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ON THE RATE OF CONVERGENCE FOR DISCRETE INITIAL-VALUE PROBLEMS

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1. Introduction.

Consider the initial-value problem for the heat equation,

$$\frac{\partial u}{\partial t} = \varrho(x)\frac{\partial^2 u}{\partial x^2}, \qquad t \geqq 0, \tag{1.1}$$

$$u(x,0) = v(x), \tag{1.2}$$

where $\varrho(x)$ is a smooth, uniformly positive and bounded function on the real line. For its numerical solution, consider the simple difference scheme

$$\begin{aligned} u_h(x,(n+1)k) &= E_k u_h(x,nk) \\ &= \lambda\varrho(x)\, u_h(x+h,nk) + (1-2\lambda\varrho(x))\, u_h(x,nk) + \\ &\quad + \lambda\varrho(x)\, u_h(x-h,nk), \\ u_h(x,0) &= v(x), \end{aligned} \tag{1.3}$$

where $\lambda = k/h^2$ is constant and so small that $1-2\lambda\varrho(x)$ is non-negative for all x.

Let $\mathscr{C}$ be the set of bounded, uniformly continuous functions of x with

$$\|u\| = \sup_x |u(x)|,$$

and let $\mathscr{C}^j$ be the set of $u \in \mathscr{C}$ with $u^{(j)} \in \mathscr{C}$. It is well known that if for the exact solution $u(x,t)$ of (1.1)-(1.2) we have

$$\sup\{\|\partial^4 u/\partial x^4(\cdot,t)\|\,;\ 0 \leqq t \leqq T\} < \infty, \tag{1.4}$$

then for $0 \leqq t \leqq T$ and $nk = t$,

$$\|u_h(\cdot,nk) - u(\cdot,t)\| = O(h^2) \quad \text{as } h \to 0. \tag{1.5}$$

To secure (1.4) one has to demand that $v \in \mathscr{C}^4$. On the other hand, if $v \in \mathscr{C}$, a density argument proves that

$$\|u_h(\cdot,nk) - u(\cdot,t)\| = o(1) \quad \text{as } h \to 0. \tag{1.6}$$

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Thus, for $v \in \mathscr{C}^4$ we have in (1.5) an estimate for the rate of convergence whereas for v only in $\mathscr{C}$ we have in (1.6) convergence but no estimate for its rate. It is natural to ask what estimate if any for the rate of convergence one can obtain by demanding that v belongs to a class smaller than $\mathscr{C}$ but larger than $\mathscr{C}^4$. An answer to this question is given by the theory of interpolation of Banach spaces; this theory will enable us to show that if v belongs to a space intermediate to $\mathscr{C}$ and $\mathscr{C}^4$ of exponent θ in the sense of interpolation theory, then the rate of convergence is $O(h^{2\theta})$ as $h \to 0$. In particular, if $v \in \mathscr{C}^s$, $s = 1,2,3$, or if $v \in \mathrm{Lip}_s$, $0 < s \leqq 1$, then the rate of convergence is $O(h^{\frac{1}{2}s})$ as $h \to 0$.

The program described above for the heat equation will be carried out in this paper generally for initial-value problems for systems of order M,

$$(1.7) \qquad \frac{\partial u}{\partial t} = P(x,D)u \equiv \sum_{|\alpha| \leqq M} P_\alpha(x) D^\alpha u, \qquad t \geqq 0 ,$$
$$u(x,0) = v(x) ,$$

which are correctly posed in L_p, $1 \leqq p < \infty$, (or in $\mathscr{C}$), and for consistent explicit difference schemes,

$$u_h(x,(n+1)k) = E_k u_h(x,nk) = \sum_\beta e_\beta(x,h)\, u_h(x+\beta h, nk) ,$$
$$u_h(x,0) = v(x), \qquad \lambda = k/h^M = \text{constant} ,$$

which are stable in $L_p(\mathscr{C})$; the convergence of u_h to u is then of course understood as convergence in $L_p(\mathscr{C})$. The restriction to t-independent coefficients $P_\alpha(x)$ and $e_\beta(x,h)$ and to explicit schemes is made to simplify the presentation; the extension to the general case would not introduce any new difficulties. In order to be able to estimate the derivatives of the exact solution $u(x,t)$ in terms of its initial-values we introduce the condition of strong correctness in $L_p(\mathscr{C})$; this is always satisfied if (1.7) has constant coefficients, if it has order one, or if it is uniformly parabolic in Petrowsky's sense. One of the results in the general case is that even if v is not smooth enough to take advantage of the full accuracy of a difference scheme, a higher order of accuracy still gives a better rate of convergence.

In our example above, it might be expected that the infinite differentiability of the solution $u(x,t)$ for $t > 0$ could help us to reduce the regularity assumptions for the initial data without losing the rate of convergence in (1.5). This will indeed be shown to be the case; it will be shown that we can almost get down to $v \in \mathscr{C}^2$ without losing the property (1.5). Also for instance for $v \in \mathrm{Lip}_s$, $0 < s \leqq 1$, we will be able to improve the above estimate of $O(h^{\frac{1}{2}s})$ to $O(h^s)$ as $h \to 0$. Generally such a reduction of the smoothness assumptions on the initial data will be shown to be possible for systems (1.7) which are what we will call strongly parabolic in $L_p(\mathscr{C})$.

The plan of the paper is as follows. In Section 2, we collect some results from the literature on interpolation of Sobolev spaces that we will need. In Section 3 we discuss the continuous initial-value problem and introduce the concepts of strong correctness and parabolicity. Finally, in Section 4 explicit difference operators are discussed, the final results on the rate of convergence are obtained, and some applications are given.

2. Sobolev spaces and interpolation.

We start by reviewing briefly some general notions from the theory of interpolation of Banach spaces. For more information on this subject see e.g. Lions and Peetre [11], Peetre [13], [14], [15], Grisvard [5].

Let X_0 and X_1 be any two Banach spaces which are both continuously imbedded in one and the same topological vector space $\mathscr{X}$. We can then form their sum

$$X_0+X_1 = \{u\,;\ u=u_0+u_1,\ u_i\in X_i,\ i=0,1\}\,,$$

which is a subspace of $\mathscr{X}$. For $u\in X_0+X_1$ and $0<t<\infty$ we set

$$K(t,u) = K(t,u;X_0,X_1) = \inf_{u=u_0+u_1}(\|u_0\|_{X_0}+t\|u_1\|_{X_1})\,. \tag{2.1}$$

If $0<\theta<1$, $1\leqq q\leqq\infty$, we denote by $X=(X_0,X_1)_{\theta,q}$ (interpolation space) the Banach space corresponding to the norm

$$\|u\|_X = \begin{cases}\left(\int\limits_0^\infty (t^{-\theta}K(t,u))^q\,dt/t\right)^{1/q}, & \text{if}\quad 1\leqq q<\infty\,,\\ \sup_{t>0} t^{-\theta}K(t,u), & \text{if}\quad q=\infty\,.\end{cases} \tag{2.2}$$

