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FAREY TRIANGLES AND FAREY QUADRANGLES IN THE COMPLEX PLANE

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1. Introduction.

An important tool in the theory of approximations of real numbers is the well-known Farey fractions. An extensive treatment of the basic properties of Farey fractions was given by Hurwitz [5]. Later they were used successfully in solving both homogeneous and inhomogeneous approximation problems by Khintchine [6] and Niven [7]. In fact, as is stressed by Niven in his interesting monograph [8], Farey fractions can for many purposes replace the application of regular continued fractions.

Inspired by some remarks made by Hurwitz at the end of the paper [5] mentioned above, Cassels, Ledermann and Mahler [2] studied, as a generalization of the real case, so-called Farey sections in the complex plane for the two imaginary quadratic number fields $\mathsf{Q}(i)$ and $\mathsf{Q}(i\,3^{\frac{1}{2}})$.

In the present paper we shall consider a generalization of Farey fractions to the complex case along rather different lines. The theory of Hurwitz and Khintchine referred to above is in fact mainly concerned with what might be called Farey intervals, i.e. closed intervals $[p'/q',p''/q'']$ with $p',p'',q',q'' \in \mathsf{Z}$, $q',q''>0$ and $p''q'-p'q''=1$, and it turns out that the proper generalization of Farey interval — at least in the cases $\mathsf{Q}(im^{\frac{1}{2}})$, $m=1,2,3,7$ — are the notions of Farey triangle and Farey quadrangle as defined in section 2. The basic properties of Farey triangles and Farey quadrangles in $\mathsf{Q}(im^{\frac{1}{2}})$, $m=1,2,3,7$, are developped in sections 3–5.

In sections 6–8 we shall apply Farey triangles and Farey quadrangles in an investigation of the approximation spectra in the cases $\mathsf{Q}(im^{\frac{1}{2}})$, $m=1,2,3,7$. Here the approximation spectrum in case $\mathsf{Q}(im^{\frac{1}{2}})$ is the set of all approximation constants $C(\xi)$, where $C(\xi)$ for any $\xi \notin \mathsf{Q}(im^{\frac{1}{2}})$ is defined as

$$(1) \qquad C(\xi) \;=\; \limsup\,(|q|\;|q\xi-p|)^{-1}\,,$$

the lim sup being taken over all algebraic integers $p,q \in \mathsf{Q}(im^{\frac{1}{2}})$, $q \neq 0$.

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For $Q(im^{\frac{1}{2}})$, $m=1,2,3,7$, we shall find all approximation constants $C(\xi)<c_m$, where

$$c_1 = 1.80\ldots, \quad c_2 = 1.733, \quad c_3 = 1.90\ldots, \quad c_7 = 1.75 ,$$

the result being

$$3^{\frac{1}{2}}, \quad 2^{\frac{1}{2}} \text{ and } 3^{\frac{1}{2}}, \quad 13^{\frac{1}{4}}, \quad 8^{\frac{1}{4}} \text{ and } 3^{\frac{1}{2}} ,$$

respectively.

For each of the approximation constants, except $3^{\frac{1}{2}}$ in case $Q(i\,2^{\frac{1}{2}})$, the set of complex numbers having the respective approximation constant consists of one single equivalence class of complex numbers, ξ being equivalent to η in case $Q(im^{\frac{1}{2}})$, when

$$\xi = \frac{a\eta+b}{c\eta+d}, \tag{2}$$

where a, b, c, d are algebraic integers in $Q(im^{\frac{1}{2}})$ and $|ad-bc|=1$.

In the case $Q(i\,2^{\frac{1}{2}})$ the set $C^{-1}(3^{\frac{1}{2}})$ consists of two distinct equivalence classes of complex numbers.

Each of the seven equivalence classes of complex numbers involved above has a simple characterization in terms of Farey triangles and Farey quadrangles.

The approximation spectra in the cases $Q(im^{\frac{1}{2}})$, $m=1,2,3,7$, have been studied previously by several authors.

The first case to be considered was $Q(i)$, where the first minimum of the spectrum was found independently by Ford [3] and Perron [9] and was shown to be isolated by Cassels [1], however without any definite lower bound for the second minimum.

In the case $Q(i\,3^{\frac{1}{2}})$ the first minimum was found by Perron [10], the second and third minima, 2 and $(\frac{32}{13}\,3^{\frac{1}{2}})^{\frac{1}{2}}$ respectively, were found and shown to be isolated by Poitou [12], so in this case our result does not contain anything new, the proof, however, is extremely simple.

