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ANGLE SUMS OF CONVEX POLYTOPES

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1. Introduction.

Let P be a d-dimensional convex polytope in d-dimensional Euclidean space E^d, and, for $0 \leqq j \leqq d-1$, let F_i^j, $i=1,\ldots,f_j(P)$, denote its j-faces. Associated with each face F_i^j is a well-defined real number called the interior angle of P at the face F_i^j, and we write $\varphi_j(P)$ for the sum of the interior angles at all the j-faces of P. The purpose of this paper is to investigate the properties of these angle-sums $\varphi_j(P)$, $j=0,\ldots,d-1$.

We begin by recalling the classical *Gram–Euler Theorem*:

(1) THEOREM. *For every convex d-polytope P,*

$$\sum_{j=0}^{d-1} (-1)^j \varphi_j(P) = (-1)^{d-1} . \tag{2}$$

This theorem was first stated and proved for $d=3$ by J. P. Gram [1] in 1874. D. M. Y. Sommerville [4] gave an invalid proof for general d which was later corrected by B. Grünbaum and appears in [2, § 14.1]. For a detailed history of the theorem and related results the reader is referred to [2, § 14.4].

Grünbaum's proof, although elementary in concept, is complicated in detail. Here, in § 2, we present a new proof of the theorem which has the merit of brevity and simplicity. It is based on proofs which were discovered independently by the two authors, and in combining these, further simplification has been possible. The only theorem on polytopes that we shall need to quote is the well-known Euler relation connecting the numbers $f_j(P)$ of j-faces of a convex d-polytope P:

$$\sum_{j=0}^{d-1} (-1)^j f_j(P) = 1+(-1)^{d-1} . \tag{3}$$

For a proof of this relation, see [2, §§ 8.1–8.2].

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In § 3 we shall prove further theorems about angle-sums of polytopes by modifying the methods of § 2. In § 4 these theorems will be applied to finding some additional linear relations (analogous to the Dehn–Sommerville relations [2, § 9.2]) that hold when the faces of the polytope are of certain prescribed combinatorial types.

In § 5, many of the results of the previous sections will be extended to convex polytopes lying in spherical space S^d, and, in particular, we will give a new proof of Sommerville's Theorem [4] [5, p. 157] concerning the angle-sums and volume of a convex spherical d-polytope. We conclude, in § 6, with some general remarks concerning extensions, generalisations and applications of our theorems.

2. Proof of the Gram-Euler Theorem.

Let S^{d-1} be the unit $(d-1)$-sphere with centre at the origin $o \in E^d$, and $x \in S^{d-1}$ be any unit vector. Let z be a relative interior point of the j-face F^j of the convex d-polytope P. Write $\chi(P,F^j,x)=1$ if the half-line $z+\lambda x$, $\lambda>0$, intersects P, and $\chi(P,F^j,x)=0$ otherwise. It is clear that the value of $\chi(P,F^j,x)$ depends only on P, F^j and x, and *not* on the choice of the point z. The *interior angle* $\varphi(P,F^j)$ of the polytope P at the face F^j may be defined by

$$(4) \qquad \varphi(P,F^j) = \frac{1}{\mu(S^{d-1})} \int_{S^{d-1}} \chi(P,F^j,x)\, d\mu(x) ,$$

where μ is the Lebesgue measure on the sphere S^{d-1}. Geometrically this means that if B is a d-dimensional ball centred at z with radius so small that the only faces of P which intersect B are those which include F^j, then $\varphi(P,F^j)$ is the ratio of the volume of $B \cap P$ to the volume of B. In particular

$$\varphi(P,F^{d-1}) = \tfrac{1}{2}$$

for every facet (face of dimension $d-1$) F^{d-1} of P. Using this notation, the angle-sums $\varphi_j(P)$, $0 \leqq j \leqq d-1$, are defined by

$$(5) \qquad \varphi_j(P) = \sum_{i=1}^{f_j(P)} \varphi(P,F_i^j) .$$

Hence, by (4),

$$(6) \qquad \sum_{j=0}^{d-1} (-1)^j \varphi_j(P) = \sum_{j=0}^{d-1} (-1)^j \sum_{i=1}^{f_j(P)} \varphi(P,F_i^j) = \frac{1}{\mu(S^{d-1})} \int_{S^{d-1}} g(x)\, d\mu(x) ,$$

where

(7) $$g(x) = \sum_{j=0}^{d-1} (-1)^j \sum_{i=1}^{f_j(P)} \chi(P, F_i^j, x) .$$

Let $T \subset S^{d-1}$ be the set of all unit vectors $x \in S^{d-1}$ which are parallel to a proper face of P. Then T is of measure zero on S^{d-1} and so may be neglected in the integrations of (4) and (6). For each $x \notin T$, let H_x be the hyperplane through o with normal x, and let P_x be the $(d-1)$-polytope that arises by orthogonal projection of P on to H_x. (Such a projection with $x \notin T$ will be called a *regular projection*.) Since $x \notin T$, each j-face F^j of P projects orthogonally in a $(1,1)$ manner onto a j-polytope F_x^j in H_x, and F_x^j will be a proper face of P_x if and only if there exists a line l parallel to x which contains a relative interior point z of F^j, but contains no interior point of P. This last assertion is true for the following reason. Firstly, if l contains an interior point of P, then F_x^j will contain an interior point of P_x, and so will not be a proper face of P_x. Secondly, if l contains no interior point of P, it will be possible to find a hyperplane H which supports P, contains l, and is such that $H \cap P = F^j$. But then $H \cap H_x$ supports P_x in H_x, and F^j projects onto the complete intersection $H \cap H_x \cap P_x$. This is a face of P_x which must therefore coincide with F_x^j. Every j-face G^j of P_x, $0 \leqq j \leqq d-2$, arises in this way from some j-face F^j of P by projection, and, in fact, from a unique such F^j, namely the inverse image of G^j under the projection.

From these facts we deduce that every j-face F^j of P, $0 \leqq j \leqq d-1$, must satisfy precisely one of the following three conditions. As above, z denotes a relative interior point of F^j.

(i) *The half-line* $z + \lambda x$, $\lambda > 0$, *contains an interior point of* P. In this case $\chi(P, F^j, x) = 1$, and the number of faces satisfying this condition is

$$\sum_{i=1}^{f_j(P)} \chi(P, F_i^j, x) .$$

(ii) *The half-line* $z + \lambda x$, $\lambda < 0$, *contains an interior point of* P. In this case $\chi(P, F^j, -x) = 1$, and the number of j-faces satisfying this condition is

$$\sum_{i=1}^{f_j(P)} \chi(P, F_i^j, -x) .$$

(iii) *The line* $z + \lambda x$, $-\infty < \lambda < \infty$, *contains no interior point of* P, and in fact, intersects P at just one point z. Then F^j projects on to a j-face of P_x and the number of j-faces satisfying this condition is $f_j(P_x)$ for $0 \leqq j \leqq d-2$. This will also be true for $j = d-1$ if we conventionally define $f_{d-1}(P_x)$ to be zero.