In view of the inequalities

$$K(t,u) \leqq K(\varkappa t,u) \leqq \varkappa K(t,u), \qquad \varkappa\geqq 1\,, \tag{2.3}$$

we can also use the following equivalent norm, namely

$$\|u\|_X^{(\varkappa)} = \begin{cases}\left(\sum_{i=-\infty}^{\infty}(\varkappa^{-i\theta}K(\varkappa^i,u))^q\right)^{1/q}, & \text{if}\quad 1\leqq q<\infty\,,\\ \sup_i \varkappa^{-i\theta}K(\varkappa^i,u), & \text{if}\quad q=\infty\,.\end{cases} \tag{2.4}$$

One can then readily prove that

$$(X_0,X_1)_{\theta,q_1} \subset (X_0,X_1)_{\theta,q_2}, \quad \text{if}\quad q_1\leqq q_2\,, \tag{2.5}$$

where inclusion means continuous imbedding.

If X and Y are two Banach spaces, $\mathscr{L}(X,Y)$ is the set of bounded linear operators from X into Y. If $A \in \mathscr{L}(X,Y)$, its norm is

$$\|A\|_{X,Y} = \sup_{0 \neq u \in X} \|Au\|_Y / \|u\|_X .$$

In addition to X_0 and X_1, let Y_0 and Y_1 be any two other Banach spaces and consider their corresponding interpolation spaces $(Y_0, Y_1)_{\theta,q}$. We then have the following interpolation theorem:

THEOREM 2.1. *Let* $A \in \mathscr{L}(X_i, Y_i)$, $i=0,1$, $0<\theta<1$, $1 \leqq q \leqq \infty$. *If*

$$X = (X_0, X_1)_{\theta,q} \quad \text{and} \quad Y = (Y_0, Y_1)_{\theta,q},$$

then $A \in \mathscr{L}(X,Y)$ *and*

$$\|A\|_{X,Y} \leqq \|A\|_{X_0,Y_0}^{1-\theta} \|A\|_{X_1,Y_1}^{\theta} .$$

Taking $Y_0 = Y_1 = Y$ and using the fact that $(Y,Y)_{\theta,q} = Y$ for all θ and q, we obtain:

COROLLARY 2.1. *Let* $A \in \mathscr{L}(X_i, Y)$, $i=0,1$, $0<\theta<1$, *and* $X = (X_0, X_1)_{\theta,\infty}$. *Then* $A \in \mathscr{L}(X,Y)$ *and*

$$\|A\|_{X,Y} \leqq \|A\|_{X_0,Y}^{1-\theta} \|A\|_{X_1,Y}^{\theta} . \tag{2.6}$$

Except in the proof of Theorem 4.4, this will be our main and essentially only tool. Therefore, for the convenience of the reader we supply a direct proof. Actually $X = (X_0, X_1)_{\theta,\infty}$ is essentially the maximal space with the properties of Corollary 2.1.

PROOF. If $u = u_0 + u_1$ we have

$$\begin{aligned} \|Au\|_Y &\leqq \|Au_0\|_Y + \|Au_1\|_Y \\ &\leqq C_0 \|u_0\|_{X_0} + C_1 \|u_1\|_{X_1} \\ &\leqq \max(C_0, C_1 t^{-1}) (\|u_0\|_{X_0} + t \|u_1\|_{X_1}) , \end{aligned}$$

where $C_i = \|A\|_{X_i,Y}$, $i=0,1$. In view of (2.1) this yields

$$\|Au\|_Y \leqq \max(C_0, C_1 t^{-1}) \, K(t,u) .$$

If $u \in X = (X_0, X_1)_{\theta,\infty}$ we then get

$$\|Au\|_Y \leqq \inf_{t>0} \max(C_0 t^{\theta}, C_1 t^{\theta-1}) \, \|u\|_X = C_0^{1-\theta} C_1^{\theta} \|u\|_X ,$$

which clearly implies (2.6).

The following theorem is often referred to as the iteration or stability theorem:

THEOREM 2.2. *Let* $0 \leqq \theta_0 < \theta_1 \leqq 1$, *and let* $\tilde{X}_i$, $i=0,1$, *denote* X_i (*if* $\theta_i = i$) *or some space* $(X_0, X_1)_{\theta_i, q_i}$ (*if* $\theta_i \neq i$). *Then, for* $\theta_0 < \theta < \theta_1$ *and* $1 \leqq q \leqq \infty$,

$$(X_0, X_1)_{\theta, q} = (\tilde{X}_0, \tilde{X}_1)_{\tilde{\theta}, q},$$

where $\theta = (1-\tilde{\theta})\theta_0 + \tilde{\theta}\theta_1$.

Combining this result with Corollary 2.1 we get:

COROLLARY 2.2. *Let* $\tilde{X}_i$, $i=0,1$, *be as above and let* $A \in \mathscr{L}(\tilde{X}_i, Y)$, $i=0,1$. *Then if* $\theta_0 < \theta < \theta_1$ *and* $X = (X_0, X_1)_{\theta, \infty}$ *we have* $A \in \mathscr{L}(X, Y)$ *and there exists a constant* C_θ *such that*

$$\|A\|_{X,Y} \leqq C_\theta \|A\|_{\tilde{X}_0, Y}^{1-\tilde{\theta}} \|A\|_{\tilde{X}_1, Y}^{\tilde{\theta}}, \qquad \theta = (1-\tilde{\theta})\theta_0 + \tilde{\theta}\theta_1 .$$

(In this paper, C always denotes a positive constant, not necessarily the same at different occurances. When desirable for clarity, subscripts will be used.)

We shall now specialize our spaces. Let $\mathscr{D}'$ be the space of N-vector valued distributions on R^d. Let L_p, $1 \leqq p \leqq \infty$, be the subspace of measurable (N-vector) functions with

$$\|u\|_{L_p} = \left(\int_{R^d} |u(x)|^p \, dx \right)^{1/p} < \infty, \qquad |u| = \left(\textstyle\sum_{i=1}^N |u_i|^2\right)^{\frac{1}{2}},$$

and let $\mathscr{C}$ be the set of uniformly continuous functions such that

$$\|u\|_{\mathscr{C}} = \sup_{R^d} |u(x)| < \infty .$$

Set $W_p = L_p$ for $1 \leqq p < \infty$ and $W_\infty = \mathscr{C}$. Further, let W_p^m (Sobolev space) be the space of $u \in \mathscr{D}'$ such that

$$D^\alpha u = i^{-|\alpha|} \partial^{|\alpha|} / \partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d} u \in W_p \quad \text{for} \quad |\alpha| \leqq m .$$

This is a Banach space with the norm

$$\|u\|_{W_p^m} = \textstyle\sum_{|\alpha| \leqq m} \|D^\alpha u\|_{W_p} .$$

Let W_p^∞ be the space of $u \in \mathscr{D}'$ such that $u \in W_p^m$ for every m. Set, in particular, $W_\infty^m = \mathscr{C}^m$ and $W_\infty^\infty = \mathscr{C}^\infty$. By a weak form of Sobolev's imbedding theorem, W_p^∞ consists of all infinitely differentiable functions $u \in \mathscr{D}'$ with $D^\alpha u \in W_p$ for all α. For all p with $1 \leqq p \leqq \infty$, W_p^∞ is dense in W_p; this is the main reason for working with W_p instead of L_p.

