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THE OBLIQUITY-TYPE OF A SET OF VECTORS

ANDREW SOBCZYK

We shall classify Euclidean congruence types of n-tuples of vectors $\{\beta_1,\ldots,\beta_n\}=\{\beta_i\}$, according to the following scheme. In Euclidean space of $2n$ dimensions E_{2n}, suppose there are n vectors $\gamma_1,\ldots,\gamma_n$ from an n-dimensional subspace E_n, and let $\varphi_1,\ldots,\varphi_n$ be orthonormal basis vectors for an orthogonal complement $E_n^{\perp}$ in E_{2n} of E_n, such that the vectors β_i may be expressed (to within Euclidean congruence of the n-tuple) in the form

$$\beta_i = \gamma_i + a_i\varphi_i, \qquad i=1,\ldots,n . \tag{1}$$

The congruence type of $\{\beta_i\}$ of course is uniquely determined by the values of the inner products (β_i,β_j). Since E_n is n-dimensional, for any $\{\beta_i\}$ a congruent n-tuple is expressible in the form (1), at least with all a_i's equal to zero. For any expression (1), we have $(\beta_i,\beta_j)=(\gamma_i,\gamma_j)$ for $i \neq j$, and

$$\|\beta_i\|^2 = (\beta_i,\beta_i) = \|\gamma_i\|^2 + a_i^2 .$$

Thus if the off-diagonal values of the inner product matrix $\{(\beta_i,\beta_j)\}$ are realized by any set of vectors $\{\gamma_i\}$, we may represent the n-tuple $\{\beta_i\}$ in the desired form provided that $\|\beta_i\| \geqq \|\gamma_i\|$ for $i=1,\ldots,n$. Accordingly we define the (*obliquity*) *type* of $\{\beta_i\}$ as the minimum possible dimension of the linear subspace $\langle\gamma_1,\ldots,\gamma_n\rangle$ spanned by the set of vectors $\{\gamma_i\}$, with respect to which a congruent n-tuple to $\{\beta_i\}$ can be expressed as $\{\gamma_i + a_i\varphi_i\}$. Thus of course in case the β_i's of an n-tuple are mutually orthogonal, the type is 0.

For any set of vectors $\{\gamma_i\}$, which is such that the inner products (γ_i,γ_j) satisfy

$$(\gamma_i,\gamma_j) = (\beta_i,\beta_j), \qquad i \neq j,\ i,j=1,2,\ldots,n , \tag{2}$$

we may replace γ_j by another vector, with satisfaction of the same condition (2), to reduce the dimension of the subspace $\langle\gamma_1,\ldots,\gamma_n\rangle$ by 1, unless γ_j is in the span of the remaining γ_i's. We state this as a Lemma.

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LEMMA. *If $\gamma_n \notin \langle\gamma_1,\dots,\gamma_{n-1}\rangle$, then without affecting the values of the off-diagonal inner products, we may replace γ_n by a vector γ_n', to reduce the dimension of $\langle\{\gamma_i\}\rangle$.*

PROOF. In case dim $\langle\gamma_1,\dots,\gamma_{n-1}\rangle=0$, the type of $\{\beta_i\}$ is zero, and we may replace γ_n by a zero vector, increasing the value of a_n to maintain congruence. Otherwise, let δ_n be a unit vector in E_n which is orthogonal to $\langle\gamma_1,\dots,\gamma_{n-1}\rangle$. Then replacing γ_n by any vector of the form $\gamma_n'=\gamma_n+d_n\delta_n$, the off-diagonal inner products are not affected. We have

$$(\gamma_n',\gamma_n') = (\gamma_n,\gamma_n) + d_n[2(\gamma_n,\delta_n)+d_n]\,.$$

The choice $d_n=-\|\gamma_n\|\cos\theta_n$, where θ_n is the angle between γ_n and δ_n, reduces (γ_n,γ_n) to its minimum possible value

$$(3)\qquad (\gamma_n',\gamma_n') = (\gamma_n,\gamma_n)(1-\cos^2\theta_n) = (\gamma_n,\gamma_n)\sin^2\theta_n\,,$$

and also subtracts off the component of γ_n in the direction of δ_n, placing γ_n' in the subspace $\langle\gamma_1,\dots,\gamma_{n-1}\rangle$.

