad \alpha>0 ,$$

then τ is called the "angular derivative" of $z(\zeta)$ at ζ_0. If $\zeta_0=\infty$ or $\zeta_0=\infty$, $z_0=\infty$, the angular derivative is defined by

$$\lim_{\zeta\to\infty} \zeta[z(\zeta)-z_0] = \tau, \quad \text{or} \quad \lim_{\zeta\to\infty} \frac{\zeta}{z(\zeta)} = \tau ,$$

respectively, the limits being attained uniformly as $\zeta \to \infty$ in each angle

$$(1)_\alpha \qquad -\tfrac{1}{2}\pi+\alpha \leqq \arg\zeta \leqq \tfrac{1}{2}\pi-\alpha, \qquad \alpha>0 .$$

We formulate our discussion for the case $\zeta_0=\infty$, $z_0=\infty$.

Valiron [4] has proved that, if $z(\zeta)/\zeta$ tends uniformly to τ^{-1}, $\tau \neq 0$, as $\zeta \to \infty$ in an angle $(1)_\alpha$, then $z'(\zeta)$ tends uniformly to τ^{-1}, as $\zeta \to \infty$ in each angle $(1)_{\alpha'}$, where $\alpha' > \alpha$.

Carathéodory [2] shows that for functions regular in the right half plane, the quotient $z(\zeta)/\zeta$ tends uniformly to a finite value as $\zeta \to \infty$ in an angle $(1)_\alpha$ if $\operatorname{Re} z(\zeta) > 0$ for $\operatorname{Re}\zeta > 0$, and Ahlfors [1] generalises this result.

In this note, we use a result in [3] to obtain a necessary condition for the existence of a finite, non-zero angular derivative, and discuss necessary conditions of Ahlfors [1], and Ferrand {[4]; [5] chap. VI} for the case when $z(\zeta)$ gives a conformal map of $\operatorname{Re}\zeta > 0$. These authors, and others, give sufficient conditions for a finite, non-zero τ, and necessary and sufficient conditions have been given by Warschawski [7] when the

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boundary of $z\{\mathrm{Re}\,\zeta > 0\}$ exhibits smooth behaviour. Recently, Warschawski [8] has given a new sufficient condition which improves previous results in the case $\mathrm{Re}\,z(\zeta) > 0$, for $\mathrm{Re}\,\zeta > 0$.

2. A necessary condition for regular functions.

We define the surface function $p(R) = p(R, \mathrm{Re}\,\zeta > 0, z)$, $R > 0$, of $z(\zeta)$ to be the finite or infinite Lebesgue integral

$$\frac{1}{2\pi}\int_0^{2\pi} n(Re^{i\theta})\,d\theta\,,$$

where $n(\omega)$ denotes the number of zeros in $\mathrm{Re}\,\zeta > 0$ of $z(\zeta) - \omega$, counted according to their multiplicity. We suppose that $z(\zeta)$ tends uniformly to infinity as ζ approaches infinity in each angle $(1)_\alpha$ and that

$$\int_{R_0}^{R_1} \frac{dR}{R\,p(R)}, \qquad p(R_0) > 0\,,$$

diverges as $R_1 \to \infty$. Write

$$s = \sigma + it = \log\zeta, \qquad X = \log|z|\,.$$

LEMMA 1 ([3, Theorem 3]). *Suppose that $F(s)$, regular in the strip $|t| < \frac{1}{2}\pi$, satisfies*

$$\overline{\lim_{\sigma\to\infty}}\,|F(\sigma)| = +\infty\,.$$

If $p(R) = p(R, |t| < \frac{1}{2}\pi, F)$ and

$$\int_{R_0}^{R_1} \frac{dR}{R\,p(R)} \to \infty \quad \text{as } R_1 \to \infty, \qquad p(R_0) > 0,\ R_0 > 0\,,$$

then

$$\sigma - \tfrac{1}{2}\int_{R_0}^{|F(s)|} \frac{dR}{R\,p(R)} \to \beta\,,$$

uniformly as $s = \sigma + it$ tends to infinity with $|t| < \frac{1}{2}\pi - \delta$, $\delta > 0$, where $-\infty < \beta \leqq +\infty$.