Since the total number of j-faces of P is $f_j(P)$, we deduce that for each j,

$$(8) \qquad \sum_{i=1}^{f_j(P)} \chi(P,F_i^j,x) + \sum_{i=1}^{f_j(P)} \chi(P,F_i^j,-x) + f_j(P_x) = f_j(P) .$$

By Euler's Theorem (3),

$$(9) \qquad \sum_{j=0}^{d-1} (-1)^j f_j(P_x) = 1 + (-1)^{d-2} ,$$

and

$$\sum_{j=0}^{d-1} (-1)^j f_j(P) = 1 + (-1)^{d-1} .$$

Hence, from (7), (8), and the above equalities,

$$\begin{aligned} g(x) + g(-x) &= \sum_{j=0}^{d-1} (-1)^j \sum_{i=1}^{f_j(P)} \chi(P,F_i^j,x) + \sum_{j=0}^{d-1} (-1)^j \sum_{i=1}^{f_j(P)} \chi(P,F_i^j,-x) \\ &= \sum_{j=0}^{d-1} (-1)^j f_j(P) - \sum_{j=0}^{d-1} (-1)^j f_j(P_x) \\ &= \big(1+(-1)^{d-1}\big) - \big(1+(-1)^{d-2}\big) \\ &= 2(-1)^{d-1} . \end{aligned}$$

This relation and (6) imply

$$\begin{aligned} \sum_{j=0}^{d-1} (-1)^j \varphi_j(P) &= \frac{1}{\mu(S^{d-1})} \int_{S^{d-1}\setminus T} g(x)\, d\mu(x) \\ &= \frac{1}{2\mu(S^{d-1})} \int_{S^{d-1}\setminus T} \big(g(x)+g(-x)\big)\, d\mu(x) \\ &= \frac{1}{\mu(S^{d-1})} \int_{S^{d-1}\setminus T} (-1)^{d-1}\, d\mu(x) \\ &= (-1)^{d-1} , \end{aligned}$$

and Theorem (1) is proved.

As remarked in the introduction, in the above proof we have only made use of the most elementary properties of polytopes. If, on the other hand, we use the properties of cell-complexes, then an even shorter proof of the theorem can be constructed in the following manner: Define

$$L_x = \{\emptyset\} \cup \{F_i^j : 0 \leqq j \leqq d-1,\ \chi(P,F_i^j,x) = 0\} ,$$

so that

$$|L_x| = \bigcup \{F_i^j : 0 \leqq j \leqq d-1,\ \chi(P,F_i^j,x) = 0\} .$$

Then L_x is a cell-complex and $|L_x|$ is mapped in a 1-1 manner onto P_x

by orthogonal projection on to H_x. Therefore, by [3, Theorem 2.3], for all x,

$$\sum_{j=0}^{d-1}(-1)^j f_j(L_x) = 1\,,$$

and the function $g(x)$, defined by (7), is given by

$$\begin{aligned} g(x) &= \sum_{j=0}^{d-1}(-1)^j\big(f_j(P)-f_j(L_x)\big) \\ &= \sum_{j=0}^{d-1}(-1)^j f_j(P) - \sum_{j=0}^{d-1}(-1)^j f_j(L_x) \\ &= 1-(-1)^d-1 \\ &= (-1)^{d-1}\,. \end{aligned}$$

Substituting this value of $g(x)$ in (6) yields (1).

3. Further theorems on angle-sums.

In this section we shall prove more general theorems on angle-sums, using the methods of § 2, and in § 4 give applications of these results.

We begin by introducing some notation. For any convex d-polytope P, define two d-vectors $\varphi(P)$ and $f(P)$ by

$$\begin{aligned} \varphi(P) &= \big(\varphi_0(P),\ldots,\varphi_{d-1}(P)\big)\,, \\ f(P) &= \big(f_0(P),\ldots,f_{d-1}(P)\big)\,. \end{aligned}$$

These will be referred to as the φ-vector and f-vector of P, respectively. Also, if P_x is a $(d-1)$-polytope that arises by regular projection of P, write

$$f(P_x) = \big(f_0(P_x),\ldots,f_{d-2}(P_x),0\big)\,.$$

Since the number of vertices of P_x cannot exceed the number of vertices of the given polytope P, the polytopes P_x are of a finite number of different combinatorial types. Hence there exists only a finite number n of distinct vectors $f(P_x)$. It will be convenient to represent these by $f(P_1),\ldots,f(P_n)$. ($P_1,\ldots,P_n$ may be regarded as convex $(d-1)$-polytopes combinatorially equivalent to some of the polytopes P_x that arise by regular projection of P.)

(10) THEOREM. *For each convex d-polytope P, the vector $f(P)-2\varphi(P)$ is a positive convex combination of the vectors $f(P_i)$, $i=1,\ldots,n$, defined above, that is to say,*

$$(11) \qquad f(P) - 2\varphi(P) = \sum_{i=1}^{n} \mu_i f(P_i),$$

where $\sum \mu_i = 1$ *and* $\mu_i > 0$ *for each* i.

PROOF. Let F^j be any j-face of P and z a relative interior point of F^j. Then, by (i) and (ii) of § 2, a necessary and sufficient condition that F^j does not project into a j-face of P_x, $x \in S^{d-1} \setminus T$, is that the line $z + \lambda x$, $-\infty < \lambda < \infty$, does not intersect the interior of the polytope P. Writing $\mathrm{cone}_z P$ for the cone spanned by P at z, we see that this condition is satisfied if and only if the line $z + \lambda x$, $-\infty < \lambda < \infty$, lies in

$$\mathrm{cone}_z P \cup \mathrm{cone}_z (2z - P) .$$

Hence F^j does not project into a face of P_x, $x \in S^{d-1} \setminus T$, if and only if

$$x \in (S^{d-1} \setminus T) \cap \big(\mathrm{cone}_o (P - z) \cup \mathrm{cone}_o (z - P)\big) .$$

Writing $H_1, \ldots, H_m$ for the hyperplanes through o parallel to the facets that are incident with z, $\mathrm{cone}_o(P - z)$ will be the intersection of m closed half-spaces bounded by $H_1, \ldots, H_m$. Since the combinatorial type of P_x is determined completely by its lattice of faces, and hence by the faces of P that project into proper faces of P_x, we deduce the following: The set of hyperplanes through o parallel to those determined by the facets of P divides S^{d-1} into a finite number of open convex spherical polytopes. For all x lying in one of these polytopes, the P_x are of the same combinatorial type. Further, the union of these spherical polytopes covers the whole of S^{d-1} except for a set of measure zero which can easily be identified with the set T.

