MATH. SCAND. 21 (1967), 219–232

ON THE STRUCTURE OF THE ORTHOGONAL GROUP

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Let V be a vector space over a field k of characteristic $\neq 2$, let Q be a non-degenerate quadratic form on V. Let $G=O(V,Q)$ denote the group of all isometries of V. We shall place a topology on $O(V,Q)$ such that $O(V,Q)$ becomes a topological group which is discrete if and only if V is finite-dimensional. Further, the classical structure theorems for $O(V,k)$ in the finite-dimensional case carry over to the infinite-dimensional situation for the topological group $O(V,k)$ (see [1, Chapter V]).

1. Preliminaries.

Let, then, (V,Q) be an arbitrary vector space over a field of characteristic $\neq 2$ with non-degenerate quadratic form Q. Call such a space a Q-space hereafter. Let $G=O(V,Q)$ be the orthogonal group of all isometries of V. Let $\mathscr{E}_0, \mathscr{E}$ be respectively the set of all finite dimensional non-singular subspaces of V, and the set of all subspaces of V. Let $\mathscr{H}$ be the set of all subgroups of G.

We define two maps: $\varrho:\mathscr{E} \to \mathscr{H}$, $\chi:\mathscr{H} \to \mathscr{E}$ as follows: If $E \in \mathscr{E}$, $H \in \mathscr{H}$, then

$$\varrho(E) = \{\sigma \in G \mid \sigma|_E = \text{identity on } E\}$$

and

$$\chi(H) = \{x \in V \mid \sigma(x)=x \text{ for all } \sigma \in H\}.$$

Some of the properties of these two maps are:

(i) If $E_1 \subseteq E_2$, $H_1 \subseteq H_2$, $E_i \in \mathscr{E}$, $H_i \in \mathscr{H}$, then $\varrho(E_1) \supseteq \varrho(E_2)$, $\chi(H_1) \supseteq \chi(H_2)$, that is, ϱ, χ are order reversing.

(ii) If $E \in \mathscr{E}$, $H \in \mathscr{H}$, then $\chi \circ \varrho(E) \supseteq E$ and $\varrho \circ \chi(H) \supseteq H$.

(iii) If $E_1 \in \mathscr{E}_0$, $E_2 \in \mathscr{E}$, $E_1 \subset E_2$, then $\varrho(E_1) \supset \varrho(E_2)$.

(iv) If $E \in \mathscr{E}$, $H \in \mathscr{H}$, then $\varrho \circ \chi \circ \varrho(E) = \varrho(E)$ and $\chi \circ \varrho \circ \chi(H) = \chi(H)$.

(v) If $E \in \mathscr{E}_0$, then $\chi \circ \varrho(E) = E$, so $\chi \circ \varrho|_{\mathscr{E}_0} = 1_{\mathscr{E}_0}$.

(vi) Let $\sigma \in G$, $E \in \mathscr{E}$, then $\varrho(\sigma(E)) = \sigma\, \varrho(E)\, \sigma^{-1}$.

Proof. We shall prove only (iii) and (v); all the others are more or less obvious.

Received June 10, 1966.

(iii): Let $V=E_1 \perp E_1^*$, where E^* denotes the space $\{y \in V \mid y \perp E\}$ for any space $E \in \mathscr{E}$, so that $E_2 \supset E_1$ implies the existence of an $x \in E_2$ such that $x \in E_1^*$. Let $\sigma = 1_{E_1} \perp (-1_{E_1^*})$, then $\sigma \in G$ and $\sigma(x) = -x$. Also $\sigma|_{E_1}=$ identity on E_1, so $\sigma \in \varrho(E_1)$, $\sigma \notin \varrho(E_2)$.

(v): Now $\chi \circ \varrho(E) \supseteqq E$ for all $E \in \mathscr{E}$ and $\varrho \circ \chi \circ \varrho(E) = \varrho(E)$ by (iv). Hence, if $\chi \circ \varrho(E) \supset E \in \mathscr{E}_0$, then $\varrho(\chi \circ \varrho(E)) \subset \varrho(E)$ contradicting (iv).

2. The Topology on $O(V)=G$.

Let $\mathscr{U} = \{U \mid U = \varrho(E),\ E \in \mathscr{E}_0\} = \varrho(\mathscr{E}_0)$. Then we have:

(i) If $U_1, U_2 \in \mathscr{U}$, there is a $U \in \mathscr{U}$ such that $U \subset U_1 \cap U_2$.
(ii) If $U \in \mathscr{U}$, then $UU = U$, $U^{-1} = U$.
(iii) If $U \in \mathscr{U}$, $\sigma \in G$, then there is a $U' \in \mathscr{U}$ such that $\sigma U' \sigma^{-1} \subseteqq U$.
(iv) $\bigcap_{U \in \mathscr{U}} U = \{1_V\}$.

PROOF.
(i) Let $U_i = \varrho(E_i)$, $E_i \in \mathscr{E}_0$, then

$$U_1 \cap U_2 = \varrho(E_1) \cap \varrho(E_2) = \{\sigma \in G \mid \sigma(x) = x \text{ for } x \in E_1 \cup E_2\}.$$

Let $E \in \mathscr{E}_0$ be such that $E \supseteqq E_1 \cup E_2$. Then

$$U = \varrho(E) \subseteqq \varrho(E_1) \cap \varrho(E_2) = U_1 \cap U_2 .$$

(ii) This is obvious.
(iii) Let $U = \varrho(E)$, $E' = \sigma^{-1}(E)$, $U' = \varrho(E')$, then $E' \in \mathscr{E}_0$ as $E \in \mathscr{E}_0$. But then

$$\varrho(\sigma(E')) = \varrho(E) = U = \sigma(\varrho(E'))\sigma^{-1} = \sigma\, U'\, \sigma^{-1} .$$

(iv) Let $\sigma \in \bigcap_{U \in \mathscr{U}} U$, then $\sigma \in \varrho(E)$, for all $E \in \mathscr{E}_0$, hence $\sigma(x) = x$ for all $x \in E$, for all $E \in \mathscr{E}_0$. But $\bigcup_{E \in \mathscr{E}_0} E = V$. Hence $\sigma = 1_V$.