In the cases $Q(i\,2^{\frac{1}{2}})$ and $Q(i\,7^{\frac{1}{2}})$ the first minima were found by Perron [11] and Hofreiter [4], but no information of the approximation spectra beyond the first minima has been known so far. The determination of the second (isolated) minima in these two cases thus represents the major contribution to the theory of approximation spectra obtained in this paper.

Finally it should be mentioned that the first (isolated) minimum in the approximation spectrum of quaternions with Hurwitz' definition of integral quaternions can be found by means of Farey simplices in a way completely analogous to the case $Q(i)$ considered in this paper. A separate paper on the approximation of quaternions will appear later in this journal.

2. Farey triangles and Farey quadrangles. Unimodular homographic maps.

In the imaginary quadratic number field $\mathsf{Q}(im^{\frac{1}{2}})$, m being a squarefree positive integer, $\mathsf{Z}(im^{\frac{1}{2}})$ denotes the ring of algebraic integers, i.e.

$$\mathsf{Z}(im^{\frac{1}{2}}) = \{a+b\omega \mid a,b \in \mathsf{Z}\}\,, \tag{3}$$

where

$$\omega = \begin{cases} im^{\frac{1}{2}}, & m \equiv 1,2 \pmod 4\,, \\ \frac{1}{2}(1+im^{\frac{1}{2}}), & m \equiv 3 \pmod 4\,. \end{cases} \tag{4}$$

Before introducing the fundamental notions of Farey triangle and Farey quadrangle, it will be convenient to consider the related concepts of Farey matrices.

DEFINITION 1. *A* 2×3 *matrix*

$$\begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{pmatrix} \tag{5}$$

is called a Farey matrix in $\mathsf{Q}(im^{\frac{1}{2}})$, *if* $p_j,q_j \in \mathsf{Z}(im^{\frac{1}{2}})$, $1 \leqq j \leqq 3$, *and*

$$|p_1q_2-p_2q_1| = |p_1q_3-p_3q_1| = |p_2q_3-p_3q_2| = 1\,. \tag{6}$$

A 2×4 *matrix*

$$\begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ q_1 & q_2 & q_3 & q_4 \end{pmatrix} \tag{7}$$

is called a Farey matrix in $\mathsf{Q}(im^{\frac{1}{2}})$, *if* $p_j,q_j \in \mathsf{Z}(im^{\frac{1}{2}})$, $1 \leqq j \leqq 4$,

$$\begin{aligned} |p_1q_2-p_2q_1| &= |p_2q_3-p_3q_2| \\ &= |p_3q_4-p_4q_3| = |p_4q_1-p_1q_4| = 1\,, \end{aligned} \tag{8}$$

and if

$$\Delta_1 = |p_1q_3-p_3q_1|, \qquad \Delta_2 = |p_2q_4-p_4q_2| \tag{9}$$

satisfy the requirements

$$\Delta_j > 0, \qquad j=1,2\,. \tag{10}$$

The Farey matrix (7) *is said to be of type* (Δ_1,Δ_2), *and is called Ptolemaic, if*

$$\Delta_1\Delta_2 = 2\,. \tag{11}$$

It should be noticed, that any permutation of the rows and columns in a 2×3 Farey matrix leaves it a Farey matrix. On the other hand the permutation of the rows but in general only those permutations of the columns of a 2×4 Farey matrix, that keep or reverse the cyclic order, will leave it a Farey matrix (of the same or the reversed type).

Similarly multiplication of each row and each column in a Farey matrix by a unit in $\mathsf{Z}(im^{\frac{1}{2}})$ leaves it a Farey matrix (of the same type).

DEFINITION 2. *Two 2×3 Farey matrices (2×4 Farey matrices) in $\mathsf{Q}(im^{\frac{1}{2}})$ are called associated, if one is obtained from the other by a permutation of the columns keeping or reversing the cyclic order together with a multiplication of each column by a unit in $\mathsf{Z}(im^{\frac{1}{2}})$.*

It is easy to see, that there are only finitely many different types of 2×4 Farey matrices. In fact, if for example $q_1=0$, then by (8), (9) and (10)

$$|p_1q_2| = |p_1q_4| = 1, \qquad \Delta_1 = |p_1q_3| > 0 ,$$

and since $p_j,q_j \in \mathsf{Z}(im^{\frac{1}{2}})$, $1\leqq j\leqq 4$, this means that

$$|p_1| = |q_2| = |q_4| = 1, \qquad \Delta_1 = |q_3| > 0 . \tag{12}$$

Hence $q_2,q_3,q_4 \neq 0$. Using (8), (9) and (12), the triangle inequality applied to the three points p_2/q_2, p_3/q_3, p_4/q_4 in the complex plane yields the inequality $\Delta_2 \leqq 2/|q_3|$, which together with (12) proves