THEOREM 1. *For each n-tuple $\{\beta_i\}$, the type exists, and its value is at most $n-1$. (For arbitrary non-zero scalars $b_1,\dots,b_n$, the type of $\{b_1\beta_1,\dots,b_n\beta_n\}$ is the same as that of $\{\beta_1,\dots,\beta_n\}$.)*

PROOF. Again consider the symmetric matrix of inner products $\{(\beta_i,\beta_j)\}$. If one of the β_i's, say β_n, is orthogonal to the span of the others, then we may replace β_n by 0 without affecting the values of the off-diagonal inner products. Then if another β_i, say β_{n-1}, is orthogonal to the span of the remaining β_i's, it also may be replaced by 0 without affecting the values of the off-diagonal inner products; and so on. An n-tuple congruent to the original β_i's may be expressed in the form $\beta_1,\beta_2,\dots,\beta_k,\ a_{k+1}\varphi_{k+1},\dots,a_n\varphi_n$. In any case of $k<n$, we have therefore that type $\{\beta_i\}\leqq k<n$.

In case no β_i is orthogonal to the span of the others, $k=n$, by the Lemma we have that the type is $\leqq n-1$. Also in the case $k<n$ of the preceding paragraph, the type is $\leqq k-1$. The process indicated in the proof of the Lemma may be continued until we have the situation that each γ_j is in the span of the other γ_i's. Let us refer to this property of the set of vectors $\{\gamma_i\}$ as the *span property.*

CONVERSE LEMMA. *If a set of vectors $\{\gamma_i\}$ has the span property, then there does not exist a congruent set of vectors $\{\gamma_i'+a_i\varphi_i\}$ with* $\dim\langle\{\gamma_i'\}\rangle<$

$\dim\langle\{\gamma_i\}\rangle$. *With the same hypothesis concerning the set* $\{\gamma_i\}$, *for any set of vectors* $\{\gamma_i'+a_i\varphi_i\}$ *such that the correspondence*

$$\gamma_i \leftrightarrow \gamma_i'+a_i\varphi_i$$

is a congruence, we have that necessarily $a_i=0$ *for* $i=1,\ldots,n$.

Proof. If a set $\{\beta_i\}$ has the span property, then of course any congruent set $\{\gamma_i'\}$ has the property. If $a_j \neq 0$, then $\gamma_j'+a_j\varphi_j$ cannot be in the span of the other $(\gamma_i'+a_i\varphi_i)$'s, because $\varphi_1,\ldots,\varphi_n$, E_n are mutually orthogonal. No linear combination of the other vectors can cancel the non-zero coefficient a_j.

Theorem 2. *For a set of vectors* $\{\beta_i\}$ *and any two sets of vectors* $\{\gamma_i+a_i\varphi_i\}$ *and* $\{\gamma_i'+a_i'\varphi_i\}$, *in which* $\{\gamma_i\}$ *and* $\{\gamma_i'\}$ *both have the span property, in case the correspondences*

$$\gamma_i+a_i\varphi_i \leftrightarrow \beta_i \quad \text{and} \quad \gamma_i'+a_i'\varphi_i \leftrightarrow \beta_i$$

are congruences, then necessarily $\gamma_i \leftrightarrow \gamma_i'$ *is a congruence, and for* $i=1,\ldots,n$, *we have* $a_i=\pm a_i'$. (Our "congruence" includes the possibility of an involutoric isometry, or "mirror image" situation, in which $\{\beta_i'\}$ could not be brought into coincidence with $\{\beta_i\}$ by an orthogonal transformation of determinant $+1$.)

