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CIRCULAR FLOWS ON S^3

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1. Introduction.

A flow on a space M is a continuous function $\Phi: R \times M \to M$ such that

i) $\Phi \mid t \times M$ is a homeomorphism onto for each $t \in R$, and
ii) $\Phi(t,\Phi(r,x)) = \Phi(t+r,x)$ for each $r,t \in R$, $x \in M$.

A point $x \in M$ is called a *fixed point* if $\Phi(t,x) = x$ for each $t \in R$. Let F denote the set of fixed points of the flow Φ. A *circular flow* (Φ,f) on S^3 is a flow on S^3 along with a continuous function $f: S^3 \setminus F \to S^1$ such that $f \circ \Phi \mid R \times x$ is a homomorphism of the additive group of real numbers onto the circle group for each $x \in S^3 \setminus F$.

In section 2 it is shown that $(S^3 \setminus F, f, S^1)$ is a fiber space whose fiber is a 2-manifold, and some general properties of F are exhibited. In section 3, it is shown that if Φ is piecewise linear, F is a tame link, and the set of tame links which can be the fixed point set of a circular flow is classified. Some examples are given in section 4 which show that F may be wild, and may fail to be a manifold.

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2. Circular flows.

Let (Φ,f) be a circular flow on S^3.

LEMMA 2.1. *For each $t \in S^1$, $f^{-1}(t)$ is a 2-manifold embedded in a locally flat manner as a closed subset of $S^3 \setminus F$.*

PROOF. F is a compact subset of S^3. Let $x \in f^{-1}(t)$ and let U be a neighborhood of x in $f^{-1}(t)$. We may assume that $\overline{U}$ contains no points of F. For each $x \in U$ there is an interval $(-\varepsilon,\varepsilon)$ such that $\Phi \mid (-\varepsilon,\varepsilon) \times x$ is a homeomorphism. Note that $\Phi(0,x) = x$. Since $\overline{U}$ is compact and is a positive distance from F, there is an $\varepsilon > 0$ such that $\Phi \mid (-\varepsilon,\varepsilon) \times x$ is a homeomorphism for each $x \in U$. Since orbits in a flow are disjoint or

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equal, $\Phi|(-\varepsilon,\varepsilon)\times U$ is one-to-one; thus it is an embedding. It is clear that $\Phi((-\varepsilon,\varepsilon)\times U)$ must contain a set open in S^3, so $\Phi((-\varepsilon,\varepsilon)\times U)$ is open in S^3, thus it is a manifold. U, being a factor of a 3-manifold by a 1-manifold, is a generalized 2-manifold [2], thus it is a 2-manifold. Since U is open in $f^{-1}(t)$, $f^{-1}(t)$ must be a 2-manifold. The embedding of $f^{-1}(t)$ in $S^3\setminus F$ is locally flat since

$$\Phi((-\varepsilon,\varepsilon)\times U) \cong (-\varepsilon,\varepsilon)\times U .$$

Lemma 2.2. *$\overline{f^{-1}(t)}\setminus f^{-1}(t)=F$ for each t and* $\dim F\leqq 1$.

Proof. Since $f^{-1}(t)$ is closed in $S^3\setminus F$,

$$\overline{f^{-1}(t)}\setminus f^{-1}(t) \subset F .$$

If F contains an open set, then $\dot F = F\setminus \mathring{F}$ is two dimensional. Since F is closed, $\dot F = \overline{S^3\setminus F}\setminus(S^3\setminus F)$ so if $x\in \dot F$, it is the limit of a sequence $\{x_n\}$, $x_n\in S^3\setminus F$. Each x_n is in a non-trivial orbit, and each such orbit intersects $f^{-1}(t)$ in y_n. Then x is the limit of $\{y_n\}$ and

$$x \in \overline{f^{-1}(t)}\setminus f^{-1}(t) .$$

Thus

$$\dot F \subset \overline{f^{-1}(t)}\setminus f^{-1}(t) .$$

Let $x\in \dot F$ and let U be a 3-cell neighborhood of x in S^3. By putting $\dot U$ in general position with respect to the locally flat 2-manifold $f^{-1}(t)$, we may assume that $\dot U\cap f^{-1}(t)$ is a countable collection of 1-manifolds. The boundary of $U\cap \dot F$ in $\dot F$ is the boundary of $\dot U\cap f^{-1}(t)$ in $\dot U$, which is countable and zero dimensional. So x has arbitrarily small neighborhoods in $\dot F$ with zero dimensional boundaries, and $\dim \dot F\leqq 1$. This is a contradiction, so

$$\dot F = F = \overline{f^{-1}(t)}\setminus f^{-1}(t) .$$

Lemma 2.3. *$S^3\setminus F$ is connected, and by a change of variables, it is possible to assume that*

$$f\circ\Phi\,|\,R\times x : R \to S^1$$

is the standard homomorphism $t\to t$ mod 1.