These facts imply that $\mathscr{U}$ may be taken as a fundamental system of neighborhoods of the identity for a Hausdorff topology on $G = O(V,Q)$. We shall call this the finite topology (see [3]).

THEOREM 1. *$G=O(V,Q)$ with the finite topology, is discrete if, and only if, V is finite-dimensional.*

PROOF. If V is finite-dimensional, then clearly G is discrete, $\{1_V\}$ is open and, hence, there is a $U \in \mathscr{U}$ such that $U \subseteqq \{1_V\}$, so

$$U = \varrho(E) = \{1_V\} \qquad \text{for some } E \in \mathscr{E}_0 .$$

Let $x \in V$ and $E_x \in \mathscr{E}_0$ be such that $E_x \supset \{x\} \cup E$. Then

$$\varrho(E_x) \subseteqq \varrho(E) = \{1_V\},$$

so $E_x = E$ by (iii) of section 1. As x was arbitrary, $E = V$. This completes the proof.

If $E \in \mathscr{E}_0$, then $\varrho(E)$ is, of course, an open subgroup and, consequently, closed. But we have more generally

PROPOSITION 1. *Let $E \in \mathscr{E}$, and let $G = O(V)$ have the finite topology, then $H = \varrho(E)$ is closed and for any $H \in \mathscr{H}$ $\varrho\chi(H) \supseteqq \bar{H}$. Hence ϱ maps $\mathscr{E}$ onto the set $\mathscr{H}$ of all closed subgroups of G.*

PROOF. Let $\sigma \in \bar{H}$, $H \in \mathscr{H}$. Then for any $E_0 \in \mathscr{E}_0$, we must have $\varrho(E_0)\sigma \cap H \neq \emptyset$. If $x \in E \subseteqq \chi(H)$, $x \neq 0$, let $E_x \in \mathscr{E}_0$ be such that $x \in E_x$. Then there is an $\eta_x \in \varrho(E_x)$ such that $\eta_x^{-1}\sigma \in H$, so that $\eta_x^{-1}\sigma|_E = 1_E$. In particular, $\sigma(x) = \eta_x(x) = x$, as $\eta_x \in \varrho(E_x)$. Thus $\sigma(x) = x$, for all $x \in E$; hence $\sigma \in H$. So $H = \bar{H}$, the closure of H. Next, since $H \subseteqq \varrho(\chi(H))$ is closed, $\bar{H} \subseteqq \varrho(\chi(H))$. This completes the proof.

Notice that if $A = \{\pm 1_V\}$, then $A = \bar{A} \subset O(V) = \varrho(\chi(A))$.

PROPOSITION 2. *If (V,Q) is infinite dimensional, then $O(V,Q)$ is a totally disconnected non-locally compact group.*

PROOF. Let $E_1 \in \mathscr{E}_0$ be such that $\dim E_1 \geqq 2$, then $V = E_1 \perp E_1^*$, $\varrho(E_1)$ is an open subgroup of G and, hence, is closed. Further, $\bigcap_{E \in \mathscr{E}_0} \varrho(E) = \{1_V\}$, so $O(V) = G$ is totally disconnected.

In order to prove that G is not locally compact, it suffices to show that G itself is non-compact since $\varrho(E') = G'$ compact would imply that $O(E'^*, Q|_{E_1^*})$, the finite topology of which is the inherited topology from $O(E,Q)$, would be compact. So we let $G_1 = \varrho(E_1)$, $\dim E_1 \geqq 2$, $E_1 \in \mathscr{E}_0$ and consider the left coset space $O(V,Q)/G_1 = G/G_1$. Suppose that G is compact, then the natural map $\pi: G \to G/G_1$ shows that G/G_1 is compact. But $G_1 = \varrho(E_1)$ is open and, hence, $\sigma G_1 = \bar{\sigma} \in G/G_1$ is also open. Thus G/G_1 is discrete and, therefore, finite. Consider next two cases:

(i) k infinite. The elements of $O(E_1)$ may be identified with the elements of the subgroup $O(E_1) \perp 1_{E_1^*}$ of G. Denote this group by G_2. Then $\tau, \tau' \in G_2$, $\tau \equiv \tau' \bmod G_1$ if, and only if, $\tau'^{-1}\tau \in G_1$. But $G_2 \cap \varrho(E_1) = \{1_V\}$, so $\tau' = \tau$. Therefore, since G_2 is infinite, G/G_1 is an infinite set. This contradicts compactness; hence G is not compact.

(ii) If k is finite, then we use lemma 1, proved in the next section, to provide us with a sequence of subspaces $F_1, F_2, \ldots$, such that $F_i \perp F_j$, $j \neq i$, and F_i are hyperbolic planes. Then construct the following maps σ_i such that $\sigma_i: F_1 \to F_i$ isometrically, and extend σ_i to V (see [4]). Choose E_1 to equal F_1. Then $\pi: \sigma_i \to \bar{\sigma}_i$, and

$$\bar{\sigma}_i = \bar{\sigma}_j \Leftrightarrow \sigma_j^{-1}\sigma_i \in G_1 = \varrho(E_1) .$$

For if $i \neq j$,

$$\sigma_j^{-1}\sigma_i : E_1 \overset{\sigma_i}{\to} F_i \overset{\sigma_j^{-1}}{\to} \sigma_j^{-1}(F_i) \neq E_1 ,$$

hence $\sigma_j^{-1}\sigma_i \notin G_1$. Thus again we have an infinite set G/G_1 which contradicts compactness.