Let P_i be one of the $(d-1)$-polytopes defined earlier, and let J_i be the union of the open spherical polytopes consisting of those $x \in S^{d-1} \setminus T$ such that $f(P_x) = f(P_i)$. Then the above reasoning shows that

$$(12) \qquad \sum_{i=1}^{n} \mu(J_i) = \mu(S^{d-1})$$

and, since each J_i is open and non-empty,

$$\mu(J_i) > 0, \qquad i = 1, \ldots, n .$$

(Notice that each region J_i is centrally symmetric in o, since $P_x = P_{-x}$.) Because $f_j(P_x)$ is constant in each J_i, we deduce

$$(13) \qquad \int_{S^{d-1} \setminus T} f_j(P_x)\, d\mu(x) = \sum_{i=1}^{n} \mu(J_i) f_j(P_i), \qquad 0 \leqq j \leqq d-1 .$$

Hence, if we integrate (8) over $S^{d-1}\setminus T$, and divide by $\mu(S^{d-1})$ we obtain, using (4) and (5),

$$2\varphi_j(P) + \frac{1}{\mu(S^{d-1})}\sum_{i=1}^{n}\mu(J_i)\,f_j(P_i) = f_j(P)\,.$$

This may be written in the form

$$f_j(P) - 2\varphi_j(P) = \sum_{i=1}^{n}\mu_i f_j(P_i)\,,$$

where $\mu_i = \mu(J_i)/\mu(S^{d-1}) > 0$, and $\sum\mu_i = 1$ by (12). Since this is true for each value of j, $0 \leqq j \leqq d-1$, we deduce that (11) holds and so Theorem (10) is proved.

From (11) it follows that

$$\varphi(P) = \tfrac{1}{2}\Big(f(P) - \sum_{i=1}^{n}\mu_i f(P_i)\Big) = \tfrac{1}{2}\sum_{i=1}^{n}\mu_i\,(f(P) - f(P_i)) \tag{14}$$

with $\sum\mu_i = 1$ and $\mu_i > 0$, $i = 1,\ldots,n$. Writing $\operatorname{relint} X$ for the relative interior of a convex set X, and $\operatorname{conv} Y$ for the convex hull of a set Y, we obtain the following alternative equivalent formulation of (10):

(15) *With the above notation,*

$$\varphi(P) \in \operatorname{relint}\operatorname{conv}\{\tfrac{1}{2}(f(P) - f(P_i)) : i = 1,\ldots,n\}$$

for any convex d-polytope P.

Now let $\mathscr{P}$ be any set of convex d-polytopes, and write $\varphi(\mathscr{P})$ for the set of vectors

$$\varphi(\mathscr{P}) = \{\varphi(P) : P \in \mathscr{P}\}\,.$$

Of particular interest will be the case where $\mathscr{P}$ consists of a polytope P and all its non-singular affine transforms. This set will be denoted by $\mathscr{A}(P)$. Since orthogonal projections are singular affine transformations, and the product of any two affine transformations is also an affine transformation, we deduce that if $Q \in \mathscr{A}(P)$, then each regular projection Q_x of Q must be affinely equivalent to some P_x, and so its f-vector must equal one of $f(P_1),\ldots,f(P_n)$. Hence (15) can be restated in the form:

(16) *With the above notation*

$$\varphi(\mathscr{A}(P)) \subset \operatorname{relint}\operatorname{conv}\{\tfrac{1}{2}(f(P) - f(P_i)) : i = 1,\ldots,n\}$$

for any d-polytope P.

The next theorem shows that (16) is, in a certain sense, the strongest possible assertion of this type.

(17) THEOREM. *For any convex d-polytope P,*

$$\operatorname{conv}\varphi(\mathscr{A}(P)) = \operatorname{relint}\operatorname{conv}\{\tfrac{1}{2}(f(P)-f(P_i)) : i=1,\ldots,n\}$$

or, equivalently,

$$\operatorname{cl}\operatorname{conv}\varphi(\mathscr{A}(P)) = \operatorname{conv}\{\tfrac{1}{2}(f(P)-f(P_i)) : i=1,\ldots,n\},$$

where $\operatorname{cl} X$ *denotes the closure of the set X.*

Because of (16), in order to prove this theorem it will suffice to show that for each i, $1 \leqq i \leqq n$, there exists a d-polytope $Q \in \mathscr{A}(P)$ such that $\varphi(Q)$ is arbitrarily close to $\frac{1}{2}(f(P)-f(P_i))$. Choose any $x \in S^{d-1} \setminus T$ such that $f(P_x)=f(P_i)$. Each point of E^d can be expressed uniquely in the form

$$y+\lambda x, \qquad y \in H_x, \quad -\infty<\lambda<\infty,$$

and so, for any $\varepsilon>0$, the equation

$$T_\varepsilon(y+\lambda x) = y+\varepsilon\lambda x$$

defines a unique non-singular linear transformation T_ε. We shall show that, by taking ε sufficiently small, $\varphi(T_\varepsilon(P))$ can be made arbitrarily close to $\frac{1}{2}(f(P)-f(P_i))$.

As $\varepsilon \to 0$, each of the $f_{d-1}(P)$ hyperplanes including the facets of $T_\varepsilon(P)$ approaches H_x. Denote by J_ε the set of all $x \in S^{d-1}$ such that

$$f\big((T_\varepsilon(P))_x\big) = f(P_i).$$

From the arguments used in the proof of (10), we deduce that as $\varepsilon \to 0$, the set J_ε tends to a set which includes the pair of open hemispheres $S^{d-1} \setminus H_x$. Thus $\mu_i=\mu(J_\varepsilon)/\mu(S^{d-1})$ may be made arbitrarily close to 1 by taking ε sufficiently small. Then (14) shows that $\varphi(T_\varepsilon(P))$ can be made arbitrarily close to $\frac{1}{2}(f(P)-f(P_i))$, and so (17) is proved.

Theorem (17) admits the following generalisation, which may be proved in a similar manner.

(18) *Let* $\mathscr{P}$ *be any set of convex d-polytopes with the property that if* $P \in \mathscr{P}$, *then* $T(P) \in \mathscr{P}$ *for every non-singular affine transformation T. Then*

(19) $$\operatorname{cl}\operatorname{conv}\varphi(\mathscr{P}) = \operatorname{cl}\operatorname{conv}\{\tfrac{1}{2}(f(P)-f(P_i)) : i \in I_P,\ P \in \mathscr{P}\},$$

where the $f(P_i)$, $i \in I_P$, *are the f-vectors of the polytopes obtained by regular projection of each* $P \in \mathscr{P}$.