Let us take $X_0 = W_p$ and $X_1 = W_p^m$. The corresponding interpolation spaces we shall denote by $B_p^{s,q}$ (Besov space) where $0 < s < m$, $1 \leqq p$, $q \leqq \infty$:

$$B_p^{s,q} = (W_p, W_p^m)_{\theta,q}, \qquad \theta = sm^{-1}.$$

Using Theorem 2.2 it can be shown that this definition is independent of m (any integer $>s$). We define for $u \in W_p$, $t>0$,

$$\omega_{1,p}(t,u) = \sup_{|\eta|\leqq t} \|T_\eta u - u\|_{W_p},$$

$$\omega_{2,p}(t,u) = \sup_{|\eta|\leqq t} \|T_\eta u - 2u + T_{-\eta} u\|_{W_p},$$

where

$$T_{\pm\eta} u(x) = u(x \pm \eta).$$

Write $s = S + \sigma$, S integer, $0 < \sigma \leqq 1$. One can then show by estimating $K(t,u)$ that $u \in B_p^{s,q}$ if and only if $u \in W_p$ and

$$\sum_{|\alpha|=S} \left(\int_0^\infty (t^{-\sigma} \omega_{1,p}(t, D^\alpha u))^q \, dt/t \right)^{1/q} < \infty, \quad \text{if} \quad 0<\sigma<1,$$

$$\sum_{|\alpha|=S} \left(\int_0^\infty (t^{-1} \omega_{2,p}(t, D^\alpha u))^q \, dt/t \right)^{1/q} < \infty, \quad \text{if} \quad \sigma=1,$$

with the usual modifications if $q = \infty$, namely

$$\sum_{|\alpha|=S} \sup_{t>0} t^{-\sigma} \omega_{1,p}(t, D^\alpha u) < \infty, \quad \text{if} \quad 0<\sigma<1,$$

$$\sum_{|\alpha|=S} \sup_{t>0} t^{-1} \omega_{2,p}(t, D^\alpha u) < \infty, \quad \text{if} \quad \sigma=1.$$

Thus, in particular, $B_p^{s,\infty}$ is defined by a kind of Lipschitz condition for the derivatives of order S (in the case $\sigma = 1$ a Zygmund type condition); these spaces are sometimes called Lipschitz spaces. For the proof of this result see Lions and Peetre [11], Peetre [13], [15]. We have

$$(2.7) \qquad B_p^{s_1,q_1} \subset B_p^{s_2,q_2} \quad \text{if} \quad s_1 > s_2 \quad \text{or} \quad s_1 = s_2,\ q_1 \leqq q_2.$$

Notice also that if s is a natural number, then

$$(2.8) \qquad B_p^{s,1} \subset W_p^s \subset B_p^{s,\infty}.$$

For more details on the spaces $B_p^{s,q}$, the reader is referred to Nikolskiĭ [12], Besov [1], Taibleson [20], Lions and Peetre [11], Peetre [13], [15], Grisvard [5].

We close this section by collecting in a theorem the special cases of the above general results on interpolation which we will need for our applications.

THEOREM 2.3. *Let $1 \leqq p \leqq \infty$. Let m be a natural number and let s be a real number with $0 < s < m$. Then for all $A \in \mathscr{L}(W_p, W_p)$,*

$$(2.9) \qquad \|A\|_{B_p^{s,\infty}, W_p} \leqq \|A\|_{W_p, W_p}^{1-\theta} \|A\|_{W_p^m, W_p}^{\theta}, \quad \theta = sm^{-1}.$$

Let s be a positive real number and let $0<\theta<1$. Then there exists a constant $C=C_\theta$ such that for all $A \in \mathscr{L}(W_p, W_p)$,

$$(2.10) \qquad \|A\|_{B_p^{\theta s,\infty}} \leqq C\|A\|_{W_p, W_p}^{1-\theta} \, \|A\|_{B_p^{s,1}, W_p}^{\theta} .$$

Let m_i, $i=1,2,3$, be non-negative integers which $m_1<m_2<m_3$. Then there exists a constant C such that for all $A \in \mathscr{L}(W_p^{m_1}, W_p)$,

(2.11)
$$\|A\|_{W_p^{m_2}, W_p} \leqq C\|A\|_{W_p^{m_1}, W_p}^{1-\theta} \, \|A\|_{W_p^{m_3}, W_p}^{\theta}, \qquad \theta=(m_2-m_1)(m_3-m_1)^{-1} .$$

Proof. Corollary 2.1 with

$$X_0 = Y = W_p, \quad X_1 = W_p^m, \quad X = B_p^{s,\infty}$$

gives (2.9) and Corollary 2.2 with

$$X_0 = \tilde{X}_0 = Y = W_p, \quad \tilde{X}_1 = B_p^{s,1}, \quad X_1 = W_p^m (m>s), \quad X = B_p^{\theta s,\infty},$$

gives (2.10). Finally, (2.11) follows by an application of (2.9) with X_0 equal to a product of as many spaces W_p as there are α with $|\alpha| \leqq m_1$, and using (2.8).

3. The initial-value problem.

Consider the intial-value problem

$$(3.1) \qquad \frac{\partial u}{\partial t} = P(x,D)u \equiv \textstyle\sum_{|\alpha|\leqq M} P_\alpha(x) D^\alpha u, \qquad t \geqq 0 ,$$

$$(3.2) \qquad u(x,0) = v(x) ,$$

where $x \in R^d$, $u=u(x,t)$ and $v=v(x)$ are complex N-vectors and $P_\alpha(x)$ are $N \times N$ matrices with elements, which we assume for simplicity to be in $\mathscr{C}^\infty$.

The intial-value problem (3.1), (3.2) is said to be correctly posed in W_p if $P=P(x,D)$ (considered as a densely defined closed operator in W_p) is the infinitesimal generator of a C_0 semi-group of operators $E(t)$ on W_p for $t \geqq 0$, that is (cf. e.g. [7]) the family of bounded operators $E(t)$, $t \geqq 0$, on W_p satisfies

$$E(0) = I = \text{the identity operator},$$
$$E(t_1+t_2) = E(t_1)\,E(t_2), \quad t_1,t_2 \geqq 0 ,$$
$$(3.3) \qquad \|E(t)v\|_{W_p} \leqq C_T \|v\|_{W_p}, \quad 0 \leqq t \leqq T, \quad v \in W_p ,$$
$$\|[k^{-1}(E(k)-I)-P]v\|_{W_p} \to 0, \quad k \to 0, \quad v \in W_p^\infty .$$

The condition (3.3) is equivalent to

$$\|E(t)v\|_{W_p} \leqq C_1 e^{C_2 t} \|v\|_{W_p}, \qquad v \in W_p .$$

The operator $E(t)$ is referred to as the solution operator connected with the initial-value problem.