$$\Delta_1\Delta_2 \leqq 2 . \tag{13}$$

However, the inequality (13) holds for any 2×4 Farey matrix, since it follows immediately from (8) and (9) using Ptolemy's inequality to the four points p_1/q_1, p_2/q_2, p_3/q_3, p_4/q_4 in the complex in case $q_j \neq 0$, $1\leqq j\leqq 4$. By (13) and

$$\Delta_j = (N(z_j))^{\frac{1}{2}}, \quad z_j \in \mathsf{Z}(im^{\frac{1}{2}}), \qquad j=1,2 , \tag{14}$$

where $N(z_j)=z_j\bar{z}_j=|z_j|^2$ denotes the norm of z_j in $\mathsf{Q}(im^{\frac{1}{2}})$, there is only a finite number of types (Δ_1,Δ_2) of 2×4 Farey matrices.

In order to give a precise description of the Farey matrices existent we shall make use of the group of unimodular linear maps

$$\Phi\colon \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \pi \\ \varrho \end{pmatrix} \tag{15}$$

of $\mathsf{Z}(im^{\frac{1}{2}})\times \mathsf{Z}(im^{\frac{1}{2}})$ onto itself, where $a,b,c,d \in \mathsf{Z}(im^{\frac{1}{2}})$ and

$$|ad-bc| = 1 . \tag{16}$$

If $\Phi((\pi,\varrho))=(p,q)$, $\Phi((\pi',\varrho'))=(p',q')$, then by (15)

$$\begin{pmatrix} p & p' \\ q & q' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \pi & \pi' \\ \varrho & \varrho' \end{pmatrix} , \tag{17}$$

hence by (16)

$$(18) \qquad |pq'-p'q| = |\pi\varrho'-\pi'\varrho| .$$

An immediate consequence of (18) is that a unimodular linear map of $\mathsf{Z}(im^{\frac{1}{2}}) \times \mathsf{Z}(im^{\frac{1}{2}})$ onto itself in a natural way maps a 2×3 Farey matrix onto a 2×3 Farey matrix and a 2×4 Farey matrix onto a 2×4 Farey matrix of the same type.

Conversely we have the following important result:

THEOREM 1. *Any 2×3 Farey matrix in $\mathsf{Q}(im^{\frac{1}{2}})$ is associated with a 2×3 Farey matrix of the form*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a unimodular matrix over $\mathsf{Z}(im^{\frac{1}{2}})$.

Any 2×4 Farey matrix in $\mathsf{Q}(im^{\frac{1}{2}})$ is associated with a 2×4 Farey matrix of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 & \pi_3 & \pi_4 \\ 0 & 1 & \varrho_3 & \varrho_4 \end{pmatrix},$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a unimodular matrix over $\mathsf{Z}(im^{\frac{1}{2}})$, and where

$$\begin{pmatrix} \pi_3 & \pi_4 \\ \varrho_3 & \varrho_4 \end{pmatrix}$$

equals one of the following matrices:

$$\begin{pmatrix} 1 & \omega \\ 1 & 1 \end{pmatrix} \textit{ or } \begin{pmatrix} 1 & \bar\omega \\ 1 & 1 \end{pmatrix}, \quad (\Delta_1,\Delta_2) = (1,1), \quad m=3,$$

$$\begin{pmatrix} 1 & 1+i \\ 1 & 1 \end{pmatrix} \textit{ or } \begin{pmatrix} 1 & 1-i \\ 1 & 1 \end{pmatrix}, \quad (\Delta_1,\Delta_2) = (1,2^{\frac{1}{2}}), \quad m=1,$$

$$\begin{pmatrix} 1 & 1+\omega \\ 1 & 1 \end{pmatrix} \textit{ or } \begin{pmatrix} 1 & 1+\bar\omega \\ 1 & 1 \end{pmatrix}, \quad (\Delta_1,\Delta_2) = (1,3^{\frac{1}{2}}), \quad m=3,$$

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad (\Delta_1,\Delta_2) = (1,2), \quad \textit{all } m,$$

$$\begin{pmatrix} 1 & 1+i \\ 1-i & 1 \end{pmatrix}, \quad (\Delta_1,\Delta_2) = (2^{\frac{1}{2}},2^{\frac{1}{2}}), \quad m=1,$$

$$\begin{pmatrix} 1 & \omega \\ \bar\omega & 1 \end{pmatrix}, \quad (\Delta_1,\Delta_2) = (2^{\frac{1}{2}},2^{\frac{1}{2}}), \quad m=2,7.$$