In this case the set $\{\frac{1}{2}(f(P)-f(P_i)) : i \in I_P,\ P \in \mathscr{P}\}$ may be finite or enumerable, according to the nature of the set $\mathscr{P}$. It will be finite if, for example $\mathscr{P}$ consists of all the projective images of a given convex d-polytope P, or of all the polytopes combinatorially equivalent to P. It will be enumerable if, for example, $d \geqq 2$ and $\mathscr{P}$ consists of all convex simplicial d-polytopes (polytopes whose proper faces are simplexes) or of all convex cubical d-polytopes (polytopes whose proper faces are combinatorially equivalent to cubes). These two latter cases will be discussed in detail in § 4, where we shall need the following corollary of (18):

(20) *Let* $\mathscr{P}$ *be any set of convex d-polytopes with the property that if* $P \in \mathscr{P}$, *then* $T(P) \in \mathscr{P}$ *for every non-singular affine transformation* T. *Then a linear relation of the type*

$$\sum_{j=0}^{d-1} \nu_j \left(f_j(P) - 2\varphi_j(P)\right) = c \tag{21}$$

holds for all $P \in \mathscr{P}$ *if and only if the corresponding relation*

$$\sum_{j=0}^{d-2} \nu_j f_j(P_i) = c \tag{22}$$

holds for all P_i, *where* $\{f(P_i) : i \in I_P, P \in \mathscr{P}\}$ *is the set of f-vectors defined in* (18).

We notice that, since ν_{d-1} does not occur in (22), its value is irrelevant. This is an immediate consequence of the identity $\varphi_{d-1}(P) = \frac{1}{2} f_{d-1}(P)$. Further, if we define $f_{-1}(P) = f_{-1}(P_i) = 1$, $i = 1, \ldots, n$, and $\varphi_{-1}(P) = 0$, then (21) and (22) can be written in the homogeneous forms

$$\sum_{j=-1}^{d-1} \nu_j \left(f_j(P) - 2\varphi_j(P)\right) = 0 \quad \text{and} \quad \sum_{j=-1}^{d-2} \nu_j f_j(P_i) = 0$$

respectively, where $\nu_{-1} = -c$.

To prove (20) we have only to observe that

(a) for each $P \in \mathscr{P}$, the vector $f(P) - 2\varphi(P)$ is a convex combination of the vectors $f(P_i)$, $i \in I_P$, by (10), and

(b) if $i \in I_P$, then by (17) the vector $f(P_i)$ can be approximated as closely as we wish by vectors of the form $f(T(P)) - 2\varphi(T(P))$, where T is a non-singular affine transformation. (Notice that $\mathscr{A}(P) \subset \mathscr{P}$ by assumption.)

Hence the vectors $\{f(P) - 2\varphi(P) : P \in \mathscr{P}\}$ belong to, and span, the minimal affine subspace of E^d containing all the vectors $\{f(P_i) : i \in I_P, P \in \mathscr{P}\}$, and (20) follows immediately.

Statement (15) also leads to a system of inequalities for the angle-sums, as follows:

(23) *Let P be a given d-polytope and j be any integer satisfying $0 \leqq j \leqq d-2$. Then, with the vectors $f(P_i)$ defined above,*

$$\tfrac{1}{2}(f_j(P) - \max_{1\leqq i\leqq n} f_j(P_i)) < \varphi_j(P) < \tfrac{1}{2}(f_j(P) - \min_{1\leqq i\leqq n} f_j(P_i))$$

if $\min_{1\leqq i\leqq n} f_j(P_i) < \max_{1\leqq i\leqq n} f_j(P_i)$, *and*

$$\varphi_j(P) = \tfrac{1}{2}(f_j(P) - f_j(P_i))$$

if all the $f_j(P_i)$, $i=1,\ldots,n$, are equal.

The proof follows from (17), since these equalities and inequalities are satisfied by the components of the vectors belonging to the set

$$\operatorname{relint} \operatorname{conv}\{\tfrac{1}{2}(f(P) - f(P_i)) : i=1,\ldots,n\}.$$

Certain polytopes (for example, the d-cubes) have the property that for $x \in S^{d-1} \setminus T$, the vectors $f(P_x)$ are all equal, so that $n=1$. Such polytopes may be called *equiprojective* and (23) enables us to calculate their φ-vectors explicitly:

$$\varphi(P) = \tfrac{1}{2}(f(P) - f(P_1)).$$

From (16), the angle-sums $\varphi_j(P)$, $0 \leqq j \leqq d-1$, of equiprojective polytopes are invariant under non-singular affine transformations of P. Further properties of equiprojective polytopes will be given in a forthcoming paper.

Let us now consider the special case of the d-dimensional simplex $P = T^d$. The $(d-1)$-polytopes that are obtained by regular projection of T^d are simplicial and have d or $d+1$ vertices. The combinatorial types of these polytopes are well-known (see [2, § 6.1]): there is one type with d vertices, namely the $(d-1)$-simplex $T^{d-1} = T_0^{\,d-1}$, and $[\frac{1}{2}(d-1)]$ types with $d+1$ vertices, namely $T_k^{\,d-1}$, $1 \leqq k \leqq [\frac{1}{2}(d-1)]$. Further, the corresponding $[\frac{1}{2}(d+1)]$ vectors $f(P_i)$ are affinely independent (see [2, Exercise 9.7.1]), and so by (17), $\operatorname{conv} \varphi(\mathscr{A}(T^d))$ has dimension $[\frac{1}{2}(d-1)]$. Substituting the known values of $f_j(P_i)$ [2, Theorems 6.1.2 and 6.1.3] and using (23) we obtain the following:

(24) *The angle-sums $\varphi_j(T^d)$ of a d-simplex T^d satisfy the inequalities*

$$m(j,d) < \varphi_j(T^d) < M(j,d) \quad \text{for } 0 \leqq j \leqq d-2,$$

where

$$m(j,d) = \tfrac{1}{2}(f_j(T^d) - f_j(T^{d-1}_{[\frac{1}{2}(d-1)]}))$$
$$= \begin{cases} 0 & \text{for } 0 \leqq j \leqq [\frac{1}{2}(d-3)], \\ \frac{1}{2}\left(\binom{[\frac{1}{2}(d+1)]}{d-j} + \binom{[\frac{1}{2}(d+2)]}{d-j}\right) & \text{for } [\frac{1}{2}(d-1)] \leqq j \leqq d-2, \end{cases}$$

and

$$M(j,d) = \tfrac{1}{2}(f_j(T^d)-f_j(T^{d-1})) = \tfrac{1}{2}\binom{d}{j} \quad \textit{for} \quad 0 \leqq j \leqq d-2 .$$

Also,

$$\varphi_{d-1}(T^d) = \tfrac{1}{2}(d+1) .$$

From the above discussion and the proof of (17), a slightly stronger statement is possible: for suitable choice of the simplex T^d, the numbers $\varphi_j(T^d)$ can *simultaneously* (that is, for all j satisfying $0 \leqq j \leqq d-2$) be made arbitrarily close to the upper bounds $M(j,d)$, or, alternatively, be made arbitrarily close to the lower bounds $m(j,d)$, defined in (24).

