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THE DECISION PROBLEM FOR SEGREGATED FORMULAS IN FIRST-ORDER LOGIC

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1. Introduction.

We consider the decision problem for satisfiability for certain first-order prenex conjunctive formulas which we call *segregated* because each disjunction has only negated atomic formulas or only unnegated atomic formulas.

Let $\mathfrak{L}$ be a given first-order predicate calculus without equality and let I be the set of positive integers. We use $\vee$, $\wedge$, and $\neg$ for the propositional connectives disjunction, conjunction, and negation, respectively; we use $\exists$ and $\forall$ for the existential and universal quantification operators, respectively; we let $P, Q_1, Q_2, \ldots$ be arbitrary (possibly empty) strings of quantifiers; and we let α_i, α_{ij}, and β_{ij}, for $i, j \in I$, be variables over the set of atomic formulas. For any $k \in I$ we let S_k^+ be the set of formulas of the form

$$P\left[\bigwedge_{i=1}^{m} \bigvee_{j=1}^{k} \alpha_{ij} \wedge \bigwedge_{r=1}^{n} \bigvee_{s=1}^{t(r)} \neg \beta_{rs}\right]$$

and we let S_k^- be the set of formulas of the form

$$P\left[\bigwedge_{i=1}^{m} \bigvee_{j=1}^{t(i)} \alpha_{ij} \wedge \bigwedge_{r=1}^{n} \bigvee_{s=1}^{k} \neg \beta_{rs}\right],$$

where $m, n \in I$ and $t \in I^I$. We show here that if $\mathfrak{L}$ has predicate letters of arbitrarily great rank then $S_2^+ \cap S_3^-$ and $S_3^+ \cap S_2^-$ are reduction classes for satisfiability (cf. page 32 of [9]). In addition we provide a decision procedure for satisfiability for the formulas of $S_1^+ \cup S_1^-$. Thus, in particular, S_1^+ is a solvable class while S_2^+ is a reduction class. Our decision procedure is a generalization of a result of J. Herbrand given on pages 44 and 45 of [4] and listed as case V' on page 256 of [2]. The result of J. Herbrand is a decision procedure for satisfiability for formulas in $S_1^+ \cap S_1^-$.

2. Reduction classes.

We first describe a simple procedure for replacing a prenex conjunctive formula with a formula of $S_3^+ \cap S_3^-$ that is equivalent to it with

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respect to satisfiability. Let X be a formula of any pure first-order language without equality and without function symbols of the form

$$Q\left[[M_1 \vee M_2] \wedge \bigwedge_{i=1}^{n} N_i\right]$$

where $n \in I$ and M_1, M_2 and N_i are disjunctions of negated and unnegated atomic formulas. Let F be a predicate letter of rank at least as great as the number of individual variables occurring in M_1 and such that F has no occurrence in X. Let $\bar{F}$ be an atomic formula obtained from F by attaching the individual variables that occur in M_1 to its argument places. We say that

$$\bar{X} = Q\left[[M_1 \vee \bar{F}] \wedge [\neg \bar{F} \vee M_2] \wedge \bigwedge_{i=1}^{n} N_i\right]$$

is obtained from X by a *subdivision of a disjunction*. We observe that X and $\bar{X}$ are equivalent with respect to satisfiability.

Suppose $\mathfrak{M}$ is a model of X (cf. page 51 of [8]). We expand $\mathfrak{M}$ to a model $\mathfrak{M}'$ for $\bar{X}$ by introducing a relation $\mathfrak{F}$ for the predicate letter F in a suitable way. We include in the relation $\mathfrak{F}$ only those ordered sets of individuals which correspond to an assignment of individuals to the free variables of $\bar{F}$ for which M_1 is not satisfied over $\mathfrak{M}$. It follows that any assignment of individuals of $\mathfrak{M}$ and $\mathfrak{M}'$ to the individual variables of $M_1 \vee M_2$ which satisfies $M_1 \vee M_2$ over $\mathfrak{M}$ also satisfies

$$[M_1 \vee \bar{F}] \wedge [\neg \bar{F} \vee M_2]$$

over $\mathfrak{M}'$. Thus any assignment of individuals of $\mathfrak{M}$ and $\mathfrak{M}'$ to the individual variables of the matrix H of X which satisfies H over $\mathfrak{M}$ is an assignment to the individual variables of the matrix $\bar{H}$ of $\bar{X}$ which satisfies $\bar{H}$ over $\mathfrak{M}'$, and we conclude that $\mathfrak{M}'$ is a model of $\bar{X}$. Conversely, suppose $\mathfrak{X}'$ is a model of $\bar{X}$ and let $\mathfrak{X}$ be the structure obtained from $\mathfrak{X}'$ by deleting the relation corresponding to the predicate letter F. The matrices H and $\bar{H}$ contain the same individual variables and by comparing their truth values over $\mathfrak{X}$ and $\mathfrak{X}'$, respectively, for assignments to these individual variables we see that $\mathfrak{X}$ is a model of X.