Let $G_F = \{\sigma \in G = O(V) \mid \sigma \in \varrho(E^*),\ E \in \mathscr{E}_0\}$ = group generated by all $\sigma \in G$ such that σ leaves E^* elementwise fixed, where $E \in \mathscr{E}_0$, $V = E \perp E^*$. We have

THEOREM 2. *Using the finite topology on $O(V)$, we find that the group G_F is dense in $O(V)$, that is, $\bar{G}_F = O(V)$.*

PROOF. Let $\sigma \in G = O(V)$. We have to show that $\sigma \in \bar{G}_F$, that is, for all $E \in \mathscr{E}_0$, there is a $\eta \in \varrho(E)$ such that $\sigma\eta \in G_F$, or that there is a $\tau \in G_F$ such that $\sigma^{-1}\tau = \eta \in \varrho(E)$. Let $E^\sigma = \sigma(E)$, then let E_1 be a finite dimensional semi-simple subspace of V such that $E_1 \supseteq E + E^\sigma$. Now $V = E_1 \perp E_1{}^*$ and $\sigma: E \to E^\sigma$, with E, E^σ both contained in E_1; hence, by Witt's Theorem, we can extend σ to σ' in $O(E_1)$. Let $\tau = \sigma' \perp 1_{E_1{}^*}$, then clearly $\tau \in G$ and $\tau \in G_F$. Further, $\sigma^{-1}\tau|_E = \sigma^{-1}\sigma|_E = 1_E$; hence $\sigma^{-1}\tau \in \varrho(E)$. This proves the theorem.

3. The structure of $O(V, Q)$.

In this section we wish to generalize, to infinite dimensional Q-spaces, a number of results which will aid us in considering further the structure of the orthogonal group $O(V)$. Some we state without proof. Many times we will use the notation: $f_Q(x,y) = x \cdot y$, for $x, y \in V$, f_Q the associated bilinear form.

LEMMA 1. *Let (V,Q) be a semi-simple Q-space over k. If* $\dim V \geqq 2$ *and k is finite, then for any $a \in k$ there is a $x \in V$ such that $f_Q(x,x) = a$. If* $\dim V \geqq 1$ *and V has isotropic vectors, then when k is any field and for any $a \in k$ we can find $x \in V$ so that $f_Q(x,x) = a$.*

PROOF. For infinite dimensional V the proof is as in the finite case (see e.g. [1]).

COROLLARY. *If (V,Q) is as in the lemma with* $\dim V \geqq 3$, *then V contains isotropic vectors if k is finite.*

Let $Z(V,Q)$ denote the center of $O(V,Q) = G$ and $O(V,Q)'$ the group theoretic commutator subgroup of $O(V,Q)$. Let $\Omega_F(V,Q) = (G_F)'$, where G_F was defined in section 2. Let $\Omega(V,Q) = \overline{O(V,Q)'}$, the closure of the group generated by the commutators.

Lemma 2. *Let (V,Q) be a semi-simple Q-space, S a two dimensional non-singular (semi-simple) subspace.*
Unless S is a hyperbolic plane and $k=\mathrm{GF}(3)$, there is a map $\sigma \in \Omega_F(V,Q)$ such that $S^* = \{x \in V \mid \sigma(x)=x\}$.

Proof. It is well known that there is a $\sigma' \in \Omega(S,Q)$ such that $\sigma'(x) \neq x$ for all $0 \neq x \in S$. Define σ as: $\sigma = \sigma' \perp 1_{S^*}$ $(V = S \perp S^*)$, then $\sigma\colon V \to V$ and is clearly in $\Omega_F(V,Q)$. This σ is the element that works.

Lemma 3. *If (V,Q) is a semi-simple Q-space with dimension $\geqq 3$, let $\sigma \in O(V,Q)$ be such that $\sigma\tau = \tau\sigma$ for all $\tau \in \Omega_F(V,Q)$.*
Then $\sigma = \pm 1_V$.

Proof. First we show that if $f_Q(x,x) \neq 0$, then $\sigma(x) = ax$, $a \in k$, $a = \pm 1$.

a) Assume $k \neq \mathrm{GF}(3)$. Let x be such that $f_Q(x,x) \neq 0$, let $S(x)$ be the space of x. So $V = S(x) \perp S(x)^*$ and $\dim S(x)^* \geqq 2$, so that there is an element $y \in S(x)^*$ such that $f_Q(y,y) \neq 0$, $x \perp y$. Let $S = S(x,y)$ be the span of x and y, S is then semi-simple. Lemma 2 implies the existence of $\varrho \in \Omega_F(V,Q)$ such that $S^* = \{z \in V \mid \varrho(z) = z\}$. Let $z \in S^*$,

$$\varrho(\sigma(z)) = \sigma\varrho(z) = \sigma(z)\,,$$

hence $\sigma(z) \in S^*$ for any $z \in S^*$; so that $\sigma(S^*) \subseteqq S^*$. Similarly, since $\sigma^{-1} = \varrho\sigma^{-1}$ for all $\varrho \in \Omega_F(V,Q)$, we get that $\sigma^{-1}(S^*) \subseteqq S^*$, and hence $\sigma(S^*) = S^*$. Consequently $\sigma(S) = S$. Now $\dim S^* \geqq 1$ and is semi-simple; so that there is a $y' \in S^*$ such that $f_Q(y',y') \neq 0$. Apply the same argument above to $S' = S(x,y')$ and get that $\sigma(S') = S'$. Hence

$$\sigma(S \cap S') = S \cap S' = S(x);$$

so that $\sigma(x) = ax$ $(a = \pm 1)$.