When P is equiprojective, the set

$$\operatorname{conv}\{\tfrac{1}{2}(f(P)-f(P_i)) : i=1,\ldots,n\}$$

has dimension zero. If this set has dimension one, then not only is $\operatorname{conv}\varphi(\mathscr{A}(P))$ an open line segment by (17), but a simple continuity argument shows that $\varphi(\mathscr{A}(P))$ is also an open line segment. For example, if $P=T^4$, then the regular projections of P are simplicial 3-polytopes with 4 or 5 vertices. In this case $\varphi(\mathscr{A}(T^4))$ is the open line segment in E^4 with end points $(0,\tfrac{1}{2},2,\tfrac{5}{2})$ and $(\tfrac{1}{2},2,3,\tfrac{5}{2})$.

If $\dim \operatorname{conv}\{\tfrac{1}{2}(f(P)-f(P_i)) : i=1,\ldots,n\} \geqq 2$, then the nature of the set $\varphi(\mathscr{A}(P))$ is more difficult to determine and we have only been able to obtain partial results in this direction. In the case of a d-simplex, we can show that $\varphi(\mathscr{A}(T^d))$ is dense in the $[\tfrac{1}{2}(d-1)]$-simplex

$$\operatorname{conv}\{\tfrac{1}{2}(f(P)-f(P_i)) : i=1,\ldots,[\tfrac{1}{2}(d+1)]\} ,$$

and suspect that, in fact, $\varphi(\mathscr{A}(T^d))$ is equal to the relative interior of this set. To establish this assertion it would suffice to show that if $\mu_1,\ldots,\mu_{[\frac{1}{2}(d+1)]}$ were any preassigned positive numbers with sum 1, then a non-degenerate simplex T^d could be found with the property that

$$\mu(J_i)/\mu(S^{d-1}) = \mu_i \quad \text{for} \quad i=1,\ldots,[\tfrac{1}{2}(d+1)]$$

(see (11) and the proof of (10)). What we have proved is that $\mu(J_i)/\mu(S^{d-1})$ can be made arbitrarily close to the given numbers μ_i, for each i, but we cannot establish equality.

In the case of a general polytope P it is trivial to show that $\varphi(\mathscr{A}(P))$ is arcwise connected, but whether it is convex, or even simply connected, are open questions.

4. Linear relations between the angle-sums of simplicial and cubical polytopes.

A *quasi-simplicial* d-polytope P is defined to be any convex d-polytope whose j-faces are simplexes for $0 \leqq j \leqq d-2$. Since, for $0 \leqq j \leqq d-2$, the j-faces of a regular projection P_x of P are the projections of j-faces of P, we deduce that for each $x \in S^{d-1} \setminus T$, all the proper faces of the $(d-1)$-polytope P_x are simplexes, and so P_x is a simplicial polytope. It is known that the f-vectors of simplicial $(d-1)$-polytopes span an affine subspace of dimension $[\frac{1}{2}(d-1)]$, namely that defined by the Dehn–Sommerville relations [2, § 9.2]

$$\text{(25 k)} \qquad \sum_{j=-1}^{d-1} b_{kj} f_j(P_x) = 0 ,$$

where k is any integer satisfying $-1 \leqq k \leqq d-1$, $f_{-1}(P_x) = 1$, and

$$\text{(26)} \qquad b_{kj} = \begin{cases} (-1)^k - (-1)^d & \text{if} \quad k=j , \\ (-1)^j \binom{j+1}{k+1} & \text{if} \quad k<j , \\ 0 & \text{if} \quad k>j . \end{cases}$$

Note that, by definition, $f_{d-1}(P_x) = 0$ and that for the value $k = -1$ equation (25k) is the same as (9). Regarded as equations in the variables $f_0(P_x), \ldots, f_{d-1}(P_x)$, (or in $f_{-1}(P_x), \ldots, f_{d-1}(P_x)$), only $[\frac{1}{2}(d+2)]$ of these are linearly independent, for example (25k) with $-1 \leqq k \leqq d-1$ and $k \equiv d-1 \pmod 2$ (see [2, § 9.2]). Using (20) we obtain the relations

$$\text{(27 k)} \qquad \sum_{j=-1}^{d-1} b_{kj} (f_j(P) - 2\varphi_j(P)) = 0 ,$$

where we define $f_{-1}(P) = 1$ and $\varphi_{-1}(P) = 0$, or, equivalently:

(28) *For every quasi-simplicial d-polytope P, the angle-sums $\varphi_j(P)$ satisfy the relations*

$$\text{(29 k)} \qquad \sum_{j=-1}^{d-1} b_{kj} \varphi_j(P) = \tfrac{1}{2} \sum_{j=-1}^{d-1} b_{kj} f_j(P) ,$$

where $-1 \leqq k \leqq d-1$, *and the coefficients* b_{kj} *are defined by* (26). *Of these equations, only* $[\frac{1}{2}(d+2)]$ *are linearly independent, for example those with* $-1 \leqq k \leqq d-1$ *and* $k \equiv d-1 \pmod 2$.

The value $k = -1$ leads again to the relation (2).

Thus all the φ-vectors of quasi-simplicial polytopes P with a given f-vector lie in an affine subspace of dimension $[\frac{1}{2}(d-1)]$ defined by

(29k), $-1 \leqq k \leqq d-1$. We do not, of course, assert that they span this space, but they will do so if, for example, P is a simplex (see § 3). Using the known results [2, § 9.5] on the solvability of the Dehn–Sommerville relations, we obtain:

(30) *For quasi-simplicial d-polytopes P, the angle sums*

$$\varphi_{[\frac{1}{2}(d-1)]}(P), \ldots, \varphi_{d-1}(P)$$

are linear functions of

$$f_{-1}(P) = 1, f_0(P), \ldots, f_{d-1}(P), \varphi_0(P), \ldots, \varphi_{[\frac{1}{2}(d-3)]}(P) \,. \tag{31}$$

To obtain these linear relations explicitly, we use the method devised by B. Grünbaum for solving the Dehn–Sommerville equations. Solving equations (27k), $-1 \leqq k \leqq d-3$, for the variables $(f_j(P) - 2\varphi_j(P))$, $j = [\frac{1}{2}(d-1)], \ldots, d-2$, is identical with the procedure described in the proof of [2, Theorem 9.5.1]. The linear relations obtained are those stated in that theorem with

$d-1$	substituted for	d ,
$[\frac{1}{2}(d-1)]$	substituted for	n,
$f_j(P) - 2\varphi_j(P)$	substituted for	$f_j(P)$, $-1 \leqq j \leqq d-2$,

and, in addition, the relation $\varphi_{d-1}(P) = \frac{1}{2} f_{d-1}(P)$.