THEOREM 1. *There exists an effective procedure for obtaining, for any first-order prenex conjunctive formula X without equality and without function symbols, a prenex formula $\bar{X}$ with the same prefix as X such that $\bar{X} \in S_3^+ \cap S_3^-$ and $\bar{X}$ is equivalent to X with respect to satisfiability.*

PROOF. The procedure consists of successively subdividing disjunctions first to reduce to ternary disjunction and then to introduce segregation, and the justification for the procedure is explained above.

THEOREM 2. *The class of first-order formulas without equality, without function symbols, and without free individual variables, in which exactly two predicate letters occur, both binary which have the form*

$$\forall a \forall b \forall c \exists d_1 \ldots \exists d_n M,$$

where M is a matrix in conjunctive normal form in which one disjunction has three terms and all other disjunctions have two terms, is a reduction class for satisfiability.

PROOF. We start with a known reduction class $\mathfrak{R}$ consisting of all first-order formulas without equality, without function symbols, and without free individual variables, in which exactly one predicate letter occurs, a binary one, which have the form $\forall a \forall b \forall c \exists d_1 \ldots \exists d_n M$ where M is a matrix in conjunctive normal form, cf. the first item in the list of reduction classes for validity which begins on page 278 of [2]. Let F be a binary predicate letter which does not occur in the formulas of $\mathfrak{R}$ and let

$$Y = \forall a \forall b \forall c \exists d \exists e[[\neg Fab \vee \neg Fbc \vee Fac] \wedge [Faa \vee Faa] \wedge \wedge [\neg Fde \vee \neg Fde]].$$

Notice that any structure $\mathfrak{X}$ is a model of Y iff its interpretation for F is a relation which is reflexive and transitive on its domain and which does not hold for some two, necessarily distinct, elements.

For any formula X in the known reduction class $\mathfrak{R}$ we will show how to obtain a formula X^* which is equivalent to X with respect to satisfiability and which has all of the properties specified in the description of the class given in the statement of the theorem. For any $X \in \mathfrak{R}$ we first determine whether it is satisfiable in a one element structure (cf. problem 3, page 70, of [8]). If X is satisfiable in a one element structure we let X^* be $\forall a \forall b \forall c[Faa \vee \neg Faa]$. For any $X \in \mathfrak{R}$ which is not satisfiable in a one element structure we first obtain a formula $\bar{X}$ by replacing each of its conjuncts which are disjunctions of the form $N_1 \vee \ldots \vee N_k$ with corresponding formulas of the form

$$\exists e_1 \ldots \exists e_{k+1}[[N_1 \vee Fe_1e_2] \wedge [N_2 \vee Fe_2e_3] \wedge \ldots \wedge [N_k \vee Fe_ke_{k+1}] \wedge \wedge [\neg Fe_1e_{k+1} \vee \neg Fe_1e_{k+1}]].$$

Then we let X^* be the result of exporting quantifiers from $\bar{X} \wedge Y$ so that X^* is logically equivalent to $\bar{X} \wedge Y$ and so that X^* has a prefix of the form $\forall a \forall b \forall c \exists d_1 \ldots \exists d_n$. Thus X^* will belong to the class of formulas described in the theorem; the only ternary disjunction of X^* will be that conjunct of Y which defines transitivity. For any $X \in \mathfrak{R}$ such that X has no one element models, if $\mathfrak{M}$ is a model of X let $\mathfrak{M}'$ be any structure obtained from $\mathfrak{M}$ by introducing a new relation for the predicate letter F which is reflexive and transitive on the domain of $\mathfrak{M}$ and which does not hold for some two elements. Then $\mathfrak{M}'$ will be a model of X^*. Conversely, for any $X \in \mathfrak{R}$ such that X has no one element models, if $\mathfrak{M}$ is a model of X^* then the structure obtained from $\mathfrak{M}$ by deleting the relation corresponding to the predicate letter F is a model of X. The procedure we use here to simplify disjunctions is exactly the method presented in Theorem 1.4 on page 319 of [1], cf. also the last paragraph on page 320 of [1].

THEOREM 3. *For any first-order predicate language without equality, without function symbols but with predicate letters of arbitrarily great rank, the classes* $S_2^+ \cap S_3^-$ *and* $S_3^+ \cap S_2^-$ *are reduction classes for satisfiability.*

PROOF. Any formula in the reduction class described in Theorem 2 above can be transformed into a formula in $S_2^+ \cap S_3^-$ (in $S_3^+ \cap S_2^-$) by successively subdividing its disjunctions as described in the first paragraph of this section. Note that a formula obtained by such a transformation will still have a prefix of the form described in Theorem 2 and it will have exactly one ternary disjunction.