b) Let $k = \mathrm{GF}(3)$, hence $k^* = \pm 1$. By lemma 1, since $\dim V \geqq 2$, for any $a \in k$ there is a vector $x \in V$ such that $f_Q(x,x) = a \in k^*$. So $V = S(x) \perp S(x)^*$, $\dim S(x)^* \geqq 2$, and hence, again there is a $y \in S(x)^*$ such that $f(y,y) = a$. Put $S = S(x,y)$. If $z = \alpha x + \beta y$, then

$$z \cdot z = \alpha^2 a + \beta^2 a = a(\alpha^2 + \beta^2)\,,$$

so

$$f(z,z) = 0 \Leftrightarrow \alpha \text{ and } \beta = 0 \text{ or } z = 0.$$

Hence S is anisotropic and not a hyperbolic plane, so that, by lemma 2, there is a $\varrho \in \Omega_F(V,Q)$ such that $S^* = \{z \mid \varrho(z) = z\}$. Then we get as before $\sigma(S) = S$.

i) If $\dim V \geqq 4$, then $\dim S^* \geqq 2$. The same method as in a) yields $\sigma(x) = c \cdot x$.

ii) If $\dim V=3$, then $\dim S^*=1$. Take $y' \in S^*$, $y' \neq 0$; semi-simplicity of S^* implies that $f(y',y') \neq 0$. Hence $f_Q(y',y') = \pm a \in k^*$. If $f_Q(y',y') = +a$, apply a previous argument to get that $\sigma(x)=c\cdot x$. If $f_Q(y',y') = -a$, then $V=S(x,y,y')$. The vectors $x \pm y, y'$ are orthogonal,

$$f_Q(x,x) = f_Q(y,y) = a = -f_Q(y',y') ,$$

and with $\varepsilon = \pm 1$

$$f_Q(x+\varepsilon y, x+\varepsilon y) = f_Q(x,x)+f_Q(y,y) = 2a = -a = f_Q(y',y') .$$

Therefore $S(x \pm y, y')$ is anisotropic, and by the previous argument we get

$$\sigma(S(x \pm y,y')) = S(x \pm y,y') .$$

Hence

$$\sigma[S(x,y) \cap S(x \pm y,y')] = S(x,y) \cap S(x \pm y,y') = S(x \pm y);$$

so we get $\sigma(x+\varepsilon y) = c_i(x+\varepsilon y)$, $i=1,2$. But $(x+y) \perp (x-y)$, and $x \pm y, y'$ is an orthogonal basis of V; so that $\sigma(y') = cy'$, hence $\sigma(x)=c\cdot x$.

Take now any finite dimensional subspace S of V. Let S' be a finite dimensional semi-simple subspace of V containing S. Let $x_1, \ldots, x_n$ be an orthogonal basis of S', then $\sigma(x_i)=c_i x_i$.

So: a) If $f_Q(x_1+x_2, x_1+x_2) \neq 0$, then

$$\sigma(x_1+x_2) = c(x_1+x_2) = c_1x_1+c_2x_2;$$

so that $c=c_1=c_2$.

b) If $f_Q(x_1+x_2, x_1+x_2)=0$, then

$$Q(x_1+x_2+x_3) = Q(x_1+x_2)+Q(x_3) = Q(x_3) \neq 0 \qquad \dim V \geqq 3 .$$

So that

$$\sigma(x_1+x_2+x_3) = c\,(x_1+x_2+x_3) = \sum_{i=1}^{3} c_i x_i ,$$

and hence $c=c_i$, $i=1,2,3$.

Thus we get $c_1=c_2=\ldots=c_n=\pm 1$. Then clearly from this we get that $\sigma = \pm 1_V$. This completes the proof.

Corollary. *$Z(V,Q)=\{\pm 1_V\}$, when (V,Q) is a semi-simple Q-space of* $\dim \geqq 3$.

For every finite dimensional semi-simple Q-space (W,Q) with associated bilinear form g_Q we have from the finite dimensional theory a homomorphism θ_w called the *spinorial norm*, $\theta_w\colon O^+(W) \to k^*/k^{*2}$, where $O^+(W)$ is the group of rotations in $O(W)$. This map is defined as follows: if $\sigma = \tau_{x_1}\tau_{x_2}\cdots\tau_{x_r}$, where τ_x is the symmetry defined by x, that is, if

$$\tau_x(y) = y - 2\frac{x\cdot y}{x\cdot x}x ,$$

then

$$\theta_w(\sigma) = x_1^2 x_2^2 \dots x_r^2 k^{*2}, \qquad x^2 = x\,x = g_Q(x,x) .$$

Some of the properties of the spinorial norm for (W,g) are as follows:

i) $\Omega(W,g_Q) \subseteqq \ker \theta_w$,

ii) θ_w maps $O^+(W)$ onto k^*/k^{*2}, when $(W,g_Q)_Q$ has isotropic vectors.

iii) If W contains non-zero isotropic vectors, then $\Omega(W,g) =$ kernel of θ_w.

iv) If $W = U \perp V$, then $\ker \theta_U = O^+(U) \cap \ker \theta_w$.

v) Suppose $\sigma = \tau_x \cdot \tau_y$,

$$\tau_x(u) = u - 2\frac{u \cdot x}{x \cdot x} x$$

$\sigma \in \ker \theta_w$, then $\sigma \in \Omega(W,g_Q)$. If $\dim W = 2$, then $\ker \theta_w = \Omega(W,g_Q)$ and each element of $\Omega(W)$ is a square of a rotation. If $\dim W = 3$, then $\ker \theta_w = \Omega(W)$ and each element of $\Omega(W)$ is the square of a rotation with the same axis.

We set down the following notation: If U is a finite dimensional semi-simple subspace of (V,Q) and $\tau \in O(U)$ we shall hereafter identify τ with $(\tau \perp 1_{U^*}) \in O(V)$.