We apply Lemma 1 to the function $F(s) = z(e^s)$, which clearly satisfies the hypotheses, since

$$p(R) = p(R, |t| < \tfrac{1}{2}\pi, F) = p(R, \mathrm{Re}\,\zeta > 0, z)\,.$$

We deduce that

$$(1) \qquad \sigma - \tfrac{1}{2} \int_{\log R_0}^{X} \frac{dT}{p(e^T)} \to \beta ,$$

uniformly as $s = \sigma + it \to \infty$ with $|t| < \frac{1}{2}\pi - \delta$, $\delta > 0$, and $-\infty < \beta \leqq +\infty$.

If we assume that $z(\zeta)$ possesses a finite, non-zero angular derivative, then $\log|z(\zeta)/\zeta| = X - \sigma$ tends to a finite limit as $s = \sigma + it \to \infty$, with $|t| < \frac{1}{2}\pi - \delta$, $\delta > 0$. Since (1) can be written

$$\sigma - X - \tfrac{1}{2} \int_{\log R_0}^{X} \frac{1 - 2p(e^T)}{p(e^T)} dT \to \beta - \log R_0 ,$$

we conclude that a necessary condition for the existence of a finite, non-zero angular derivative is that

$$\int_{R_0}^{R_1} \frac{1 - 2p(R)}{R\,p(R)} dR \to \gamma \quad \text{as } R_1 \to \infty ,$$

where $-\infty \leqq \gamma < +\infty$. Thus we may state

THEOREM 1. *If* $z(\zeta)$, *regular for* $\operatorname{Re}\zeta > 0$, *has a surface function* $p(R)$ *such that*

$$\int_{R_0}^{R_1} \frac{dR}{R\,p(R)} \to \infty \quad \text{as } R_1 \to \infty, \qquad p(R_0) > 0,\ R_0 > 0 ,$$

then a necessary condition for $z(\zeta)$ *to possess a finite, non-zero angular derivative is that*

$$\int_{R_0}^{R_1} \frac{1 - 2p(R)}{R\,p(R)} dR \to \gamma \quad \text{as } R_1 \to \infty ,$$

where $-\infty \leqq \gamma < +\infty$.

3. The case of conformal maps.

Suppose now that $z(\zeta)$ maps $\operatorname{Re}\zeta > 0$ conformally onto a domain G so that $z = \infty$, $\zeta = \infty$ correspond. Suppose also that $z(\zeta)/\zeta$ tends uniformly to a finite, non-zero limit as $\zeta \to \infty$ with $-\frac{1}{2}\pi + \alpha \leqq \arg\zeta \leqq \frac{1}{2}\pi - \alpha$, $\alpha > 0$. By translating and rotating G, if necessary, we may suppose that the angular derivative is real, and also that $z = 0$ is a boundary point of G. Ahlfors observes that the argument principle allows us to deduce that G contains all points z with $-\frac{1}{2}\pi + \varepsilon \leqq \arg z \leqq \frac{1}{2}\pi - \varepsilon$, and $|z| > r_0(\varepsilon)$, for arbitrary $\varepsilon > 0$. Therefore G contains the half line $\operatorname{Im} z = 0$, $\operatorname{Re} z > R_0'$, say. For $R > R_0'$, let $\theta(R)$ denote the angular measure of the arc of $|z| = R$, which lies in G and meets the real z-axis. In the notation of Lemma 1, $2\pi p(R)$ denotes the total angular measure of all arcs of $|z| = R$ which lie in G. However, for this special case, there is no difficulty in modifying the proof of Lemma 1 so that $\theta(R)/2\pi$ can replace $p(R)$ in the conclusion. Theorem 1 is altered accordingly so that a necessary geometrical condition on G for $z(\zeta)$ to possess a finite, non-zero angular derivative is that

$$\int_{R_0}^{R_1} \frac{\pi - \theta(R)}{R\,\theta(R)}\,dR \to \gamma' \quad \text{as } R_1 \to \infty, \qquad \theta(R_0) > 0,\, R_0 > R_0',$$

where $-\infty \leqq \gamma' < +\infty$.