If P is simplicial (instead of quasi-simplicial), then the assertions (28) and (30) still hold, but in this case (29k) may be simplified since, again by the Dehn–Sommerville relations, the right side is equal to $(-1)^{d-1} f_k(P)$.

(32) *For every simplicial d-polytope P, the angle-sums* $\varphi_j(P)$ *satisfy the relations*

$$\sum_{j=-1}^{d-1} b_{kj} \varphi_j(P) \;=\; (-1)^{d-1} f_k(P) \,, \tag{33 k}$$

where $-1 \leqq k \leqq d-1$, $\varphi_{-1}(P) = 0$, *and the coefficients* b_{kj} *are defined by* (26). *Of these equations* $[\frac{1}{2}(d+2)]$ *are linearly independent, for example those with* $k \equiv d-1 \pmod 2$.

When $k = -1$, equation (33k) coincides with (2).

Statement (30) also holds for simplicial polytopes, but in this case the numbers $f_{[\frac{1}{2}d]}(P), \ldots, f_{d-1}(P)$ can be expressed as linear functions of $f_{-1}(P) = 1, f_0(P), \ldots, f_{[\frac{1}{2}(d-2)]}(P)$.

Analogous results can be obtained in a similar manner when P is a *quasi-cubical* d-polytope, that is, a convex d-polytope whose j-faces are combinatorially equivalent to j-cubes for $0 \leqq j \leqq d-2$. The polytopes P_x

are *cubical*, that is, all their proper faces are combinatorially equivalent to cubes, and the numbers $f_j(P_x)$ are known to satisfy relations (25k) with $-1 \leqq k \leqq d-1$ and coefficients b_{kj} defined by

$$(34) \qquad b_{kj} = \begin{cases} (-1)^k - (-1)^d & \text{if} \quad k=j \\ (-1)^j 2^{j-k}\binom{j}{k} & \text{if} \quad 0 \leqq k < j \\ (-1)^j & \text{if} \quad -1 = k < j \\ 0 & \text{if} \quad k > j \end{cases}$$

(see [2, § 9.4]). Hence we obtain analogues of (28) and (32):

(35) *For every quasi-cubical d-polytope P, the angle-sums* $\varphi_j(P)$ *satisfy relations* (29 k) *where* $-1 \leqq k \leqq d-1$ *and the coefficients* b_{kj} *are defined by* (34). *Of these equations, only* $[\frac{1}{2}(d+2)]$ *are linearly independent, for example those with* $k \equiv d-1 \pmod 2$.

When $k = -1$, equation (29 k) coincides with (2).

(36) *For every cubical d-polytope P, the angle sums* $\varphi_j(P)$ *satisfy relations* (33 k) *where* $-1 \leqq k \leqq d-1$ *and the coefficients* b_{kj} *are defined by* (34). *Of these equations, only* $[\frac{1}{2}(d+2)]$ *are linearly independent, for example those with* $k \equiv d-1 \pmod 2$.

When $k = -1$, equation (33 k) coincides with (2).

The analogue of (30) holds for quasi-cubical and cubical polytopes. In the latter case, we can also deduce from the proof of [2, Theorem 9.4.1] that the numbers $f_{[\frac{1}{2}d]}, \ldots, f_{d-1}(P)$ are linear functions of $f_{-1}(P) = 1$, $f_0(P), \ldots, f_{[\frac{1}{2}(d-2)]}(P)$.

5. Spherical polytopes.

We begin this section with a proof of Sommerville's Theorem concerning the angle-sums and volume of a spherical polytope, and then extend some of the results of § 3 and § 4 to spherical polytopes.

Let $C \subset E^{d+1}$ be a pointed $(d+1)$-dimensional convex polyhedral cone with vertex C^0 at the origin. Thus: C is the intersection of a finite number of closed half-spaces; $\lambda x \in C$ for all $x \in C$ and all $\lambda \geqq 0$; and C possesses a supporting hyperplane H such that $H \cap C = C^0$. For $1 \leqq j \leqq d$, the j-faces of C will be denoted by C_i^j, $i = 1, \ldots, f_j(C)$. A *convex spherical d-polytope P* is defined as the intersection $C \cap S^d$ of such a convex cone C with the unit d-sphere S^d centred at the origin o. For $0 \leqq j \leqq d-1$, the j-faces of P are defined as the intersections of the $(j+1)$-faces of C with S^d, and we write

$$F_i^j = C_i^{j+1} \cap S^d, \qquad i = 1, \ldots, f_j(P) = f_{j+1}(C).$$

The functions $\chi(C,C^j,x)$, $\varphi(C,C^j)$ and $\varphi_j(C)$ are defined as at the beginning of § 2, with $d-1$ replaced by d. No modifications of these definitions are required because C is an unbounded polyhedral set instead of a polytope. The *interior angle* $\varphi(P,F^j)$ of the spherical polytope P at the face F^j is defined to be equal to the angle $\varphi(C,C^{j+1})$, where $F^j=C^{j+1}\cap S^d$. Geometrically, $\varphi(P,F^j)$ has an analogous interpretation to that in the Euclidean case mentioned at the beginning of § 2. If B is a $(d+1)$-dimensional ball centred at an interior point of F^j, with radius so small that the only faces of P which intersect B are those which include F^j, then $\varphi(P,F^j)$ is the ratio of the d-volume of $B\cap P$ to the d-volume of $B\cap S^d$. The angle-sums $\varphi_j(P)$ of P are defined by (5), so that, for $0\leqq j\leqq d-1$,

$$\varphi_j(P) = \varphi_{j+1}(C) \quad \text{with} \quad \varphi_{d-1}(P) = \tfrac{1}{2}f_{d-1}(P) = \tfrac{1}{2}f_d(C) .$$

We also define

$$\varphi_{-1}(P) = \varphi_0(C) = \varphi(C,C^0) = \mu(P)/\mu(S^d) ,$$

where μ is the d-dimensional Lebesgue measure on S^d. It is convenient also to put $\varphi_d(P)=1$, which is in accordance with our definition of $\varphi_j(P)$ for $j<d$, since

$$\varphi_d(P) = \varphi_{d+1}(C) = \varphi(C,C) = \mu(S^d)/\mu(S^d) = 1 .$$

(37) THEOREM. *For any convex spherical d-polytope P,*

$$\sum_{j=0}^{d}(-1)^j\,\varphi_j(P) = \big(1+(-1)^d\big)\varphi_{-1}(P) . \tag{38}$$

This is Sommerville's Theorem. When d is even it gives an expression for $\varphi_{-1}(P)$ (and therefore for the volume $\mu(P)$ of P) in terms of the angle-sums of P.

PROOF. For $-1\leqq j\leqq d-1$, from the definitions,

$$\varphi(P,F^j) = \varphi(C,C^{j+1}) = \frac{1}{\mu(S^d)}\int_{S^d}\chi(C,C^{j+1},x)\,d\mu(x) , \tag{39}$$

where $F^j=S^d\cap C^{j+1}$.