LEMMA 4. *Let (V,f) be any Q-space. Suppose that k is a finite field and τ is an isometry of a subspace U_1 of V onto a finite dimensional subspace U_2 of V. If U_2 is contained in a finite dimensional semi-simple Q-space W of V with* codim $W \geqq 2$, *then we can extend τ to an element $\varrho \in \Omega_F(V,Q)$.*

PROOF. Let W_2 be a finite dimensional semi-simple Q-subspace of V containing U_1, U_2 and W such that $\dim(W_2/W) \geqq 2$. Then we can extend τ to a rotation σ of W_2, since we can always multiply by a reflection of $W^* \cap W_2$. Now if ϱ_1 is a rotation of $W^* \cap W_2$, then $\varrho_1 \cdot \sigma$ also extends τ. But, since $\dim(W^* \cap W_2) \geqq 2$ and k is finite, we know $W^* \cap W_2$ contains elements with arbitrary squares. Hence there is a rotation $\varrho_1 \in O(W^* \cap W_2)$ such that $\theta_{w_2}(\varrho_1) = \theta_{w_2}(\sigma)$, thus $\theta_{w_2}(\varrho_1 \cdot \sigma) = 1$, and hence $\varrho_1 \cdot \sigma \in \Omega(W_2, Q|W_2)$. Since $\dim W_2 \geqq 3$, then W_2 contains non-zero isotropic vectors; so that $\Omega(W_2, Q|W_2') = \ker \theta_{w_2}$. Let $\varrho = \varrho_1 \cdot \sigma \perp 1_{w_2^*}$, then $\varrho \in \Omega_F(V,Q)$.

COROLLARY. *If k is finite,* $\dim V \geqq 4$, *and x,y are non-zero isotropic vectors, then there is a $\lambda \in \Omega_F(V,Q)$ such that $\lambda x = y$.*

PROOF. x is in some hyperbolic plane, as is y. Let τ map the first hyperbolic plane onto the other isometrically and such that $\tau x = y$. Then extend τ by the lemma.

LEMMA 5. *Let U_1, U_2 be semi-simple isometric subspaces of (V,Q), (V,Q) a semi-simple Q-space. Suppose that U_2 is finite dimensional and contains isotropic lines. Then there is a $\lambda \in \Omega_F(V,Q)$ such that $U_2 = \lambda(U_1)$.*

PROOF. Let W_1 be a semi-simple finite dimensional subspace containing U_1 and U_2. By multiplying by an appropriate symmetry, we can find a rotation σ of W_1, such that $\sigma U_1 = U_2$. Again, we can follow this by any rotation ϱ of U_2. By Lemma 1, we can achieve: $\theta_{w_1}(\varrho) = \theta_{w_1}(\sigma)$, since U_2 has isotropic lines. Hence $\lambda_1 = \varrho \cdot \sigma$ is such that $\theta_{w_1}(\lambda_1) = 1$, so that $\lambda_1 \in \Omega(W_1)$ and $U_2 = \lambda(U_1)$. Then $\lambda = \lambda_1 \perp 1_{w_1*}$ is the desired map.

COROLLARY. *If $\dim V \geqq 3$ and x,y are non-zero isotropic vectors, then there is an element $\beta \in k^*$ such that for any $\alpha \in k^*$, there is an element $\lambda \in \Omega_F(V,Q)$ such that $\lambda(x) = \beta\alpha^2 y$.*

PROOF. Let y,y' be a hyperbolic pair, and let W be a semi-simple subspace of finite dimension, such that x,y,y' are in W. Choose a rotation $\sigma \in O(W)$ such that $\sigma x = y$ (possible since $\dim V \geqq 3$). Let $\theta_w(\sigma) = \beta k^{*2}$, and let ϱ be a rotation of $S(y,y')$ such that $\varrho(y) = \beta\alpha^2 y$. Then $\theta_w(\varrho) = \theta_w(\sigma)$ and $\lambda = \varrho\sigma$ is the desired element of $\Omega_F(V,Q)$ (identified with $1_{w*} \perp \lambda$).

LEMMA 6. *Let (V,f) be a semi-simple Q-space with $\dim V \geqq 5$. Let $P = S(x,y)$ be a singular plane (that is, $\operatorname{rad} P \neq \{0\}$) where $x^2 = y^2 \neq 0$. Then there is $\lambda \in \Omega_F(V,Q)$ such that $\lambda y = y$, $\lambda x = z$, where $S(x,z)$ is semi-simple.*

PROOF. Let $S(u)$ be the radical of P, then $P = S(y,u)$ and $x = \alpha y + \beta u$, $\beta \neq 0$. But $x^2 = y^2$, hence

$$x^2 = \alpha^2 y^2 + 2\alpha\beta uy + \beta^2 u^2 = \alpha^2 y^2 .$$

So that $\alpha = \pm 1$, and we can replace αy by y, βu by u, to obtain $x = y + u$. Let $H = S(y)^*$ be the hyperplane orthogonal to y, since $u \in H$, there is an isotropic vector $v \in H$, such that $u \cdot v = -x^2$. Then we have two cases:

a) k is a finite field: $\dim H \geqq 4$, so that there is a $\lambda_1 \in \Omega_F(H)$ such that

$$\lambda_1 u = v, \qquad \lambda = 1_{H*} \perp \lambda_1 \in \Omega_F(V,Q) .$$

Hence

$$\lambda(y) = y, \qquad \lambda x = \lambda(y+u) = y+v = z ,$$

and

$$x \cdot z = (y+u)\cdot(y+v) = y^2 + y\cdot v + y\cdot u + u\cdot v = 0 ,$$

$$z \cdot z = x^2 = y^2 \neq 0 .$$

Therefore $S(x,z)$ is semi-simple.

b) k is an infinite field; thus there is a $\lambda \in \Omega_F(H, f|H)$ such that

$$\lambda u = \alpha\beta^2 v, \ \lambda y = y ,$$

α, β chosen as follows:

$$\lambda x = \lambda(y+u) = y+\alpha\beta^2 v = z .$$

Then

$$z^2 = y^2 = x^2 ,$$

$$x\cdot z = x\cdot y+\alpha\beta^2 x\cdot v = (y+u)\cdot y+\alpha\beta^2(y+u)v = y^2-\alpha\beta^2 x^2 .$$

Then $S(x,z)$ is non-singular when we can choose β such that

$$x\cdot z = y^2-\alpha\beta^2 x^2 \neq \pm x^2 .$$

Since k is infinite, one can clearly find an appropriate β and α.