In the sequel we shall demand not only that the initial-value problem be correctly posed, but that it satisfies the stronger requirement of the following definition. We say that the initial-value problem is strongly correctly posed in W_p if for any $m > 0$, $v \in W_p^m$ implies $E(t)v \in W_p^m$ and there is a constant $C_{m,T}$ such that for all $v \in W_p^m$,

$$\|E(t)v\|_{W_p^m} \leqq C_{m,T} \|v\|_{W_p^m}, \qquad 0 \leqq t \leqq T . \tag{3.4}$$

In particular, this definition implies that $E(t)\, W_p^\infty \subseteq W_p^\infty$.

There are certain cases when correctness in W_p automatically implies strong correctness in W_p. We shall look at some examples.

EXAMPLE 3.1. Assume that $P(x,D) = P(D)$ has constant coefficients and that the corresponding initial-value problem (3.1), (3.2) is correctly posed in W_p so that

$$\|E(t)v\|_{W_p} \leqq C_T \|v\|_{W_p}, \quad 0 \leqq t \leqq T, \quad v \in W_p . \tag{3.5}$$

Since $P(D)$ commutes with D^α for any α one can easily prove that this holds also for $E(t)$ so that

$$D^\alpha v \in W_p \quad \text{implies} \quad D^\alpha E(t)v = E(t) D^\alpha v \in W_p$$

for $t \geqq 0$ and thus by (3.5),

$$\|D^\alpha E(t) v\|_{W_p} \leqq C_T \|D^\alpha v\|_{W_p}, \qquad 0 \leqq t \leqq T .$$

Hence by summation over $|\alpha| \leqq m$ we obtain

$$\|E(t) v\|_{W_p^m} \leqq C_T \|v\|_{W_p^m}, \qquad 0 \leqq t \leqq T ,$$

which proves that the initial-value problem is strongly correctly posed in W_p.

EXAMPLE 3.2. Consider the case of a first order system,

$$\frac{\partial u}{\partial t} = P(x,D)u \equiv \sum_{j=1}^{d} P_j(x) \frac{\partial u}{\partial x_j} + P_0(x)u, \qquad t \geqq 0 , \tag{3.6}$$

$$u(x,0) = v(x) . \tag{3.7}$$

and assume that this initial-value problem is correctly posed in W_p; let us say then that the system is hyperbolic in W_p. For $p = 2$, cf. Kreiss

[9] and Strang [19]; for $p=\infty$, hyperbolicity is a quite strong demand (cf. Brenner [2]).

We will show that a hyperbolic system in W_p with coefficients in $\mathscr{C}^\infty$ is automatically strongly correctly posed in W_p. By formal differentiation of (3.6) we obtain for $|\alpha| \leqq m$,

$$\frac{\partial}{\partial t} D^\alpha u = P(x,D) D^\alpha u + Q_\alpha(x,D) u ,$$

where

$$Q_\alpha(x,D) = \textstyle\sum_{|\beta| \leqq |\alpha|} Q_{\alpha\beta}(x)\, D^\beta u$$

is a differential operator of order $\leqq m$ with coefficients in $\mathscr{C}^\infty$. If $u(x,t)$ is the solution of (3.6), (3.7) it is therefore natural to try to determine the $D^\alpha u(x,t)$ for $t>0$ as the solutions to the following initial-value problem in the set of u_α, $|\alpha| \leqq m$, namely

$$\frac{\partial u_\alpha}{\partial t} = P(x,D) u_\alpha + \textstyle\sum_{|\beta| \leqq |\alpha|} Q_{\alpha\beta}(x) u_\beta, \qquad t \geqq 0 , \tag{3.8}$$

$$u_\alpha(x,0) = D^\alpha v(x), \qquad |\alpha| \leqq m .$$

Let $\mathscr{W}_p^m$ be the set of vectors U with components $u_\alpha \in W_p$, $|\alpha| \leqq m$. Then (3.8) can be written

$$\frac{\partial U}{\partial t} = \tilde{\mathscr{P}}(x,D)\, U ,$$

where $\tilde{\mathscr{P}} = \tilde{\mathscr{P}}(x,D)$ operates in $\mathscr{W}_p^m$. Let us denote the corresponding operator with $Q_{\alpha\beta}(x)=0$ by $\mathscr{P}=\mathscr{P}(x,D)$ so that

$$(\mathscr{P}(x,D)\, U)_\alpha = P(x,D)\, u_\alpha .$$

Clearly $\mathscr{P}$ which like $\tilde{\mathscr{P}}$ acts in $\mathscr{W}_p^m$ is the infinitesimal generator of the C_0 semi-group $\mathscr{E}(t)$ in $\mathscr{W}_p^m$ defined by

$$(\mathscr{E}(t)U)_\alpha = E(t) u_\alpha .$$

Since $\mathscr{P}$ and $\tilde{\mathscr{P}}$ differ only by a bounded operator in $\mathscr{W}_p^m$, $\tilde{\mathscr{P}}$ is also the infinitesimal generator of a C_0 semi-group $\tilde{\mathscr{E}}(t)$ in $\mathscr{W}_p^m$. Let $\mathscr{D}_m$ be the (closed) operator which takes $u \in W_p^m$ into the vector in $\mathscr{W}_p^m$ with components $D^\alpha u$, $|\alpha| \leqq m$. Then (3.8) can be written

$$\mathscr{D}_m P(x,D)\, u = \tilde{\mathscr{P}}(x,D)\, \mathscr{D}_m u .$$

From this we easily obtain

$$\mathscr{D}_m E(t) v = \tilde{\mathscr{E}}(t)\, \mathscr{D}_m v ,$$

from which the strong correctness follows at once.

EXAMPLE 3.3. Assume that the system (3.1) is uniformly parabolic in Petrowsky's sense, so that the eigenvalues $\lambda(x,\xi)$ of $\sum_{|\alpha|=M} P_\alpha(x)\xi^\alpha$ satisfy

$$\sup\{\operatorname{Re}\lambda(x,\xi)\,;\, x,\xi\in R^d, |\xi|=1\} < 0 .$$

The initial-value problem (3.1), (3.2) is then correctly posed in W_p for any p with $1 \leqq p \leqq \infty$; this follows for instance from well-known estimates for the fundamental solution (cf. e.g. Friedman [3]). To see that such a system is indeed strongly correctly posed in W_p we only have to notice that for any $v \in W_p$ the solution $u(x,t)=E(t)v$ is infinitely differentiable for $t>0$. For $v \in W_p^m$ it follows with the notation of Example 3.2 that $\mathscr{D}_m u = \mathscr{D}_m E(t)v$ satisfies a system which is also uniformly parabolic in Petrowsky's sense and which has initial-values $\mathscr{D}_m v$ in $\mathscr{W}_p^m$. This proves (3.4).