Although it is known that a ternary disjunction is necessary to define transitivity, cf. [6], it is not known whether a ternary disjunction is necessary in these reduction classes. Thus it is not known whether $S_2^+ \cap S_2^-$ is a reduction class for satisfiability. However, by properties of the negation of formula (1) on page 233 of [2], it is known that the decision problem for satisfiability for $S_2^+ \cap S_2^-$ is not finitely reducible, cf. page 69 of [7]. In [5] a decision procedure is provided for a class of formulas in which all disjunctions are binary but the class does not include all of $S_2^+ \cap S_2^-$.

3. A decision procedure.

We will use, in a limited way, the proof theory of Linear Reasoning introduced by W. Craig in [3], but we will restate all of the definitions and facts that we need from that theory. From the eleven one-premise rules of inference used in Linear Reasoning we need only the four that we list below. To state these rules we let $M_1, M_2, \ldots$ be the arbitrary quantifier free formulas, we let x and y be arbitrary individual variables, and we let t be an arbitrary individual term.

Duplication: $$\frac{P[Q_1M_1\wedge\ldots\wedge Q_mM_m\wedge\ldots\wedge Q_nM_n]}{P[Q_1M_1\wedge\ldots\wedge Q_mM_m\wedge Q_mM_m\wedge\ldots\wedge Q_nM_n]}.$$

$\exists$-exportation: $$\frac{P[Q_1M_1\wedge\ldots\wedge\exists xQ_mM_m(x)\wedge\ldots\wedge Q_nM_n]}{P\,\exists y[Q_1M_1\wedge\ldots\wedge Q_mM_m(y)\wedge\ldots\wedge Q_nM_n]},$$

where (1) $Q_mM_m(y)$ is the result of substituting y for the free occurrences of x in $Q_mM_m(x)$, (2) y is free at the free occurrences of x in $Q_mM_m(x)$, and (3) y does not occur free in Q_kM_k, $k\neq m$, nor in $\exists xQ_mM_m(x)$.

$\forall$-exportation: same as $\exists$-exportation, with $\forall x$ and $\forall y$ in place of $\exists x$ and $\exists y$, respectively.

$\forall$-instantiation: $$\frac{Q_1\forall yQ_2M(y)}{Q_1Q_2M(t)},$$

where (1) $Q_2M(t)$ is the result of substituting t for the free occurrences of y in $Q_2M(y)$ and (2) t is free at the free occurrences of y in $Q_2M(y)$.

For any formulas X and Y, an *L^s-deduction* of Y from X is an ordered n-tuple $\langle X_1,\ldots,X_n\rangle$, where $X_1=X$ and $X_n=Y$, together with a specification of how, for any m, $1\leq m<n$, X_{m+1} results from X_m by an application of one of the four L-rules listed above.

LEMMA 1. *For any prenex formula X of $\mathfrak{L}$, X is not satisfiable iff there exists an L^s-deduction from X of a prenex formula Y with a truth functionally inconsistent matrix.*

PROOF. By the consistency of the L-rules, X is not satisfiable whenever such an L^s-deduction exists.

Conversely, suppose that X is a prenex formula which is not satisfiable. Let C be $\alpha\wedge\neg\alpha$ where α is any predicate letter of rank zero which does not occur in X. By Theorem 2 of [3] (a completeness theorem for Linear Reasoning) there exists a symmetric L-deduction $\mathfrak{D}$ of C from X. A symmetric L-deduction consists of a succession of applications of eleven one-premiss rules of inference in a specified order. This specified order requires that all applications of a rule called *matrix change* occur together and in the middle of the deduction. The rule *matrix change* is a rule that allows one to infer PM' from PM where M,M' are quantifier free and $\neg M\wedge M'$ is tautologous. Since all applications of matrix change occur together and the prefix is left unchanged with this rule, we may assume that there is exactly one application of matrix change in $\mathfrak{D}$. We report also that matrix change is the only one of the eleven L-rules which can be applied to a premiss in such a way as to introduce a new predicate letter that does not occur in the premiss or in such a way as to exclude all occurrences of a predicate letter that does occur in the premiss.

Let PM' be the formula obtained from PM by the application of matrix change in $\mathfrak{D}$. Then M and M' have no predicate letters in common and M' is not tautologous so M must be contradictory.

We report further that the only L-rules that can occur in a symmetric L-deduction before the application of matrix change are the four rules listed above for our definition of L^s-deduction and an equivalence rule, $\forall$-vacuous introduction, which consists of introducing a vacuous universal quantifier into the prefix. We obtain an L^s-deduction from$X$ of a prenex formula Y with a truth functionally inconsistent matrix by modifying the initial part of $\mathfrak{D}$ to the point of the application of matrix change by omitting any steps justified with the rule $\forall$-vacuous introduction and by deleting corresponding occurrences of universal quantifiers in succeeding formulas.