Lemma 7. *Let (V,Q) be a semi-simple Q-space, $x \in V$ such that $f_Q(x,x) \neq 0$. Suppose σ is an isometry of V which keeps fixed every line L generated by a vector y such that*

$$y^2 = f_Q(y,y) = x^2 .$$

Then $\sigma = \pm 1_V$ if $\dim V \geqq 4$ *for any k.*

Proof. Since $f_Q(\sigma x, \sigma x) = f_Q(z,x) \neq 0$, we may assume that $\sigma x = x$ by replacing σ by $-\sigma$ if necessary. Let $H = S(x)^*$. We have two cases:

a) H contains non-zero isotropic vectors: Let $u \in H$ be such that $u^2=0$, $u \neq 0$. Since $(x+u)^2 = x^2$, it follows that $\sigma(x+u) = \varepsilon(x+u)$ where $\varepsilon = \pm 1$. But

$$\sigma(u) = \sigma(x+u-x) = \varepsilon(x+u)-x$$

is isotropic, hence $\varepsilon = +1$. So

$$\sigma(u) = (x+u)-x = u .$$

Let $y \in H$. Then y is in some hyperbolic plane of H. But σ equals the identity on any hyperbolic plane of H so that $\sigma(y) = y$, $\sigma(x) = x$ implies that $\sigma = 1_V$, hence the original $\sigma = \pm 1_V$.

b) H is anisotropic. Let $z \neq 0$, $z \in H$. If there are at least six rotations of the plane $P = S(x,z)$, then they carry $S(x)$ into three distinct lines which are kept fixed by σ, which gives us, by use of the properties of σ and $\sigma x = x$, that $\sigma z = z$. But if k is finite, then since $\dim H \geqq 3$, by the corollary to lemma 1, H has non-zero isotropic vectors, so that we are in case a). And if k is infinite, then it is known that there are more than six rotations of the plane P when P is either hyperbolic or anisotropic. Of course, P is semi-simple.

This completes the proof.

Now we generalize to infinite dimensions a proposition which is practically the result we want (see [1]).

PROPOSITION 3. *Suppose* (V,Q) *is a semi-simple Q-space with* $\dim V \geqq 5$. *Let H be a subgroup of* $O(V)=G$ *which enjoys the following properties*:

a) *H is invariant under transformation by elements of* $\Omega_F(V,Q)$; *that is, if* $\sigma \in \Omega_F(V,Q)$, *then* $\sigma H \sigma^{-1} \subseteqq H$.

b) *H is not contained in the center of* $G=O(V)$.

Then H will contain an element $\sigma \neq 1_V$ *which is the square of a three dimensional rotation; that is, it is the square of a rotation arising from a three-dimensional space.*

PROOF. Pick $\sigma \in H$ such that $\sigma \neq \pm 1_V$ (corollary to lemma 3). Then σ must move some non-isotropic line $S(x)$, otherwise $\sigma = \pm 1_V$, by lemma 7. Now we can choose this σ such that the plane $P=S(x,\sigma(x))$ is semi-simple, as follows: Suppose P is singular. Using $\sigma(x)^2=x^2$, it follows by lemma 6 that, there is a $\lambda \in \Omega_F(V)$ such that $\lambda(\sigma(x))=\sigma(x)$, $\lambda x=z$, with $S(x,z)$ semi-simple. Take $\varrho=\lambda\sigma^{-1}\lambda^{-1}\sigma$. This is in H, by hypothesis a), and $\varrho(x)=z$, hence $S(x,\varrho(x))$ is semi-simple. Let σ be this ϱ. So we may assume that $P=S(x,\sigma(x))$ is a semi-simple plane. Using this σ we claim that there is a $\varrho \neq \pm 1_V$ of H which keeps some non-isotropic line fixed. To see this we may assume that σ moves every non-isotropic line, otherwise take $\varrho=\sigma$. Suppose there were a $\lambda \in \Omega_F(V,Q)$ such that it keeps every vector of P fixed and does not commute with σ, then $\varrho=\lambda^{-1}\sigma^{-1}\lambda$. $\sigma \in H$, by hypothesis a) and $\varrho(x)=\lambda^{-1}(x)=x$. Thus $\varrho \neq -1_V$, and $\lambda\sigma \neq \sigma\lambda$ implies that $\varrho \neq 1_V$. Thus we would have the desired ϱ. To prove the existence of the above λ consider the cases:

i) $\sigma P \neq P$. Let $u \in P$ such that $\sigma u \notin P$, so that $\sigma u=v+w$, $v \in P$, $w \in P^*$, with $w \neq 0$. But the $\dim P^* \geqq 3$. Then one can pick a three dimensional semi-simple subspace of P^* containing w, say W, and then find $\lambda \in \Omega(W)$ which moves w (see [1, p. 105]). Extend λ to V, and get

$$\lambda\sigma(u) = \lambda(v+w) = v+\lambda(w) \neq v+w = \sigma u = \sigma\lambda(u)\,,$$

hence $\lambda\sigma \neq \sigma\lambda$ and $\lambda \in \Omega_F(V)$.

ii) $\sigma P=P$. Then $\sigma P^*=P^*$. Let $\tau=\sigma|_{P^*}$. Suppose $\tau\varrho=\varrho\tau$ for all $\varrho \in \Omega_F(P^*)$, then by lemma 3 we know that $\tau=\pm 1_{P^*}$. But this contradicts the assumption that σ moves the non-isotropic lines. Hence there is a $\varrho' \in \Omega_F(P^*)$ such that $\tau\varrho' \neq \varrho'\tau$. Extending ϱ' to V, by $\varrho=1_P \perp \varrho'$, we have that $\sigma\varrho \neq \varrho\sigma$, $\varrho|_P=1_P$.