The systems in the latest example actually have a stronger property than strong correctness of the corresponding initial-value problem; the solutions are smooth for $t>0$. We shall generally say that the system (3.1) is strongly parabolic of order b in W_p if the initial-value problem (3.1), (3.2) is correctly posed in W_p, if $v \in W_p$ implies $D^\alpha E(t)v \in W_p$ for all α when $t>0$, and if

$$\|E(t)v\|_{W_p^m} \leqq C_{m,T}\, t^{-(m-j)/b} \|v\|_{W_p^j}, \qquad 0<t\leqq T,\ j\leqq m . \tag{3.9}$$

We shall show that the order b of strong parabolicity is at most M. Assume the contrary and let $\gamma = M/b < 1$. Then for $v \in W_p^\infty$ we have

$$\|E(t)v\|_{W_p^M} \leqq Ct^{-\gamma}\|v\|_{W_p}, \qquad 0<t\leqq T , \tag{3.10}$$

$$\|E(t)v\|_{W_p^{2M}} \leqq Ct^{-\gamma}\|v\|_{W_p^M}, \qquad 0<t\leqq T . \tag{3.11}$$

Since

$$(E(t)-I)v = \int_0^t P(\cdot,D)\, E(s)v\, ds ,$$

we obtain by (3.11), for $0<t\leqq T$,

$$\|(E(t)-I)v\|_{W_p^M} \leqq C_1 \int_0^t \|E(s)v\|_{W_p^{2M}}\, ds \leqq C_2 t^{1-\gamma}\, \|v\|_{W_p^M} . \tag{3.12}$$

By the identity

$$v = t^{-1}\int_0^t E(s)v\, ds - t^{-1}\int_0^t (E(s)-I)v\, ds ,$$

we obtain by (3.10) and (3.12), for $0<t\leqq T$,

$$\|v\|_{W_p{}^M} \leqq C_3\{t^{-\nu}\|v\|_{W_p} + t^{1-\nu}\|v\|_{W_p{}^M}\},$$

and since t is arbitrarily small we obtain for $v \in W_p^\infty$,

$$\|v\|_{W_p{}^M} \leqq C_4\|v\|_{W_p},$$

which is impossible. This proves that $b \leqq M$.

EXAMPLE 3.4. Systems which are parabolic in Petrowsky's sense are strongly parabolic of order M in W_p for any p with $1 \leqq p \leqq \infty$; by Example 3.3, and by (2.8) and (2.9) in Theorem 2.3 is sufficient to prove (3.9) for $j=0$. But for that case it follows for instance from the properties of the fundamental solution (cf. e.g. [3, p. 260ff.]).

EXAMPLE 3.5. Systems with constant coefficients which are parabolic in L_2 of order b in the sense of [21] are also strongly parabolic in $L_2 = W_2$ of order b in the present sense. Indeed, by Parseval's relation, the inequality (3.9) with $j=0$, $p=2$ is equivalent to

$$|(1+|\xi|)^m \exp(tP(\xi))| \leqq C_{m,T}\, t^{-m/b}, \qquad 0<t\leqq T, \tag{3.13}$$

for all real ξ. On the other hand, it easily follows from [18] that if (3.1) is parabolic of order b in L_2, then for all real ξ,

$$|\exp(tP(\xi))| \leqq C\exp(-tC_1|\xi|^b + C_2 t), \qquad t \geqq 0,$$

which clearly implies (3.13).

4. The discrete problem and its rate of convergence.

Assume that the initial-value problem (3.1), (3.2) is correctly posed in W_p. For its approximate solution we consider explicit operators E_k, approximating the solution operator $E(k)$, of the form

$$E_k v(x) = \sum_\beta e_\beta(x,h)\, v(x+\beta h).$$

Here $h>0$ is a small parameter related to k by $k/h^M = \lambda =$ constant where M is the order of the system (3.1). Further $\beta = (\beta_1,\ldots,\beta_d)$ with β_j integer, $e_\beta(x,h)$ are $N \times N$ matrices which are polynomials in h with bounded coefficients, and the summation is over a finite set of β.

The operator E_k is said to be consistent with $E(k)$ if for any sufficiently smooth solution $u(x,t)$ of (3.1),

$$u(x,t+k) = E_k u(x,t) + o(k), \quad h \to 0;$$

more precisely, E_k is said to approximate $E(k)$ with order of accuracy (at least) μ if for any sufficiently smooth solution of (3.1),

$$u(x,t+k) = E_k u(x,t) + kO(h^\mu), \quad h \to 0 . \tag{4.1}$$

When (3.1), (3.2) is strongly correctly posed in W_p, this local condition implies the following global estimate:

THEOREM 4.1. *Assume that the initial-value problem* (3.1), (3.2) *is strongly correctly posed in W_p and that E_k approximates $E(k)$ with order of accuracy μ. Then there exists a constant C such that for any $v \in W_p^{M+\mu}$,*

$$\|(E_k - E(k))v\|_{W_p} \leqq C h^{M+\mu} \|v\|_{W_p{}^{M+\mu}}. \tag{4.2}$$

PROOF. We may assume that $v \in W_p^\infty$. For any β we can expand

$$v(x+\beta h) = T_{\beta h} v(x)$$

in a Taylor series with respect to h and obtain for any natural number ϱ,

$$v(x+\beta h) = \sum_{|\alpha|<\varrho} \frac{\beta^\alpha}{\alpha!} (hi)^{|\alpha|} D^\alpha v(x) + R_\varrho^{(\beta)}(v) ,$$

where $R_\varrho^{(\beta)}(v) = O(h^\varrho)$ as $h \to 0$; more exactly we have for $\varrho > 0$,

$$R_\varrho^{(\beta)}(v) = \sum_{|\alpha|=\varrho} \frac{\beta^\alpha}{\alpha!} \varrho i^\varrho \int_0^h (h-s)^{\varrho-1} D^\alpha v(x+\beta s)\, ds . \tag{4.3}$$

It follows that E_k has the form

$$E_k v(x) = \sum_{j+|\alpha|<M+\mu} \varphi_{j,\alpha}(x)\, h^{j+|\alpha|} D^\alpha v(x) + \tilde{R}(v) , \tag{4.4}$$

$$\tilde{R}(v) = \sum_{\beta,\varrho \leqq M+\mu} \varphi_{\beta,\varrho}(x,h)\, h^{M+\mu-\varrho} R_\varrho^{(\beta)}(v) , \tag{4.5}$$

with bounded coefficients $\varphi_{j,\alpha}(x)$ and $\varphi_{\beta,\varrho}(x,h)$.