COROLLARY. *The types* (Δ_1,Δ_2), $\Delta_1 \leqq \Delta_2$, *of* 2×4 *Farey matrices actually occurring, are the following*:

$$\text{(19)}\qquad \begin{cases} m = 1: & (1,2^{\frac12}),\ (1,2),\ (2^{\frac12},2^{\frac12})\,, \\ m = 2,7: & (1,2),\ (2^{\frac12},2^{\frac12})\,, \\ m = 3: & (1,1),\ (1,3^{\frac12}),\ (1,2)\,, \\ m \neq 1,2,3,7: & (1,2)\,. \end{cases}$$

PROOF. Let the 2×3 Farey matrix given have the form (5). Then

$$\begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix}^{-1} \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \pi \\ 0 & 1 & \varrho \end{pmatrix},$$

where by definition 1 and (18) $|\pi| = |\varrho| = 1$, that is, π,ϱ are units in $\mathsf{Z}(im^{\frac12})$, and hence the Farey matrix (5) is associated with

$$\begin{pmatrix} p_1\pi & p_2\varrho & p_3 \\ q_1\pi & q_2\varrho & q_3 \end{pmatrix} = \begin{pmatrix} p_1\pi & p_2\varrho \\ q_1\pi & q_2\varrho \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Let the 2×4 Farey matrix given have the form (7). Then

$$\text{(20)}\qquad \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix}^{-1} \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ q_1 & q_2 & q_3 & q_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \pi & \pi' \\ 0 & 1 & \varrho & \varrho' \end{pmatrix},$$

where by definition 1 and (18)

$$\text{(21)}\qquad |\pi| = |\varrho'| = 1, \quad |\varrho| = \Delta_1, \quad |\pi'| = \Delta_2, \quad |\pi\varrho' - \pi'\varrho| = 1\,.$$

Since any 2×4 Farey matrix is associated with a 2×4 Farey matrix of type (Δ_1,Δ_2), $\Delta_1 \leqq \Delta_2$, we may as well assume that (7) satisfies this condition.

First let $\Delta_1 = 1$, then by (20) and (21) the Farey matrix (7) is associated with

$$\begin{pmatrix} p_1\pi & p_2\varrho & p_3 & p_4\varrho\varrho'^{-1} \\ q_1\pi & q_2\varrho & q_3 & q_4\varrho\varrho'^{-1} \end{pmatrix} = \begin{pmatrix} p_1\pi & p_2\varrho \\ q_1\pi & q_2\varrho \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & \alpha \\ 0 & 1 & 1 & 1 \end{pmatrix},$$

where, by (18), $\alpha = \pi^{-1}\varrho\varrho'^{-1}\pi'$ satisfies the conditions $|\alpha| = \Delta_2$, $|\alpha - 1| = 1$, and since $\Delta_2 = \Delta_1\Delta_2 \leqq 2$ by (13), it follows that $\alpha = \pi_4$ in one of the first four cases listed in the theorem.

Secondly let $\Delta_1 = 2^{\frac12}$, then also $\Delta_2 = 2^{\frac12}$ by (13) and our assumption $\Delta_1 \leqq \Delta_2$, and hence, by (14), $m = 1$, 2 or 7.

$m = 1$. By (20) and (21) the Farey matrix (7) is associated with

$$\begin{pmatrix} p_1\pi & p_2\varrho(1-i)^{-1} & p_3 & p_4\varrho(1-i)^{-1}\varrho'^{-1} \\ q_1\pi & q_2\varrho(1-i)^{-1} & q_3 & q_4\varrho(1-i)^{-1}\varrho'^{-1} \end{pmatrix} = \begin{pmatrix} p_1\pi & p_2\varrho(1-i)^{-1} \\ q_1\pi & q_2\varrho(1-i)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & \beta \\ 0 & 1 & 1-i & 1 \end{pmatrix},$$

where, by (18), $\beta = \pi^{-1}\varrho(1-i)^{-1}\varrho'^{-1}\pi'$ satisfies the conditions $|\beta| = 2^{\frac12}$, $|(1-i)\beta - 1| = 1$, and hence $\beta = 1 + i$.