Thus we have an element $\sigma \neq \pm 1_V$ in H such that σ keeps a certain non-isotropic line $S(x)$ fixed. Then, by lemma 7, we have again that there must be a vector y such that $y^2=x^2$, but $S(y)$ is moved by σ. Let $\sigma y=z$, then $S(y)\neq S(z)$. Let τ_x be the symmetry defined by the hyperplane $S(x)^*$, that is

$$\tau_x(u) = u - 2\frac{x\cdot u}{x\cdot x}x .$$

Let W be a semi-simple subspace of finite dimension containing x and y. Then Witt's theorem says that there is an element $\mu \in O(W)$ and hence an element $\mu \in O_F(V)=G_F$, such that $\mu(x)=y$. Now $\mu\tau_x\mu^{-1}=\tau_{\mu(x)}=\tau_y$, therefore

$$\lambda = \tau_y\tau_x = \mu\tau_x\mu^{-1}\tau_x^{-1} \in \Omega_F(V,Q) .$$

Again, $\sigma\tau_x\sigma^{-1}=\tau_{\sigma x}=\tau_{\pm x}=\tau_x$, and $\sigma\tau_y\sigma^{-1}=\tau_{\sigma y}=\tau_z$. Therefore

$$\sigma\lambda\sigma^{-1} = \sigma\tau_y\tau_x\sigma^{-1} = \sigma\tau_y\sigma^{-1}\sigma\tau_x\sigma^{-1} = \tau_z\tau_x .$$

Hence $\varrho=\sigma\lambda\sigma^{-1}\lambda^{-1}=\tau_z\tau_x\tau_x\tau_y=\tau_z\tau_y$. But $\varrho \in H$, and $\varrho\neq 1_V$, since $S(z)\neq S(y)$. Further $\varrho \in \Omega_F(V,Q)$, (ref. V) on the spinorial norm. Finally, let $P=S(y,z)$, then $\operatorname{rad} P=P^*\cap P=\operatorname{rad} P^*$. Since $y^2\neq 0$, $\dim \operatorname{rad} P \leqq 1$, let W be any finite dimensional semi-simple subspace of $\dim = n_w \geqq 5$ containing P. Then $P^*\cap W$ contains a semi-simple subspace W_1 of dimension n_w-3. Let U be a three-dimensional semi-simple subspace orthogonal to W_1 in W. So that ϱ is a rotation of U, since $\varrho=\tau_z\tau_y$ leaves each element of P^*, and hence those of W_1, fixed. But $\varrho \in \Omega(W)\subseteqq \ker\theta_w$, and by properties iv) and v) of θ_w, we get

$$\varrho \in \ker\theta_u = O^+(U) \cap \ker\theta_w, \qquad \ker\theta_u = \Omega(U) .$$

So that $\varrho \in \Omega(U)$ and ϱ is a square of a rotation of U.

This completes the proof, since $\varrho\neq 1_V$, $\varrho \in H$, and ϱ is the square of a rotation of U a three-dimensional semi-simple subspace of V; hence $\varrho \in \Omega_F(V,Q)$.

Suppose now that (V,Q) contains isotropic vectors. Let U be a three-dimensional semi-simple subspace of V, and ϱ the square of a rotation of U. Suppose U is anisotropic. Then the axis of rotation of ϱ is not isotropic and thus ϱ is the square of a rotation of an anisotropic plane P. We prove that any anisotropic plane P can be imbedded in a three-dimensional semi-simple subspace U' of V containing isotropic vectors, as follows: Select $u\neq 0$, such that $u^2=0$ and u is not orthogonal to P. This is possible as follows: Let $S(u,v)$ be a hyperbolic plane, $P=S(y,z)$. Then

$$(u+\tfrac{1}{2}y^2v)^2 = 2\tfrac{1}{2}y^2(u\cdot v) = y^2 .$$

Let $\tau: u+\frac{1}{2}y^2v \to y$, extend τ to V. Then $\tau: S(u,v) \to S(t,s)$ a hyperbolic plane containing y, so that

$$\tau : u+\tfrac{1}{2}y^2\cdot v \to t+\tfrac{1}{2}y^2\cdot s = y, \qquad t(t+\tfrac{1}{2}y^2 s) = \tfrac{1}{2}y^2 \neq 0 .$$

Hence this t works. Let u then be such that $u^2=0$ and not $u \perp P$, set $U' = S(y,z,u)$. If U' were singular, and $S(v)$ its radical, then $S(v)$ would be the only isotropic line of U', since P is anisotropic. So that $S(v)=S(u)$ and this implies that $u \in P^*$, contradicting the fact that not $u \perp P$. Hence U' is semi-simple. This proves our assertion, so that we may assume that ϱ is the square of rotation of U which contains isotropic vectors in the first place. This also proves that the generators (see [1, p. 135]) $(\tau_x\tau_y)^2$ of $\Omega_F(V)$ are squares of rotations of three-dimensional subspaces U which contain isotropic lines. We will use this remark below.