Set $u(x,t) = E(t)v$. Since $v \in W_p^\infty$, this function is then in W_p^∞ for $t \geqq 0$ by the strong correctness. Let $\mu = (\nu-1)M + \varkappa$, $0 < \varkappa \leqq M$. We have by a Taylor expansion with respect to k,

(4.6)

$$u(x,k) = \sum_{j=0}^{\nu} \frac{k^j}{j!} \frac{\partial^j u}{\partial t^j}(x,0) + R_{\nu+1}(v) = \sum_{j=0}^{\nu} \frac{h^{jM}}{j!} \lambda^j P^j v(x) + R_{\nu+1}(v) ,$$

where $R_{\nu+1}(v) = O(k^{\nu+1})$ when $k \to 0$; more precisely

$$R_{\nu+1}(v) = \int_0^k \frac{(k-s)^\nu}{\nu!} \frac{\partial^{\nu+1} u}{\partial t^{\nu+1}}(x,s)\, ds = \int_0^k \frac{(k-s)^\nu}{\nu!} P^{\nu+1} E(s) v\, ds . \tag{4.7}$$

Together, (4.4) and (4.6) imply

$$(E_k - E(k))v = \sum_{|\alpha|, j < M+\mu} \psi_{j,\alpha}(x)\, h^j D^\alpha v(x) + \tilde{R}(v) - R_{\nu+1}(v) ,$$

where the $\psi_{j,\alpha}(x)$ are bounded. Since the $D^\alpha v(x)$ are arbitrary, and since for any $v \in W_p^\infty$ the two remainder terms are $O(h^{M+\mu})$ as $h \to 0$, the condition (4.1) implies $\psi_{j,\alpha}(x) = 0$. The theorem therefore follows by (4.3), (4.5), and (4.7) from the following two lemmas:

LEMMA 4.1 *For the remainder* $\tilde{R}(v)$ *defined by* (4.3), (4.4), *and* (4.5) *there is a constant* C *such that for* $v \in W_p^{M+\mu}$,

$$\|\tilde{R}(v)\|_{W_p} \leqq C h^{M+\mu} \|v\|_{W_p{}^{M+\mu}} .$$

PROOF. By (4.3) and (4.5) it is sufficient to prove that for $u \in W_p$, $\varrho > 0$, and arbitrary β,

$$\left\| \int_0^h (h-s)^{\varrho-1} T_{\beta s} u \, ds \right\|_{W_p} \leqq \frac{h^\varrho}{\varrho} \|u\|_{W_p} . \tag{4.8}$$

But by Minkowsky's inequality we have

$$\begin{aligned} \left\| \int_0^h (h-s)^{\varrho-1} T_{\beta s} u \, ds \right\|_{W_p} &\leqq \int_0^h (h-s)^{\varrho-1} \|T_{\beta s} u\|_{W_p} \, ds \\ &= \int_0^h (h-s)^{\varrho-1} \, ds \, \|u\|_{W_p} = \frac{h^\varrho}{\varrho} \|u\|_{W_p} , \end{aligned}$$

and the result follows.

LEMMA 4.2. *Under the assumptions of Theorem* 4.1, *for the remainder* $R_{\nu+1}(v)$ *defined by* (4.7), *there is a constant* C *such that for* $v \in W_p^{M+\mu}$,

$$\|R_{\nu+1}(v)\|_{W_p} \leqq C h^{M+\mu} \|v\|_{W_p{}^{M+\mu}} .$$

PROOF. It follows immediately from (4.7), using at the last step the strong correctness, that

$$\begin{aligned} \|R_{\nu+1}(v)\|_{W_p} &\leqq \int_0^k \frac{(k-s)^\nu}{\nu!} \|P^{\nu+1} E(s) v\|_{W_p} \, ds \\ &\leqq C k^{\nu+1} \max_{0 \leqq s \leqq k} \|E(s) v\|_{W_p{}^{(\nu+1)M}} \\ &\leqq C h^{(\nu+1)M} \|v\|_{W_p{}^{(\nu+1)M}} . \end{aligned} \tag{4.9}$$

On the other hand, we clearly have

$$R_{\nu+1}(v) = R_\nu(v) - \frac{h^{\nu M}}{\nu!} \lambda^\nu P^\nu v ,$$

and so as in (4.9),

(4.10) $$\|R_{\nu+1}(v)\|_{W_p} \leqq Ch^{\nu M} \|v\|_{W_p{}^{\nu M}} .$$

The lemma now follows from (4.9), (4.10) by (2.11) in Theorem 2.3.

The operator E_k is said to be stable in W_p if for any $T > 0$ there is a constant $C = C_T$ such that

$$\|E_k{}^n v\|_{W_p} \leqq C \|v\|_{W_p}, \qquad nk \leqq T,\ v \in W_p .$$

It is well known that for consistent operators E_k stability is the necessary and sufficient condition for the convergence of the solution of the discrete initial-value problem to the solution of the continuous initial-value problem in the sense that for any $v \in W_p$, $nk \leqq T$,

(4.11) $$\|(E_k{}^n - E(nk))v\|_{W_p} \to 0, \quad k \to 0 ,$$

(Lax' equivalence theorem [16]).

We now easily obtain the following estimate for the rate of convergence:

THEOREM 4.2. *Assume that the initial-value problem* (3.1), (3.2) *is strongly correctly posed in W_p and that E_k is stable in W_p and approximates $E(k)$ with order of accuracy μ. Then there is a constant $C = C_T$ such that for any $v \in W_p^{M+\mu}$, $nk \leqq T$,*

$$\|(E_k{}^n - E(nk))v\|_{W_p} \leqq Ch^{\mu} \|v\|_{W_p{}^{M+\mu}} .$$

PROOF. We have

(4.12) $$(E_k{}^n - E(nk))v = \sum_{j=0}^{n-1} E_k^{n-1-j}(E_k - E(k))\, E(jk)v ,$$

and so by the stability of E_k, Theorem 4.1, and the strong correctness,

$$\|(E_k{}^n - E(nk))v\|_{W_p} \leqq C \sum_{j=0}^{n-1} kh^{\mu} \|E(jk)v\|_{W_p{}^{M+\mu}} \leqq Cnkh^{\mu} \|v\|_{W_p{}^{M+\mu}} ,$$

which proves the theorem.