$m=2$. By (20) and (21) the Farey matrix (7) is associated with

$$(22)\qquad \begin{pmatrix} p_1\pi & p_2\varrho\bar{\omega}^{-1} & p_3 & p_4\varrho\bar{\omega}^{-1}\varrho'^{-1} \\ q_1\pi & q_2\varrho\bar{\omega}^{-1} & q_3 & q_4\varrho\bar{\omega}^{-1}\varrho'^{-1} \end{pmatrix} = \begin{pmatrix} p_1\pi & p_2\varrho\bar{\omega}^{-1} \\ q_1\pi & q_2\varrho\bar{\omega}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & \gamma \\ 0 & 1 & \bar{\omega} & 1 \end{pmatrix},$$

where, by (18), $\gamma=\pi^{-1}\varrho\bar{\omega}^{-1}\varrho'^{-1}\pi'$ satisfies the conditions $|\gamma|=2^{\frac{1}{2}}$, $|\bar{\omega}\gamma-1|=1$, and hence $\gamma=\omega$.

$m=7$. By (20) and (21) the Farey matrix (7) is associated with (22) or with

$$\begin{pmatrix} p_1\pi & p_2\varrho\omega^{-1} & p_3 & p_4\varrho\omega^{-1}\varrho'^{-1} \\ q_1\pi & q_2\varrho\omega^{-1} & q_3 & q_4\varrho\omega^{-1}\varrho'^{-1} \end{pmatrix} = \begin{pmatrix} p_1\pi & p_2\varrho\omega^{-1} \\ q_1\pi & q_2\varrho\omega^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & \delta \\ 0 & 1 & \omega & 1 \end{pmatrix}.$$

In the first case $\gamma=\pi^{-1}\varrho\bar{\omega}^{-1}\varrho'^{-1}\pi'$ satisfies the conditions $|\gamma|=2^{\frac{1}{2}}$, $|\bar{\omega}\gamma-1|=1$, and hence $\gamma=\omega$. In the second case $\delta=\pi^{-1}\varrho\omega^{-1}\varrho'^{-1}\pi'$ satisfies the conditions $|\delta|=2^{\frac{1}{2}}$, $|\omega\delta-1|=1$, and hence $\delta=\bar{\omega}$. However, since the Farey matrix

$$\begin{pmatrix} 1 & 0 & 1 & \bar{\omega} \\ 0 & 1 & \omega & 1 \end{pmatrix},$$

is associated with

$$\begin{pmatrix} \bar{\omega} & -1 & 0 & 1 \\ 1 & 0 & 1 & \omega \end{pmatrix} = \begin{pmatrix} \bar{\omega} & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & \omega \\ 0 & 1 & \bar{\omega} & 1 \end{pmatrix},$$

the result indicated in the theorem follows in both cases.

This completes the proof of theorem 1.

The corollary, which follows immediately from theorem 4, shows the interesting fact, that non-Ptolemaic 2×4 Farey matrices occur only for $m=1,3$.

THEOREM 2. *In a Ptolemaic 2×4 Farey matrix the following norm relations hold*

$$(23)\qquad \Delta_1^2\{N(q_2)+N(q_4)\} = 2\{N(q_1)+N(q_3)\},$$

$$(24)\qquad \Delta_2^2\{N(q_1)+N(q_3)\} = 2\{N(q_2)+N(q_4)\}.$$

PROOF. Of course, by the defining relation $\Delta_1\Delta_2=2$ of a Ptolemaic 2×4 Farey matrix the two relations (23) and (24) are equivalent, so we need only prove one of these relations.

Suppose first that for example $N(q_1)=0$, then by (12)

$$N(q_2)+N(q_4) = 2, \qquad \Delta_1^2 = N(q_3),$$

which proves (23).

Suppose next that $q_j \neq 0$, $1\leqq j\leqq 4$, then the quadrangle in the complex plane with vertices p_1/q_1, p_2/q_2, p_3/q_3, p_4/q_4 is convex and inscribable in a circle (or equals a line segment) by Ptolemy's theorem, which is applicable since $\Delta_1\Delta_2 = 2$. Hence we get the following two expressions for $2\cos A_2$, where A_2 is the angle in this quadrangle at p_2/q_2:

$$\frac{\Delta_1^2 N(q_4)}{|q_1q_3|} - \left(\left|\frac{q_1}{q_3}\right| + \left|\frac{q_3}{q_1}\right|\right) = \left(\left|\frac{q_1}{q_3}\right| + \left|\frac{q_3}{q_1}\right|\right) - \frac{\Delta_1^2 N(q_2)}{|q_1q_3|},$$

whence (23).

DEFINITION 3. *A Farey triangle* FT$(p_1/q_1, p_2/q_2, p_3/q_3)$ *in the complex plane in the case* $\mathrm{Q}(im^{\frac{1}{2}})$ *is the convex hull of three points* p_j/q_j, $q_j \neq 0$, $1 \leqq j \leqq 3$, *such that the corresponding matrix* (5) *is a* 2×3 *Farey matrix in* $\mathrm{Q}(im^{\frac{1}{2}})$.