Let $T\subset S^d$ be the set of all unit vectors $x\in S^d$ which are parallel to a proper face of C. Then T is of measure zero on S^d, and so may be ignored in the integration of (39). For any $x\in S^d\setminus T$, let H_x be the hyperplane through o with normal x, and let C_x be the set obtained by projecting C orthogonally on to H_x. (As before, such a projection will be called *regular*.) Two cases arise:

CASE I: $x\in \operatorname{relint}P\cup\operatorname{relint}-P$, *or, equivalently,*

$$\chi(C,C^0,x) + \chi(C,C^0,-x) = 1 . \tag{40}$$

If z is any relative interior point of a j-face C^j of C, then it is easy to see that one of the two open half-lines $z+\lambda x$, $\lambda > 0$, and $z+\lambda x$, $\lambda < 0$, lies in the interior of C, and the other does not intersect C. We deduce that, for $1 \leqq j \leqq d$,

$$\chi(C,C^j,x) + \chi(C,C^j,-x) = 1 ,$$

and so

$$\sum_{i=1}^{f_j(C)} (\chi(C,C_i^j,x) + \chi(C,C_i^j,-x)) = f_j(C) .$$

Further, the regular projection C_x, in this case, is the whole hyperplane H_x, so that $f_j(C_x) = 0$, $j = 0,\ldots,d$, and the above relation may be written, for $1 \leqq j \leqq d$,

$$\sum_{i=1}^{f_j(C)} (\chi(C,C_i^j,x) + \chi(C,C_i^j,-x)) + f_j(C_x) = f_j(C) . \tag{41}$$

CASE II: $x \in S^d \setminus (T \cup P \cup -P)$, *or, equivalently,*

$$\chi(C,C^0,x) + \chi(C,C^0,-x) = 0 . \tag{42}$$

In this case C_x is a pointed convex $(d-1)$-cone in H_x, since any supporting hyperplane H of C that contains x and intersects C in C^0 projects into a supporting hyperplane of C_x intersecting C_x only in C^0. Let H' be a hyperplane parallel to H which intersects the interior of C. Then $H' \cap C$ is a convex d-polytope Q, and $H' \cap C_x = Q_x$ is a regular projection of Q on to $H' \cap H_x$. For $0 \leqq j \leqq d-1$, $G^j = C^{j+1} \cap H'$ is a j-face of Q, and the faces of Q_x are precisely the intersections of H' with the faces of C_x. We deduce that

$$f_j(Q) = f_{j+1}(C), \qquad f_j(Q_x) = f_{j+1}(C_x) ,$$

and since, for $j > 0$, the points z used in defining the function χ may be chosen as points of H',

$$\chi(Q,G^j,x) = \chi(C,C^{j+1},x) .$$

From these facts, and (8) applied to Q, we deduce that for $1 \leqq j \leqq d$, equation (41) holds in this case also.

Relation (41) is the analogue of (8) and we shall use it in a similar manner.

By Euler's Theorem applied to the cone C and the regular projection C_x, we deduce

$$\sum_{j=1}^{d} (-1)^{j-1} f_j(C) = 1 + (-1)^{d-1} , \tag{43}$$

and

$$\sum_{j=1}^{d}(-1)^{j-1}f_j(C_x) = \begin{cases} 0 & \text{if } x \in \operatorname{relint} P \cup \operatorname{relint} -P\,, \\ 1+(-1)^d & \text{if } x \in S^d \setminus (T \cup P \cup -P)\,, \end{cases}$$

or, using (40) and (42),

$$(44) \quad \sum_{j=1}^{d}(-1)^{j-1}f_j(C_x) = \bigl(1+(-1)^d\bigr)\bigl(1-\chi(C,C^0,x) - \chi(C,C^0,-x)\bigr)\,.$$

Taking the alternating sum of (41), and using (43) and (44), we obtain

$$\sum_{j=1}^{d}(-1)^{j-1}\sum_{i=1}^{f_j(C)}\bigl(\chi(C,C_i^j,x)+\chi(C,C_i^j,-x)\bigr) + \\ + \bigl(1+(-1)^d\bigr)\bigl(1-\chi(C,C^0,x)-\chi(C,C^0,-x)\bigr) = 1+(-1)^{d-1}\,.$$

Integrating over $S^d\setminus T$, dividing by $\mu(S^d)$ and using (39) and (5), we obtain

$$\sum_{j=1}^{d}(-1)^{j-1}\bigl(2\varphi_j(C)\bigr) + \bigl(1+(-1)^d\bigr)\bigl(1-2\varphi_0(C)\bigr) = 1+(-1)^{d-1}\,,$$

which, by the definition of $\varphi_j(P)$ is identical with (38) and Theorem (37) is proved.

For a given convex spherical d-polytope P, and $x \in S^d\setminus T$, define $(d+1)$-vectors as follows:

$$(45) \quad \begin{aligned} f(P) &= \bigl(1,f_0(P),\ldots,f_{d-1}(P)\bigr) = \bigl(f_0(C),\ldots,f_d(C)\bigr)\,, \\ \varphi(P) &= \bigl(\varphi_{-1}(P),\varphi_0(P),\ldots,\varphi_{d-1}(P)\bigr) = \bigl(\varphi_0(C),\ldots,\varphi_d(C)\bigr)\,, \end{aligned}$$

and

$$f(C_x) = \bigl(f_0(C_x),\ldots,f_{d-1}(C_x),0\bigr)\,.$$

As in § 3, the number of combinatorial types of the regular projections C_x is finite, and so there are only a finite number of distinct vectors $f(C_x)$. It will be convenient to represent these by $f(C_0), f(C_1),\ldots,f(C_n)$, choosing $f(C_0)=o$, which is the f-vector $f(C_x)$ when $x \in \operatorname{relint} P \cup \operatorname{relint} -P$ (see Case I above). With this notation, we can now prove the analogue of (10) for spherical polytopes.