Proof. Since $\dim F\leqq 1$, the set F cannot separate S^3, so $S^3\setminus F$ is arcwise connected. If x_0 is not a fixed point, then

$$f\circ\Phi\,|\,R\times x_0 : R \to S^1$$

is some non-trivial homomorphism, and a change of variable in R will change it to the standard homomorphism. If $x\in S^3\setminus F$ and α is an arc from x_0 to x, then $f\circ\Phi\,|\,R\times\alpha$ is a homotopy between $f\circ\Phi\,|\,R\times x_0$ and

$f \circ \Phi | R \times x$. Since the set of homomorphisms $R \to S^1$ is discrete in the space of continuous functions and each stage of the homotopy $f \circ \Phi | R \times \alpha$ is a homomorphism, $f \circ \Phi | R \times x$ is also the standard homomorphism.

THEOREM 2.1. *f: $S^3 \setminus F \to S^1$ is a locally trivial fiber space.*

PROOF. For each t,

$$\Phi | t \times f^{-1}(0) : f^{-1}(0) \to f^{-1}(t \bmod 1)$$

is a homeomorphism with inverse

$$\Phi | (-t) \times f^{-1}(t \bmod 1)$$

because

$$f \circ \Phi(t \times f^{-1}(0)) = t \bmod 1 .$$

Since $f \circ \Phi(1 \times f^{-1}(0)) = 0$, $S^3 \setminus F$ is $f^{-1}(0) \times [0,1]$ with $f^{-1}(0) \times 0$ identified to $f^{-1}(0) \times 1$ by

$$\Phi | 1 \times f^{-1}(0).$$

Henceforth we will write S_t for $f^{-1}(t)$.

THEOREM 2.2. *S_0 has a finite number of components, all homeomorphic. Moreover, there is a circular flow (Φ, f') which has the same fixed point set and $S_0' = f'^{-1}(0)$ is homeomorphic to one component of S_0.*

PROOF. The map $\Phi | 1 \times S_0$ is a permutation of the components of S_0. This permutation cannot be written as a product of more than one disjoint cycle because this would imply that $S^3 \setminus F$ is not connected. There cannot be more than a countable number of components, and there is no permutation of $\{1,2,3,\ldots\}$ which is not the product of disjoint cycles, so the number n of components of S_0 is finite. Let S_0' be one component of S_0 and let

$$f'(x) = \frac{p}{n} + \frac{f(x)}{n} \quad \text{for} \quad x \in \Phi([p, p+1] \times S_0') .$$

Since

$$\Phi([0,n] \times S_0') = \Phi([0,1] \times S_0) = S^3 \setminus F ,$$

the domain of f' is $S^3 \setminus F$. Since $f \circ \Phi | R \times x$ is a homomorphism, so is $f \circ \Phi | R \times x$.

COROLLARY 2.1. *For the purpose of identifying F and S_0, one may assume that S_0 is connected and that*

$$f \circ \Phi \,|\, R \times x : R \to S^1$$

is the standard homomorphism.

THEOREM 2.3 *F is not splitable.*

PROOF. Let S^2 be a 2-sphere in $S^3 \setminus F$. Then $f(S^2)$ is inessential in S^1, so S^2 is homotopic in $S^3 \setminus F$ to a subset of S_0. Since F must be non-empty and we may assume that S_0 has one component, S_0 cannot contain a 2-sphere, so S^2 is homotopic to a point in S_0. Then S^2 must bound a 3-cell in $S^3 \setminus F$, and F is in only one complementary domain of S^2.

It follows immediately that F is not the union of more than one compact disjoint subset, one of which is cellular. Of course, F is not cellular because of the exactness of the homotopy sequence of the fiber space $(S^3 \setminus F, f, S^1)$. We will show in section 4 that F may be somewhat pathological.

3. Piecewise Linear Circular Flows.

A circular flow (Φ, f) on S^3 will be called piecewise linear (PL) if Φ is piecewise linear.