A Farey quadrangle FQ$(p_1/q_1, p_2/q_2, p_3/q_3, p_4/q_4)$ *in the complex plane in the case* $\mathrm{Q}(im^{\frac{1}{2}})$ *is the convex hull of four points* p_j/q_j, $q_j \neq 0$, $1 \leqq j \leqq 4$, *such that the corresponding matrix* (7) *is a* 2×4 *Farey matrix in* $\mathrm{Q}(im^{\frac{1}{2}})$, *and such that the polygonal line* p_1/q_1, p_2/q_2, p_3/q_3, p_4/q_4, p_1/q_1 *is the frontier of the convex hull (counted twice when the convex hull degenerates to a line segment).*

By definitions 1, 2 and 3 a FT corresponds to a number of classes of associated 2×3 Farey matrices having no zero in the second row, and conversely every class of associated 2×3 Farey matrices having no zero in the second row defines a FT.

Similarly a FQ corresponds to a number of classes of associated 2×4 Farey matrices having no zero in the second row. The converse holds only partially in this case due to the convexity property posed on a FQ (for approximation reasons that become clear in sections 6–8). However, it should be noticed that every class of associated Ptolemaic 2×4 Farey

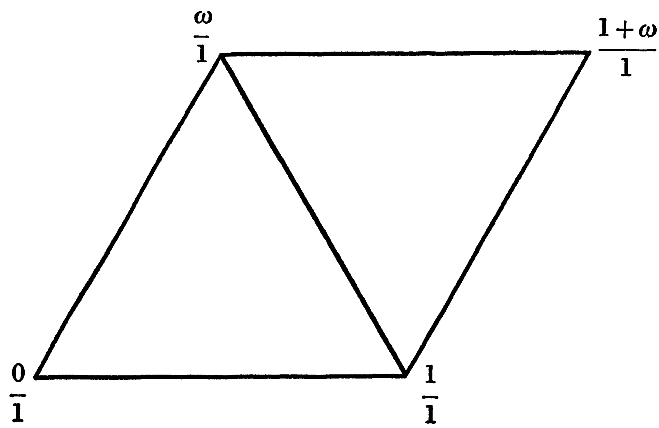

Fig. 1.

matrices having no zero in the second row defines a FQ, since the convexity property follows automatically in the Ptolemaic case.

We shall now point out, as was mentioned already in the introduction, why the cases $m=1,2,3,7$ are of particular importance.

In figs. 1–2 the fundamental parallelograms in the complex plane spanned by 1 and ω are subdivided into 4 FT's in the case $m=1$ and into 2 FT's in the case $m=3$, and hence the whole complex plane can be subdivided into FT's in these two cases.

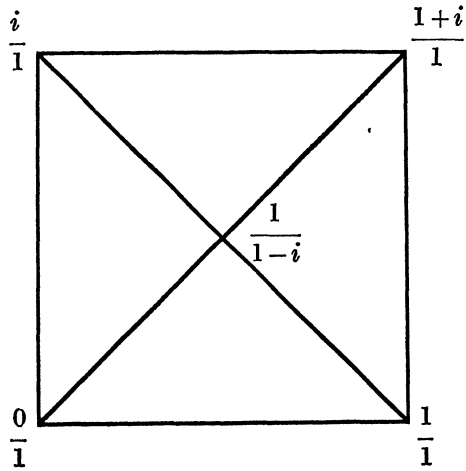

Fig. 2.

Similarly in figs. 3–4 the fundamental parallelograms are subdivided into 6 FT's and 4 FQ's of type $(2^{\frac12}, 2^{\frac12})$ in the case $m=2$ and into 2 FT's and 2 FQ's of type $(2^{\frac12}, 2^{\frac12})$ in the case $m=7$, and hence the whole complex plane can be subdivided into FT's and FQ's of type $(2^{\frac12}, 2^{\frac12})$ in these two cases.

By theorem 1, any FT corresponds to a 2×3 Farey matrix of the form

$$\begin{pmatrix} p_1 & p_2 & p_1+p_2 \\ q_1 & q_2 & q_1+q_2 \end{pmatrix}.$$

From definition 1, p_1/q_1, p_2/q_2, $(p_1+p_2)/(q_1+q_2)$ are in lowest terms (even in the case, where $\mathsf{Z}(im^{\frac12})$ is no unique factorization ring). Now the representation in lowest terms of a number in Q as a fraction p/q with $p,q \in \mathsf{Z}(im^{\frac12})$ is unique apart from multiplying the numerator and the denominator by the same unit in $\mathsf{Z}(im^{\frac12})$, especially if $m \neq 1,3$, the only units in $\mathsf{Z}(im^{\frac12})$ being ± 1, a representation in lowest terms must have $p,q \in \mathsf{Z}$.