Proof. It follows from our hypothesis that the correspondence $\gamma_i \leftrightarrow \gamma_i'+(a_i'\pm a_i)\varphi_i$ is a congruence. By the Converse Lemma, for each $i=1,\ldots,n$ we must have $a_i'\pm a_i=0$, and therefore that $\gamma_i \leftrightarrow \gamma_i'$ is a congruence.

Corollary. *Given a set of vectors* $\{\beta_i\}$, *for any expression of the vectors in the form* $\beta_i=\gamma_i+a_i\varphi_i$, $i=1,\ldots,n$, *in which the set* $\{\gamma_i\}$ *has the span property, we have that the dimension of* $\langle\gamma_1,\ldots,\gamma_n\rangle$ *is as small as possible, so that the type of* $\{\beta_i\}$ *is equal to that dimension.*

Theorem 3. *In case* $\text{type}\{\beta_i\}=\dim\langle\gamma_1,\ldots,\gamma_n\rangle=m$, *the inner product matrix* $\{(\gamma_i,\gamma_j)\}$ *(which agrees off the diagonal with* $\{(\beta_i,\beta_j)\}$*) is of the form* CC^{T}, *where* C *is an* n *by* m *matrix, and* C^{T} *is its transposed matrix.*

Proof. We may choose an orthonormal basis $\delta_1,\ldots,\delta_n$ for E_n, such that $\langle\delta_1,\ldots,\delta_m\rangle=\langle\gamma_1,\ldots,\gamma_n\rangle$. Then

$$\gamma_1=c_{11}\delta_1+\ldots+c_{1m}\delta_m, \ \ldots\ ,\gamma_n=c_{n1}\delta_1+\ldots+c_{nm}\delta_m;$$

the matrix of coefficients $C=\{c_{ij}\}$ is the required matrix.

Representing vectors by their coefficients with respect to an orthonormal basis in E_n, the set of vectors

$$\begin{aligned} \gamma_1 &= (1,1,1,\ldots,1,0) \\ \gamma_2 &= (1,0,0,\ldots,0,0) \\ \gamma_3 &= (0,1,0,\ldots,0,0) \\ &\cdots\cdots\cdots\cdots \\ \gamma_n &= (0,0,0,\ldots,1,0) \end{aligned}$$

has the span property, with

$$\dim\langle\gamma_1,\ldots,\gamma_n\rangle = n-1 .$$

If A is any linear transformation of rank k on $\langle\gamma_1,\ldots,\gamma_n\rangle$, then the set of transforms $A\gamma_1,\ldots,A\gamma_n$ has the span property. Also it is geometrically obvious that there are sets of any number n of vectors in the plane, or in a line, which have the span property. Therefore for each integer k between 0 and $n-1$, inclusive, there exists a linearly independent set $\{\beta_i\}$ which is of obliquity type k.

The author has in mind application of the obliquity type to classification of convex polytopes. At each vertex of a polytope, consider the set of edge vectors originating at the vertex. In case of an n-simplex, the type can be zero for at most one vertex. If the type is 0 at one vertex, then necessarily it is 1 at all other vertices. The type of an equilateral n-simplex is 1 at each vertex. This follows from the fact that the equilateral n-simplex may be congruently embedded in E_{n+1} with its vertices at $(a,0,\ldots,0),\ldots,(0,\ldots,0,a)$. Translation through say $(-a,0,\ldots,0)$ places one vertex at the origin, and the set of vectors from the origin to the other vertices clearly is of type 1. Similarly, it may be seen that a simplex which is of type 1 at one vertex, must be of type $\leqq 2$ at all of its other vertices; and that for any Euclidean simplex, if the minimum type among the vertices is k, then each vertex is of type either k or $k+1$.

CLEMSON UNIVERSITY, CLEMSON, SOUTH CAROLINA, U.S.A.