We state the following three necessary conditions of Ahlfors [1] for the existence of a finite, non-zero τ.

1. G contains an angle grater than $\pi - \varepsilon$ (near infinity), for arbitrary $\varepsilon > 0$, but no angle greater than π (near infinity), that is no set of points

$$\{z : \ |\arg z| > \pi + \delta;\ |z| > \varrho\}$$

for any $\varrho, \delta > 0$.

2. $$\varlimsup_{R_1 \to \infty} \int_{R_0}^{R_1} \frac{\pi - \theta(R)}{R\,\theta(R)}\,dR < +\infty.$$

3. If $\bar{G}$ is the largest domain contained in G which is symmetrical about the real axis, and if $\bar{G}$ is described by a function $\bar{\theta}(R)$ $R > R_0$, for which $\bar{\theta}^2(R)$ has finite variation in $[R_0, \infty]$, then

$$\varliminf_{R_1 \to \infty} \int_{R_0}^{R_1} \frac{\pi - \bar{\theta}(R)}{R\bar{\theta}(R)}\,dR > -\infty .$$

Thus we can sharpen 2 and 3, and restate Theorem 1 as

THEOREM 1a. *With the above notation, a necessary condition for $z(\zeta)$ to possess a finite, non-zero τ is that*

$$\lim_{R_1 \to \infty} \int_{R_0}^{R_1} \frac{\pi - \theta(R)}{R\,\theta(R)}\,dR = \gamma', \qquad \theta(R_0) > 0\,,$$

where $-\infty \leqq \gamma' < +\infty$.

If the symmetrical domain $\bar{G} \subset G$ is such that $\bar{\theta}^2(R)$ has bounded variation in $[R_0, \infty]$, then it is necessary that

$$\lim_{R_1 \to \infty} \int_{R_0}^{R_1} \frac{\pi - \bar{\theta}(R)}{R\,\bar{\theta}(R)}\,dR \qquad \textit{exists and is finite.}$$

Ahlfors' condition 2 has been improved by Ferrand {[4]; [5, Chap. VI]} who replaces $\theta(R)$ by a smaller function $\hat{\theta}(R)$ defined as follows. Let $\{R_n\}_1^\infty$ be an increasing unbounded sequence of numbers such that

$$\text{(A)} \qquad \sum_1^\infty \left(\log \frac{R_{n+1}}{R_n}\right)^2 < +\infty, \qquad R_1 > R_0',$$

and set

$$\theta_n = \min_{R_n \leqq R < R_{n+1}} \theta(R), \qquad n = 1, \ldots .$$

Then $\hat{\theta}(R) = \theta_n$, $R_n \leqq R < R_{n+1}$, and the necessary condition is

4. For all sequences $\{R_n\}_1^\infty$ satisfying (A), it is necessary that

$$\overline{\lim_{n\to\infty}} \sum_{i=1}^n (\pi - \theta_i)\left(\log \frac{R_{i+1}}{R_i}\right) < +\infty,$$

or equivalently,

$$\overline{\lim_{\hat{R}\to\infty}} \int_{R_1}^{\hat{R}} \frac{\pi - \hat{\theta}(R)}{R} dR < +\infty .$$

The proof of Lemma 1 is based on a generalisation of Ahlfors' well-known distortion theorem ["Erste Hauptungleichung"], and a comparable distortion theorem involving the function $\hat{\theta}(R)$ is established in [4, p. 184, equation (6)]. Basing ourselves on this equation we find that Lemma 1 can be proved in the case of conformal maps in terms of the function $\hat{\theta}(R)$. This enables us to state

THEOREM 1b. *A necessary condition for $z(\zeta)$ to possess a finite non-zero τ is that*

$$\lim_{\hat{R}\to\infty} \int_{R_1}^{\hat{R}} \frac{\pi - \hat{\theta}(R)}{R} dR = \gamma'' \quad \textit{as } \hat{R} \to \infty,$$

where $-\infty \leqq \gamma'' < +\infty$.

We remark that $\gamma' = -\infty$, $\gamma'' > -\infty$ can occur for the same G.

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