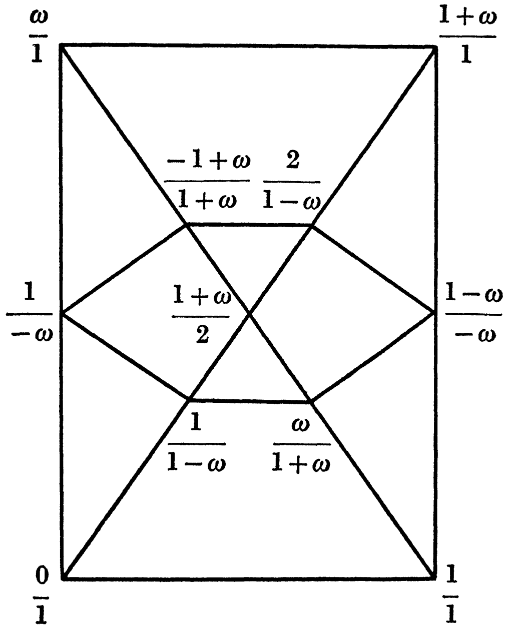

Fig. 3.

It follows from these remarks that every FT in case $Q(im^{1/2})$, $m \neq 1,3$, having two vertices on the real axis degenerates to a line segment. Consequently the fundamental parallelograms cannot be subdivided into FT's for $m \neq 1,3$.

Further, for $m \neq 1,2,3,7$ every FQ is af type $(1,2)$ by the corollary of theorem 1 and hence subdivisible into two FT's by drawing the 1-diagonal. Consequently the fundamental parallelograms cannot be subdivided into FT's and FQ's for $m \neq 1,2,3,7$, since there are no subdivisions into FT's.

It might be worth-while finding a substitute of FT's and FQ's for $m \neq 1,2,3,7$, but we shall not make any attempt in this direction here.

The unimodular homographic map

$$\varphi\colon\ w = (az+b)(cz+d)^{-1} \tag{25}$$

corresponding to the unimodular linear map Φ defined in (15), is a 1–1 map of the extended complex z-plane onto the extended complex w-plane.

Also, since $\varphi(\pi/\varrho) = p/q$ when $\pi, \varrho, p, q \in Z(im^{1/2})$ are related by (15), φ is a 1–1 map of the set of irreducible fractions π/ϱ with $\pi, \varrho \in Z(im^{1/2})$ onto itself. ($\varepsilon/0$, $|\varepsilon| = 1$ is counted among the irreducible fractions.)

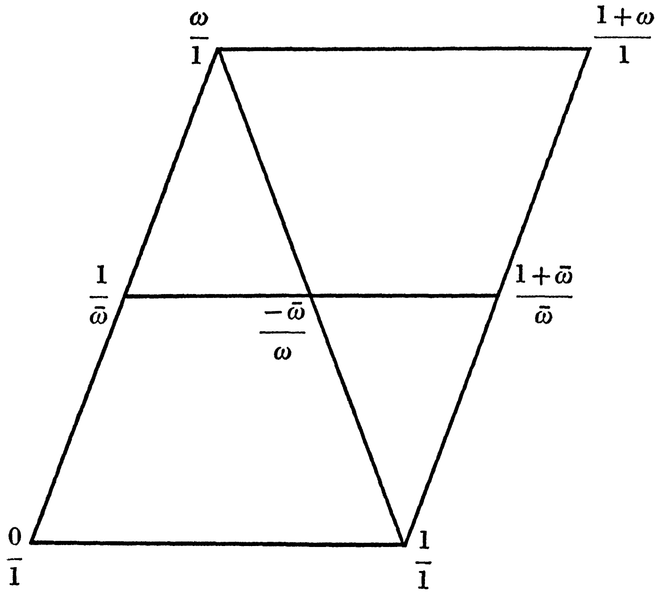

Fig. 4.

Further, if $\varphi(1/0)=a/c$, $c\neq 0$ and $\varphi(\pi/\varrho)=p/q$, $q\neq 0$, the following relations follow easily from (15), (16) and (25):

$$|q| = |c|\,|\varrho|\,|\pi/\varrho+d/c|\,, \tag{26}$$

$$|qw-p| = |\varrho z-\pi|/|cz+d|\,, \tag{27}$$

$$|cw-a| = 1/|cz+d|\,. \tag{28}$$

These properties of a unimodular homographic map together with the well-known properties of being conformal and mapping circles or straight lines onto circles or straight lines allows one to investigate FT's and FQ's by means of their inverse images by suitably chosen unimodular homographic maps φ, e.g. obtained in letting the corresponding Farey matrices be of the form indicated in theorem 1.