Now let (V,Q) be a semi-simple Q-space of dimension $\geqq 5$, containing isotropic vectors and H a subgroup of $O(V)$ as in the proposition above. Let

$$\varrho \in H \cap \Omega_F(V), \qquad \varrho \neq 1_V ,$$

and such that ϱ is the square of a rotation of a three-dimensional semi-simple subspace U containing isotropic vectors.

i) Suppose k contains more than three elements. Then from the finite dimensional theory we know that $\Omega(U) \approx PSL_2(k)$, the projective special linear group, is simple. $H \cap \Omega(U)$ contains ϱ and is an invariant subgroup of $\Omega(U)$, hence $\Omega(U) \subseteqq H$, by simplicity of $\Omega(U)$. The subspace U contains a hyperbolic plane P and therefore H contains $\Omega(P)$. Let U' be a subspace, such as U, P' a hyperbolic plane of U'. By lemma 5, there is a $\lambda \in \Omega_F(V)$ such that $P' = \lambda P$. So that $\Omega(P') = \lambda\Omega(P)\lambda^{-1}$, hence $\Omega(P') \subseteqq H$. The group $\Omega(U')$ is simple and $\Omega(U') \cap H \supseteqq \Omega(P')$, a non-trivial group. But $\Omega(U') \cap H$ is invariant in $\Omega(U')$, hence $\Omega(U') \subseteqq H$. This implies that H contains all generators of $\Omega_F(V)$, and hence H contains $\Omega_F(V,Q)$.

ii) Suppose $k = \mathrm{GF}(3)$. We use the fact that a finite dimensional Q-space over a finite field contains a proper subspace of arbitrary prescribed geometry (by use of lemma 1 repeatedly). Since $\dim V \geqq 5$, we can find a semi-simple subspace V_0' of dimension 4 which is of index 1. Then the subspace V_0' contains a three-dimensional subspace U' isometric to U, by use of the same fact. So by extending this isometry of U' with U to V, U is contained in a 4-dimensional semi-simple subspace V_0 of V with index 1. And again by the finite dimensional theory the group $\Omega(V_0)$ $(\approx PSL_2(kd^{\frac{1}{2}}))$ is simple. Further $\Omega(V_0) \cap H$ contains ϱ. Hence $\Omega(V_0) \subseteqq H$. Then, as before, with V_1 any 4-dimensional semi-simple subspace of index 1, we have again $\Omega(V_1) \subseteqq H$. If U' is any subspace such as U we can imbed it in such a space V_1. Thus we can get $\Omega(U') \subseteqq H$. And, as in i) we thus have $\Omega_F(V) \subseteqq H$. We have proved

THEOREM 3. *Suppose (V,Q) is a semi-simple quadratic space of dimension $\geqq 5$, and that V contains isotropic vectors. Let H be a subgroup of $O(V)$ having the following properties:*

i) H is invariant under transformation by $\Omega_F(V,Q)$; that is to say, if $\tau \in \Omega_F(V,Q)$, then $\tau H \tau^{-1} \subseteqq H$.

ii) H is not contained in $Z(V,Q)$.

Then $\Omega_F(V,Q) \subseteqq H$.

COROLLARY. *Let (V,Q) be a semi-simple quadratic Q-space of dimension $\geqq 5$, containing isotropic vectors. Let H be a closed normal subgroup of $O(V,Q)$, provided with the finite topology, such that $H \nsubseteqq Z(V,Q)$. Then H contains the closure of $\Omega_F(V,Q)$.*

If H is also in $\overline{\Omega_F(V)}$, then $H = \overline{\Omega_F(V)}$.

Now since G_F is dense in $G = O(V)$, then by continuity of multiplication $\overline{\Omega_F(V,Q)} = \overline{O(V,Q)'} = \Omega(V,Q)$, the commutator subgroup of $O(V)$. Hence we have

THEOREM 4. *Let (V,Q) be a semi-simple quadratic space of dimension $\geqq 5$ containing isotropic vectors. Then the group $\Omega(V,Q)/Z(V,Q) \cap \Omega(V,Q)$, where $\Omega(V,Q)$ = closure of the group of commutators of $O(V,Q)$, and $Z(V,Q)$ = center of $O(V,Q)$, is a simple topological group.*

REMARK 1. If we provide the general linear group of an infinite dimensional linearly topologized vector space with the finite topology, we get a topological group. Using the same type of definitions and analysis with the aid of the problems in Bourbaki [2, pp. 97–99], we can prove easily a similar theory for the projective special linear group of the general linear group.

REMARK 2. For the spaces of the type considered in theorem 4 we have a new way to prove that $O(V,Q)$ is not compact; namely, if $O(V,Q)$ is compact, then, since it is also totally disconnected, there exist arbitrarily small open normal subgroups. But this contradicts theorem 3, and hence $O(V,Q)$ could not be compact.

Finally, let us consider when $\Omega(V,Q) = O(V,Q)'$. Since $\overline{\Omega_F(V,Q)} = \Omega(V,Q)$, it suffices to know when $\Omega_F(V,Q)$ is dense in $O(V,Q)$. Again, since $O(V,Q) = \overline{G_F}$, it suffices to ask when $\overline{\Omega_F(V,Q)} \geqq G_F$. That is, for any $\sigma \in G_F$, and any $E \in \mathscr{E}_0$ does there exist an $\eta \in \varrho(E)$ such that $\sigma \cdot \eta \in \Omega_F$. It is clear that such η must exist in order that $\Omega(V,Q) = O(V,Q)$; also if it does exist, then $\eta \in G_F \cap \varrho(E)$. Hence, it has a norm, $\theta_w(\eta) = \theta(\eta)$, for some $W \in \mathscr{E}_0$. Further, we have $\theta(\sigma\eta) = \theta(\tau) = 1$, since $\sigma\eta = \tau \in \Omega_F(V,Q)$; so that $\theta(\sigma) = \theta(\eta)$. Now $\eta \in O(E^*) \cap G_F$ and we may always pick η such that $\theta(\sigma) = \theta(\eta)$ if E^* has isotropic vectors (if it does not, we may not be able to find η such that $\theta(\eta) = \theta(\sigma)$). Thus we have

THEOREM 5. *Let (V,Q) be a semi-simple quadratic space of dimension $\geqq 5$ containing infinitely many linearly independent isotropic vectors. Then we have the diagram*

$$\underbrace{\{1_V\} \subseteqq Z_V \cap}_{\text{A group of order 1 or 2.}} \underbrace{\Omega(V,Q) \subseteqq \Omega(V,Q)}_{\text{A simple topological group.}} = O(V,Q).$$

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