THEOREM 3.1. *If (Φ, f) is a* PL *flow on S^3, then F is a tame link and $S_0 \cup F$ is a compact manifold with boundary F.*

PROOF. We will assume that Φ is simplicial. Since

$$f \circ \Phi \,|\, R \times x : R \to S^1$$

is a homomorphism, f is also simplicial under some subdivision of S^1. If t is a non-vertex of S^1, the argument in [3] shows that S_t is a finite open complex. In as much as F is $\overline{S_t} \setminus S_t$, it must be a finite 1-complex. Let v be a vertex of F and let N be the star of v in S^3. Since Φ is simplicial and v is a fixed point, Φ must take $\dot{N}$ onto $\dot{N}$. Thus

$$(\Phi \,|\, R \times \dot{N}, f \,|\, \dot{N} \setminus (F \cap \dot{N})) = (\psi, g)$$

is a circular flow on the 2-sphere $\dot{N}$. The arguments of the previous section apply to show that $g^{-1}(0)$ is an open arc. Since the boundary of this open arc in $\dot{N}$ cannot be a single point (from the fiber homotopy exact sequence), it must be two points. In other words, $F \cap \dot{N}$ is exactly two points, and F is a compact 1-manifold. This makes F a tame link. This argument also shows that $S_0 \cap \mathring{N}$ is a finite collection of open disks D_i with

$$\overline{D_i} \cap \mathring{N} = F \cap \mathring{N}\,.$$

If F_i is a component of F, the intersection of S_0 with a regular neighborhood of F_i is a finite collection $\{A_n\}$ of annuli, $A_n=(0,1]\times S^1$, with $\overline{A_n}\setminus A_n=F_i$ for each n. If F_i is the only component and is unknotted, S_0 must be a disk and there must be only one annulus A_1 since $\pi_1(S_0)=0$ from the fiber homotopy exact sequence. Otherwise $1\times S^1\subset A_n$ must represent a non-zero element of $\pi_1(S^3\setminus F)$. Since

$$\pi_1(S_0)\to\pi_1(S^3\setminus F)$$

is a monomorphism, $1\times S^1\subset A_n$ will represent a non-zero element of $\pi_1(S_0)$. If there are more than one annuli, A_1 and A_2, $1\times S_1\subset A_1$ and $1\times S^2\subset A_2$ must represent the same element of $\pi_1(S^3\setminus F)$, thus also the same element of $\pi_1(S_0)$. Therefore they bound an annulus in S_0, which must be all of S_0 since it is a component. This annulus will disconnect $S^3\setminus F$, which is impossible since it is the total space of a fiber space over S^1. There is, thus, only one annulus A_1, and F is the boundary of $S_0\cup F$.

THEOREM 3.2. *The following classes of tame links are identical:*

a) *Tame links which are fixed point sets of circular flows.*
b) *Tame links whose complements fiber over S^1 such that the boundary of the fiber is the link.*
c) *Strongly indecomposable links whose augmentation subgroups are free of finite rank* [1].

PROOF. A link which satisfies a) also satisfies b) by Theorem 2.1. If L is a link and $S^3\setminus L$ fibers over S^1 as specified, the fiber space may be understood as $F\times I$ where F is the fiber and $F\times 0$ is identified with $F\times 1$ by some homeomorphism h. Define $\Phi\colon R\times S^3\to S^3$ by

$$\begin{aligned}\Phi(t,x) &= h^n(x)\times(t-n) && \text{if } n\leqq t<n+1,\ x\in S^3\setminus L\,,\\ \Phi(t,x) &= x && \text{if } x\in L\,.\end{aligned}$$

Then if $f\colon S^3\setminus L\to S^1$ is given by

$$\begin{aligned}f(x,t) &= t \quad \text{if } 0\leqq t<1\,,\\ f(x,1) &= 0\,,\end{aligned}$$

(Φ,f) is a circular flow. By Theorem 2.2, we may assume that F is a connected surface. Since

$$0\to\pi_1(F)\to\pi_1(S^3\setminus L)\to\pi_1(S^1)\to 0$$

is exact, L is a strongly indecomposable link whose augmentation subgroup is free of finite rank [1].

If L satisfies c), then we have an exact sequence

$$0 \to E \to \pi_1(S^3 \setminus L) \to Z \to 0 ,$$

where E is the augmentation subgroup. Since L is strongly indecomposable, E is the fundamental group of a connected spanning surface. Stallings' construction [3] gives a fibration of $S^3 \setminus L$ over S^1 which in turn gives a circular flow by the construction above, and L satisfies a).

4. Examples.

EXAMPLE 1. To show that the class of tame links described in Theorem 3.2 is non-trivial, note that tho trefoil knot and

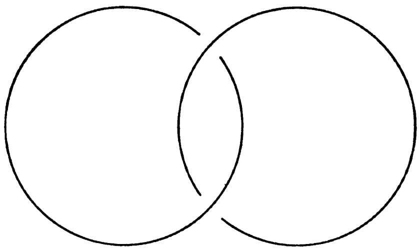

Fig. 1.

both satisfy the conditions.