The situation is thus that for initial-values in W_p we have in (4.11) convergence without any added information on its rate, and if the initial-values are known to be in $W_p^{M+\mu}$ we can conclude that the rate of convergence is $O(h^{\mu})$ as $h \to 0$. It is natural to ask what one can say if the initial data belong to a space intermediate to W_p and $W_p^{M+\mu}$. The interpolation theory is Section 2 gives us the tools to prove:

THEOREM 4.3. *Assume that the initial-value problem* (3.1), (3.2) *is strongly correctly posed in W_p and that E_k is stable in W_p and approxi-*

mates $E(k)$ with order of accuracy μ. Then for $0<s<M+\mu$ there is a constant $C=C_{s,T}$ such that for any $v \in B_p^{s,\infty}$, $nk \leqq T$,

$$(4.13) \qquad \|(E_k^n - E(nk))v\|_{W_p} \leqq C h^{s\gamma} \|v\|_{B_p^{s,\infty}}, \qquad \gamma = \mu(M+\mu)^{-1}.$$

PROOF. This follows at once from Theorems 2.3 and 4.2 if we observe that by the stability of E_k and the correctness of (3.1), (3.2), for $v \in W_p$, $nk \leqq T$,

$$\|(E_k^n - E(nk))v\|_{W_p} \leqq C\|v\|_{W_p}.$$

Notice that $\gamma=\mu(M+\mu)^{-1}$ grows with μ and $\lim_{\mu\to\infty}\gamma=1$. This means that the estimate (4.13) becomes increasingly better for fixed s when μ grows. In other words, if for a given strongly correctly posed initial-value problem one can construct stable difference schemes of arbitrarily high order of accuracy, then given any $s>0$ one can obtain rates of convergence arbitrarily close to $O(h^s)$ as $h \to 0$ for all initial values in $B_p^{s,\infty}$.

EXAMPLE 4.1. Consider a symmetric hyperbolic system

$$\frac{\partial u}{\partial t} = \sum_{j=1}^d P_j(x)\frac{\partial u}{\partial x_j}, \qquad P_j^*(x) = P_j(x).$$

The corresponding initial-value problem is correctly posed in L_2 as is well known (cf. Friedrichs [4]). Consider Friedrichs' scheme defined by

$$E_k v = \sum_{j=1}^d \{(d^{-1}I + \lambda P_j)T_j v + (d^{-1}I - \lambda P_j)T_{-j}v\},$$

where

$$T_{\pm j}v(x) = v(x \pm he_j).$$

This operator is known to approximate $E(k)$ with order of accuracy 1 and to be stable in L_2 for sufficiently small λ so that for such λ, (4.13) reads

$$\|(E_k^n - E(nk))v\|_{L_2} \leqq Ch^{\frac{1}{2}s}\|v\|_{B_2^{s,\infty}}, \qquad 0<s<2.$$

Consider now an operator $\tilde{E}_k$ which approximates $E(k)$ with order of accuracy 2 and which is stable in L_2. Such operators can for instance easily be constructed by the methods of Kreiss [9], e.g.

$$\tilde{E}_k v(x) = [I + \lambda \textstyle\sum_j P_j(x)\Delta_j + \frac{1}{2}\lambda^2(\sum_j P_j(x)\Delta_j)^2 - \sigma \sum_j \tilde{\Delta}_j^2]\, v(x),$$

where

$$\Delta_j = \tfrac{1}{2}(T_j - T_{-j}), \qquad \tilde{\Delta}_j = 2I - T_j - T_{-j},$$

with appropriate choices of σ and λ. (Cf. also Lax and Wendroff [10], Strang [18].) For such an operator, (4.13) reads

$$\|(\tilde{E}_k^n - E(nk))v\|_{L_2} \leqq Ch^{2s/3}\|v\|_{B_2^{s,\infty}}, \qquad 0<s<3.$$

In particular, if $v \in W_2^2$ then for E_k the rate of convergence is $O(h)$ but for $\tilde{E}_k$ we get $O(h^{4/3})$. The method of estimating the rate of convergence for $\tilde{E}_k$ when $v \in W_2^2$ by just considering the operator $\tilde{E}_k$ as approximating $E(k)$ with order of accuracy 1 and using Theorem 4.2 gives only $O(h)$.

It is natural to ask if for a parabolic system the smoothing property of the solution operator can be used to reduce the regularity demands on the initial data in Theorems 4.2 and 4.3. This shall indeed be shown to be the case; we have:

THEOREM 4.4. *Assume that the system* (3.1) *is strongly parabolic of order* b *in* W_p *and that* E_k *is stable in* W_p *and approximates* $E(k)$ *with order of accuracy* μ. *Then there is a constant* $C = C_{q,T}$ *such that for* $v \in B_p^{M+\mu-b,q}$ *and* $nk \leqq T$,

$$\|(E_k{}^n - E(nk))v\|_{W_p} \leqq Ch^{\mu}(\log h^{-1})^{1-q^{-1}} \|v\|_{B_p{}^{M+\mu-b,q}} . \tag{4.14}$$

Further, for $0 < s < M + \mu - b$ *there is a constant* $C = C_{s,T}$ *such that for* $v \in B_p^{s,\infty}$, $nk \leqq T$,

$$\|(E_k{}^n - E(nk))v\|_{W_p} \leqq Ch^{s\gamma} \|v\|_{B_p{}^{s,\infty}}, \qquad \gamma = \mu(M+\mu-b)^{-1} . \tag{4.15}$$

PROOF. Using (2.10) in Theorem 2.3 we see that (4.15) follows from (4.14) with $q=1$. It remains to prove (4.14). As in the proof of Theorem 4.2 we use the identity (4.12). Set

$$s_0 = M+\mu, \qquad s_1 = M+\mu-b, \qquad \sigma_i = s_i/b, \qquad i = 0,1 ,$$

and

$$F_{k,j}v = E_k^{n-1-j}(E_k - E(k))E(jk)v .$$

We have by the stability and by Theorem 4.1

$$\|F_{k,j}v\|_{W_p} \leqq Ch^{s_0} \|E(jk)v\|_{W_p{}^{s_0}} . \tag{4.16}$$

For $j=0$, (2.9) in Theorem 2.3 yields

$$\|F_{k,0}v\|_{W_p} \leqq Ch^{s_1} \|v\|_{B_p{}^{s_1,\infty}} ,$$

or, in view of (2.7) since $\mu \leqq s_1$,

$$\|F_{k,0}v\|_{W_p} \leqq Ch^{\mu} \|v\|_{B_p{}^{s_1,q}} . \tag{4.17}$$

For $j=1,\ldots,n-1$, we use the strong parabolicity and get

$$\|F_{k,j}v\|_{W_p} \leqq Ch^{s_0}(jk)^{-\sigma_0} \|v\|_{W_p} . \tag{4.18}$$

Let $v = v_0 + v_1$. Then applying (4.18) to v_0 and (4.16) to v_1 we obtain

$$\|F_{k,j}v\|_{W_p} \leqq Ch^{s_0}(jk)^{-\sigma_0}(\|v_0\|_{W_p} + (jk)^{\sigma_0}\|v_1\|_{W_p{}^{s_0}}) .$$

This again yields

$$\|F_{k,j}v\|_{W_p} \leqq Ch^{s_0}(jk)^{-\sigma_0} K((jk)^{\sigma_0}, v;\, W_p, W_p^{s_0}) .$$