An L^s-deduction will be said to be of *order* h in case in includes exactly h applications of duplication.

LEMMA 2. *Let X and Y be prenex formulas such that the matrix of X is in conjuctive normal form and such that there exists an L^s-deduction of Y from X. Then the matrix N of Y is in conjunctive normal form and for any h conjuncts $C_1, \ldots, C_h$ of N there exists an L^s-deduction from X of order $\leq h$ of a prenex conjunctive formula Y' with a matrix in which $C_1, \ldots, C_h$ occur as conjuncts.*

PROOF. We will say that a formula is *conjunctive* in case the formula obtained from it by deleting all of its quantifiers is in conjunctive normal form. We observe that an application of any of the four L^s-rules to a conjunctive formula produces a conjunctive formula. It follows that if there is an L^s-deduction from a prenex conjunctive formula X of a prenex formula Y, then Y is conjunctive.

Let $\mathfrak{D}$ be any L^s-deduction of a prenex conjunctive formula Y from a prenex conjunctive formula X. For each occurrence of a conjunct in the matrix of Y we will define a unique *predecessor* in each preceeding formula of $\mathfrak{D}$. For any occurrence of a formula X_i of $\mathfrak{D}$ which is introduced by an application of duplication, the occurrence of the connective $\wedge$ between the two identical conjuncts produced in the duplication will be called the *center* of X_i. The *predecessor* in a formula X_j of $\mathfrak{D}$ of an occurrence of a conjunct in the next formula, X_{j+1}, of $\mathfrak{D}$ is the occurrence of the conjunct of X_j in the same relative position where in the case of an application of duplication the position is determined by counting from the left for an occurrence of a conjunct left of the center and by counting from the right for an occurrence of a conjunct right of the center. The *predecessor* in any earlier formula of an occurrence of a conjunct is defined as required so that the relation of predecessor is the smallest transitive relation including the elements just described.

For any h conjuncts $C_1, \ldots, C_h$ of the matrix of Y we will show how to modify $\mathfrak{D}$ to obtain an L^s-deduction from X of order $\leqq h$ of a prenex formula Y' with a matrix in which $C_1, \ldots, C_h$ occur as conjuncts. To obtain the required modification of $\mathfrak{D}$, first specify particular occurrences of $C_1, \ldots, C_h$ in Y and then omit any application of duplication in $\mathfrak{D}$ unless the two identical subformulas produced each contain a predecessor of one of the specified occurrences of $C_1, .., C_h$. When an application of duplication is to be omitted select one of the two identical subformulas produced by it to be deleted and delete in all later formulas any subformula whose predecessor is in this selected subformula and also delete any quantifiers exported from deleted subformulas. If just one of two identical subformulas produced in an application of duplication contains a predecessor of a specified occurrence of one of $C_1, \ldots, C_h$, then select the other of the two identical subformulas to be deleted in omitting that application of duplication.

THEOREM 4. *There is a positive solution to the decision problem for satisfiability for formulas of* $S_1^+ \cup S_1^-$ *in any first order language without equality.*

PROOF. A quantifier free formula M in S_1^+ (in S_1^-) is contradictory iff it contains as one conjunct a disjunction of negated atomic formulas (of atomic formulas) and as other conjuncts each of these (the negation of each of these) atomic formulas. Thus a quantifier free formula M in $S_1^+ \cup S_1^-$ for which the largest number of terms in any of its disjunctions is h, is contradictory iff some subconjunction of $\leqq h+1$ of its conjuncts is contradictory.

Let X be any prenex conjunctive formula in S_1^+ (in S_1^-) and let h be the largest number of terms that occur in any disjunction in X. By properties of the L^s-rules, if there is an L^s-deduction from X of a prenex formula Y, then Y is in S_1^+ (in S_1^-) and h is the largest number of terms that occur in any disjunction in Y. By Lemmas 1 and 2, X is not satisfiable iff there exists an L^s-deduction of order $\leqq h+1$ of a prenex formula with a truth functionally inconsistent matrix. But for any prenex formula X and any $h \in I$, one can easily determine whether there exists an

L^s-deduction of order $\leqq h+1$ from X of a prenex formula whose matrix is contradictory. One could make a finite list of L^s-deductions and be sure that any L^s-deduction of order $\leqq h+1$ from X would be essentially the same as a member of the list. An inessential difference in L^s-deductions as far as determining whether the matrix of the resulting formula is contradictory is in different choices of individual variables introduced in some applications of $\forall$-instantiation. Except for this inessential difference there is an easily obtained finite number of different sequences of applications of the given rules of inference starting from a given formula X and including no more than $h+1$ applications of duplication.

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