(46) *For each convex spherical d-polytope P, the vector* $f(P)-2\varphi(P)$ *is a positive convex combination of the vectors* $f(C_i)$, $i=0,\ldots,n$, *that is to say,*

$$(47) \quad f(P)-2\varphi(P) = \sum_{i=0}^{n}\mu_i f(C_i)\,,$$

where $\sum\mu_i=1$ and $\mu_i>0$ *for each* i. *Further,*

$$(48) \quad \mu_0 = 2\varphi_{-1}(P) = 2\mu(P)/\mu(S^d)\,.$$

PROOF. For $0 \leqq i \leqq n$, let $J_i \subset S^d$ be the set of all $x \in S^d \setminus T$ such that $f(C_x) = f(C_i)$. Then, as in § 3, for each i, $\mu(J_i) > 0$, $\sum \mu(J_i) = \mu(S^d)$, and from (40) and (42),

(49)

$$\frac{\mu(J_0)}{\mu(S^d)} = \frac{1}{\mu(S^d)} \int_{S^d \setminus T} (\chi(C, C^0, x) + \chi(C, C^0, -x))\, d\mu(x) = 2\varphi_0(C) = 2\varphi_{-1}(P) .$$

Also

$$\int_{S^d \setminus T} f_j(C_x)\, d\mu(x) = \sum_{i=0}^{n} \mu(J_i) f_j(C_i)$$

(compare (13)). Integrating (41) over $S^d \setminus T$ and dividing by $\mu(S^d)$ we obtain

$$2\varphi_j(C) + \sum_{i=0}^{n} \frac{\mu(J_i)}{\mu(S^d)} f_j(C_i) = f_j(C) ,$$

or, writing $\mu_i = \mu(J_i)/\mu(S^d)$, $\varphi_j(C) = \varphi_{j-1}(P)$, $f_j(C) = f_{j-1}(P)$,

$$f_{j-1}(P) - 2\varphi_{j-1}(P) = \sum_{i=0}^{n} \mu_i f_j(C_i) . \tag{50}$$

This holds for $j = 1, \ldots, d$, and also trivially for $j = 0$ since $\varphi_{-1}(P) = \frac{1}{2}\mu_0$ by (49), $f_{-1}(P) = 1$, $f_0(C_i) = 1$ for $i = 1, \ldots, n$, $f_0(C_0) = 0$, and

$$\sum \mu_i = \sum \mu(J_i)/\mu(S^d) = 1 .$$

But (50) clearly implies (47), and Theorem (46) is established.

The next assertion will enable us to deduce homogeneous linear relations between the components of the vector $f(P) - 2\varphi(P)$ that hold when the faces of P are of some prescribed type (compare (20)).

(51) *Let $P = C \cap S^d$ be a given convex spherical d-polytope. If a homogeneous linear relation*

$$\sum_{j=-1}^{d-1} \nu_j f_{j+1}(C_i) = 0, \quad i = 1, \ldots, n , \tag{52}$$

holds for all the regular projections C_i of C, then

$$\sum_{j=-1}^{d-1} \nu_j (f_j(P) - 2\varphi_j(P)) = 0 . \tag{53}$$

It should be noticed that the value of ν_{d-1} is arbitrary since $f_d(C_i) = 0$ for all i, and therefore ν_{d-1} does not enter relation (52). Also, relation (52) is trivially satisfied for $i = 0$, since $f_{j+1}(C_0) = 0$ for all $j \geqq -1$. Hence it is sufficient to require that (52) holds only for the *pointed* regular projections.

To prove (51), we multiply (50) by ν_{j-1} and sum from $j=0$ to d. Using (52) we obtain (53) and the assertion is proved.

If P is quasi-simplicial, that is, every $(d-2)$-face of P is a spherical $(d-2)$-simplex, then each $(d-1)$-face of C has exactly $d-1$ edges, or, equivalently, every bounded cross-section Q of C is a quasi-simplicial convex d-polytope. From the discussion of Case II on p. 214, we see that each pointed regular projection C_x, $x \in S^{d-1} \setminus (T \cup P \cup -P)$, is the join of C^0 to a regular projection Q_x of some Q, and Q_x is a simplicial $(d-1)$-polytope. Hence,

$$\sum_{j=-1}^{d-1} b_{kj} f_{j+1}(C_x) = \sum_{j=-1}^{d-1} b_{kj} f_j(Q_x) = 0 ,$$

where k is any integer satisfying $-1 \leqq k \leqq d-1$ and the coefficients b_{kj} are defined in (26). (When $k=d-1$ we obtain $f_d(C_x)=0$.) From (51), with b_{kj} written in place of ν_j, we deduce

(54) *For every quasi-simplicial spherical d-polytope P, the angle-sums* $\varphi_j(P)$ *satisfy relations* (29 k) *where* $-1 \leqq k \leqq d-1$ *and the coefficients* b_{kj} *are defined by* (26).

This is the analogue of (28), and similar assertions regarding the linear independence of these equations hold. When $k=-1$, with $\varphi_{-1}(P) = \mu(P)/\mu(S^d)$ and $f_{-1}(P)=1$, (29 k) is identical with Sommerville's relation (38).

In a similar manner, statements (30), (32), (35) and (36) have exact analogues for spherical polytopes. The same assertions hold, and the proofs are identical except that we make use of (51) in place of (20).

6. Remarks.

The results of this paper may be generalised in the following manner. In the proofs we have never used the fact that the measure μ used in the definition of $\varphi(P,F^j)$ (see (4)) is the Lebesgue measure on S^{d-1}. Let $\mathscr{S}^{d-1}$ be the Boolean algebra of subsets of S^{d-1} generated by the hemispheres, and let ν be a finitely additive real or complex valued set function defined on $\mathscr{S}^{d-1}$, that satisfies

(a) ν is invariant under reflection in the origin o, so that $\nu(A)=\nu(-A)$ for each set $A \in \mathscr{S}^{d-1}$,

(b) if H is any hyperplane of E^d through o, and $A \in \mathscr{S}^{d-1}$, $A \subset H$, then $\nu(A) = 0$, and

(c) $\nu(S^{d-1}) \neq 0$.

Then all the results of § 2 and § 4, as well as the sufficiency part of statement (20), remain true with angles $\varphi(P, F^j)$ defined by (4) with μ replaced by ν. In fact, more generally, using the fact proved at the end of § 2, that for all $x \in S^{d-1}$, $g(x) = (-1)^{d-1}$, it can be shown that the Gram–Euler relation (2) holds even if the set function ν used in defining the angles of the polytope satisfies only (c) and not (a) and (b) above. In § 3 we require the further condition that ν is real-valued, and

(d) if $A \subset S^{d-1}$ is a non-empty open set, $A \in \mathscr{S}^{d-1}$, then $\nu(A) > 0$.

This is needed to ensure the positivity of the coefficients μ_i in (11). For Theorems (17), (18) and the necessity statement in (20), we must also assume that ν is countably additive. The results of § 5 admit a similar generalisation. From a geometrical point of view, of course, the case where the angles are defined in terms of the Lebesgue measure is the most interesting.

Theorems (10) and (46) admit an obvious generalisation to arbitrary polyhedral sets, and many of our other assertions can be extended in a similar manner.

Finally we remark that although the results of this paper are all of a metrical character, they have applications of a purely combinatorial-geometric type. Theorem (10), for example, can be used to give information about the possible combinatorial types of $(d-1)$-polytopes that can occur as facets of a d-dimensional convex polytope. Details appear in the authors' paper *Facets and nonfacets of convex polytopes*, Acta Math. 119 (1967), 113–145.

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