Setting

$$F_k = \sum_{j=1}^{n-1} F_{k,j}$$

we now get

$$\|F_k v\|_{W_p} \leqq Ch^{s_0} \sum_{j=1}^{n-1} (jk)^{-\sigma_0} K((jk)^{\sigma_0}, v) .$$

We majorize the sum by

$$\sum_i \sum_{2^{i-1}<jk\leqq 2^i} (jk)^{-\sigma_0} K((jk)^{\sigma_0}, v) ,$$

where i runs through all integers such that $k \leqq 2^i \leqq 2T$. In view of the inequalities (2.3), each term of the inner sum can be estimated by $2^{\sigma_0} 2^{-i\sigma_0} K(2^{i\sigma_0}, v)$. Since obviously

$$\sum_{2^{i-1}<jk\leqq 2^i} 1 \leqq 2^i k^{-1} ,$$

we thus get, noticing that $\sigma_0 - \sigma_1 = 1$ and $h^M = \lambda k$,

$$\|F_k v\|_{W_p} \leqq Ch^{\mu} \sum_{k\leqq 2^i\leqq 2T} 2^{-i\sigma_1} K(2^{i\sigma_0}, v) . \tag{4.19}$$

By Hölder's inequality and the equivalence of the norms (2.2) and (2.4) (with $\varkappa = 2^{\sigma_0}$) we have

$$\begin{aligned} \sum_{k\leqq 2^i\leqq 2T} 2^{-i\sigma_1} K(2^{i\sigma_0}, v) &\leqq \left(\sum_{k<2^i\leqq 2T} 1\right)^{1-q^{-1}} \left(\sum_{i=-\infty}^{\infty} \left(2^{-i\sigma_1} K(2^{i\sigma_0}, v)\right)^q\right)^{q^{-1}} \\ &\leqq C(\log h^{-1})^{1-q^{-1}} \|v\|_{B_p{}^{s_1}, q} . \end{aligned}$$

Thus inserting this in (4.19) we get

$$\|F_k v\|_{W_p} \leqq Ch^{\mu}(\log h^{-1})^{1-q^{-1}} \|v\|_{B_p{}^{s_1}, q} . \tag{4.20}$$

Together, (4.17) and (4.20) imply (4.14), which completes the proof.

Remark. If we were only interested in the spaces $B_p^{s,\infty}$, the proof of (4.14) could be somewhat simplified. However, to be able to prove (4.15) we need the case $q = 1$, too. A result reminiscent of (4.14) with $q = \infty$ has been obtained in a similar situation by Saul'yev (cf. [17, p. 83]).

Example 4.2. Consider again the initial-value problem (1.1), (1.2) in the introduction. This problem is correctly posed in $\mathscr{C}$, the equation is parabolic in Petrowsky's sense, and so strongly parabolic in $\mathscr{C}$ of order 2. The difference operator E_k defined by (1.3) is stable in $\mathscr{C}$ and approximates $E(k)$ with order of accuracy 2. The conclusions of Theorem 4.4 read in this case

$$\|(E_k{}^n - E(nk))v\|_{\mathscr{C}} \leqq \begin{cases} Ch^2(\log h^{-1})^{1-q^{-1}} \|v\|_{B_\infty{}^2, q} , & \\ Ch^s \|v\|_{B_\infty{}^s, \infty} , & 0 < s < 2 . \end{cases}$$

For the case $\varrho(x) = \text{constant}$, G. Hedstrom [6] has been able to remove the log h^{-1} when $s=2$, $q=\infty$. For a similar result for the homogeneous initial-boundary value problem for (1.1), see Juncosa and Young [8].

REFERENCES

1. O. V. Besov, *Investigation of a family of function spaces in connection with imbedding and continuation theorems*, Trudy Mat. Inst. Steklov. 60 (1961), 42–81. (Russian.)
2. Ph. Brenner, *The Cauchy problem for symmetric hyperbolic systems in L_p*, Math. Scand. 20 (1967), 27–37.
3. A. Friedman, *Partial differential equations of parabolic type*, Prentice–Hall, Englewood Cliffs, New Jersey, 1964.
4. K. O. Friedrichs, *Symmetric hyperbolic linear differential equations*, Comm. Pure Appl. Math. 7 (1954), 345–392.
5. P. Grisvard, *Commutativité de deux foncteurs d'interpolation et applications*, J. Math. Pures Appl. 45 (1966), 143–266.
6. G. W. Hedstrom, *The rate of convergence of some difference schemes*. J. SIAM Numer. Anal. 5 (1968), 363–406.
7. E. Hille and R. S. Phillips, *Functional analysis and semigroups*. Providence, Rhode Island, 1957.
8. M. L. Juncosa and D. M. Young, *On the order of convergence of solutions of a difference equation to a solution of the diffusion equation*. J. Soc. Indust. Appl. Math. 1 (1953), 111–135.
9. H. O. Kreiss, *On difference approximations of the dissipative type for hyperbolic differential equations*, Comm. Pure Appl. Math. 17 (1964), 335–353.
10. P. D. Lax and B. Wendroff, *Difference schemes for hyperbolic equations with high order of accuracy*, Comm. Pure Appl. Math. 17 (1964), 381–398.
11. J. L. Lions and J. Peetre, *Sur une classe d'espaces d'interpolation*, Inst. Hautes Études Sci. Publ. Math. No. 19 (1964), 5–68.
12. S. M. Nikolskiĭ, *On imbedding, continuation, and approximation theorems for differentiable functions of several variables*, Usephi Mat. Nauk 16 (1961), no. 5 (101), 63–114. (Russian.) (Also Russian Math. Surveys 16 (1961), 55–104.)
13. J. Peetre, *Espaces d'interpolation, généralisations, applications*, Rend. Sem. Mat. Fis. Milano 34 (1964), 133–164.
14. J. Peetre, *Espaces d'interpolation et théorème de Soboleff*, Ann. Inst. Fourier (Grenoble) 16 (1966), 279–317.
15. J. Peetre, *Interpolation dans le cas des semi-groupes*, To appear.
16. R. D. Richtmyer and K. W. Morton, *Difference methods for initial-value problems*, Second edition, Interscience, New York, 1967.
17. V. K. Saul'yev, *Integration of equations of parabolic type by the method of nets*, Pergamon Press, Oxford, London, Edinburgh, New York, Paris, Frankfurt, 1964.
18. G. Strang, *Accurate partial difference methods*. I, *Linear Cauchy problems*, Arch. Rational Mech. Anal. 12 (1963), 392–402.
19. G. Strang, *On strong hyperbolicity*, J. Math. Kyoto Univ. 6 (1967), 397–417.
20. M. H. Taibleson, *On the theory of Lipschitz spaces of distributions in Euclidean n-space*. I, *Principal properties*, J. Math. Mech. 13 (1964), 407–479.
21. V. Thomée, *Parabolic difference operators*, Math. Scand. 20 (1967), 77–107.

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