EXAMPLE 2. *The fixed point set F of a circular flow may be a wild knot.* The complement of the trefoil fibers over S^1. The intersection of a small cell neighborhood of a point on the knot with the fiber is a disk. It follows that we may fiber $C_i \setminus k_i$ for each i, where C_i is a cell and k_i is the trefoil in Figure 2:

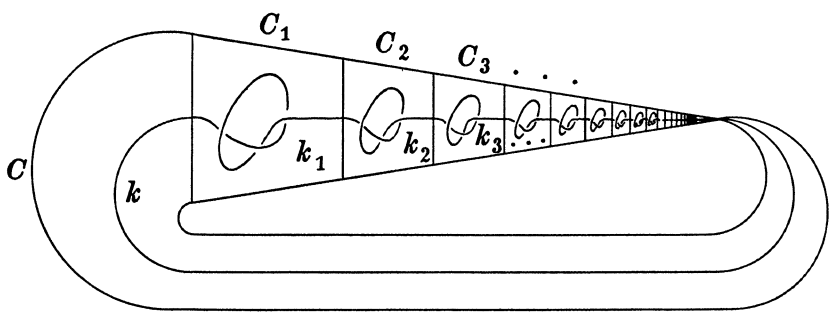

Fig. 2.

By fibering $C \setminus k$ trivially, we have a fibering of the complement of a wild knot F in a pinched solid torus T. T is embedded in a trivial manner in S^3, so we may extend this fibration to $S^3 \setminus F$ using disks sewn to the boundary of T. This fibration gives a circular flow.

EXAMPLE 3. *The fixed point set F of a circular flow may not have a finite number of components.* Consider the figures 3, 4 and 5.

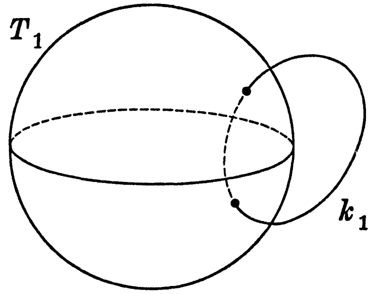

Fig. 3.

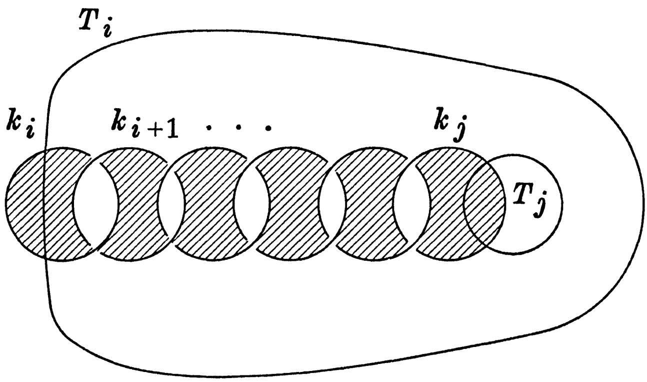

Fig. 4.

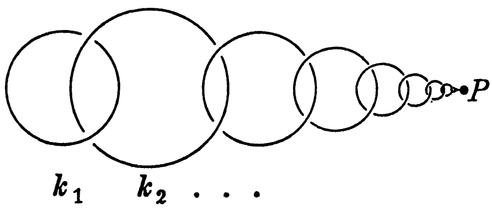

Fig. 5.

$T_1, T_2, \ldots$ are polyhedral cells and $k_1, k_2, \ldots$ are unknotted simple closed curves in S^3. The set

$$T_1 \setminus (T_1 \cap k_1)$$

may be fibered over S^1 in a trivial fashion where the closure of the fiber is a disk intersecting k_1 in $T_1 \cap k_1$. The set

$$S^3 \setminus (k_i \cup \ldots \cup k_j)$$

may be fibered using the shaded surface and Stallings' theorem. This shows how to fiber

$$T_i \setminus (T_j \cup k_i \cup \ldots \cup k_j) .$$

This construction gives a sequence of fiberings

$$\begin{array}{ccccccccc} T_1 \setminus k_1 & \to & S^3 \setminus (k_1 \cup k_2 \cup \mathring{T}_2) & \to & \ldots & \to & S^3 \setminus (k_1 \cup \ldots \cup k_n \cup \mathring{T}_n) & \to & \ldots \\ \downarrow & & \downarrow & & & & \downarrow & & \\ S^1 & \to & S^1 & \to & \ldots & \to & S^1 & \to & \ldots \end{array}$$

where the horizontal maps are inclusions. The union of this sequence is a fibration of

$$S^3 \setminus (k_1 \cup k_2 \cup \ldots \cup P)$$

over S^1 (Figure 5). Then $F = k_1 \cup \ldots \cup P$ and the fibering gives a circular flow whose fixed point set is F. Note that F is zero dimensional at P. By constructing such a fibration at each point of a countable dense set of an unknotted simple closed curve, one gets a circular flow whose fixed point set is not a 1-manifold at any point.

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