3. Fundamental properties of Farey triangles and Farey quadrangles in $Q(im^{\frac{1}{2}})$, $m=1,2,3,7$.

In the first theorem to be considered we shall make use of the fact that $Z(im^{\frac{1}{2}})$ is a principal ideal ring for $m=1,2,3,7$.

THEOREM 3. *Let ξ be any complex number satisfying the inequality*

$$\left|\xi-\frac{p_0}{q_0}\right| \leq \frac{1}{a_m|q_0|^2}, \tag{29}$$

where $p_0, q_0 \in \mathsf{Z}(im^{\frac{1}{2}})$, $q_0 \neq 0$, $m = 1, 2, 3$ *or* 7, *and* p_0/q_0 *is irreducible, the constants* a_m *being*

$$a_1 = 2^{\frac{1}{2}}, \quad a_2 = 3^{\frac{1}{2}}, \quad a_3 = 2/3^{\frac{1}{2}}, \quad a_7 = 4/7^{\frac{1}{2}} . \tag{30}$$

Then ξ *belongs to*

1) *a Farey triangle in* $\mathsf{Q}(im^{\frac{1}{2}})$ *having* p_0/q_0 *as one of its vertices in case* $m = 1$ *or* 3,

2) *a Farey triangle or a Farey quadrangle of type* $(2^{\frac{1}{2}}, 2^{\frac{1}{2}})$ *in* $\mathsf{Q}(im^{\frac{1}{2}})$ *having* p_0/q_0 *as one of its vertices in case* $m = 2$ *or* 7.

The constants a_m, $m = 1$, 2, 3, *or* 7, *in* (30) *are smallest possible.*

REMARK. The corresponding theorem with $\mathsf{Z}(im^{\frac{1}{2}})$, a_m and Farey triangle or Farey quadrangle replaced by Z, 1 and Farey interval, respectively, was proved by Hurwitz [5].

Since the proofs of the four cases $m = 1, 2, 3, 7$ are quite similar, we shall give only the detailed proof for $m = 1$. In this case we shall need the following

LEMMA 1. *Let* Γ *be a circle in the complex plane with radius* $2^{\frac{1}{2}}$ *and an arbitrary centre* O.

Then one can select 4, 6 *or* 8 *points from* $\mathsf{Z}(i)$, *all of them different from* O *and lying inside or on the boundary of* Γ, *such that the selected points*

$$z_1 = z_{n+1}, z_2, \ldots, z_n, \qquad n = 4, 6 \text{ or } 8 ,$$

(*with a suitable notation*) *satisfy the following conditions*:

(i) $|z_{j+1} - z_j| = 1$, $1 \leqq j \leqq n$,

(ii) *the polygon* $z_1 z_2 \ldots z_n z_1$ *is a rectangle with* O *as an interior point,*

(iii) *the circles through* O, z_j, z_{j+1}, $1 \leqq j \leqq n$, *are inside or on the boundary of* Γ.

PROOF. Condition (iii) means that the circles through O, z_j, z_{j+1} have diameters $d_j \leqq 2^{\frac{1}{2}}$. Let α_j, $0 < \alpha_j < \pi$, be the angle $z_j O z_{j+1}$. Then

$$d_j = |z_{j+1} - z_j| / \sin \alpha_j ,$$

hence $d_j = 1/\sin \alpha_j$ if condition (i) is satisfied. Thus, in fact condition (iii) amounts to

$$\tfrac{1}{4}\pi \leqq \alpha_j \leqq \tfrac{3}{4}\pi, \qquad 1 \leqq j \leqq n .$$

Now we distinguish between three cases according to the position of O in the lattice $\mathsf{Z}(i)$:

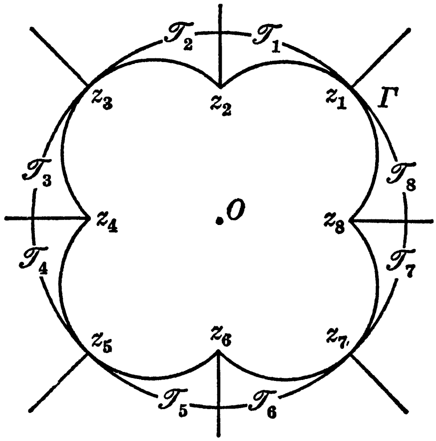

Fig. 5a.

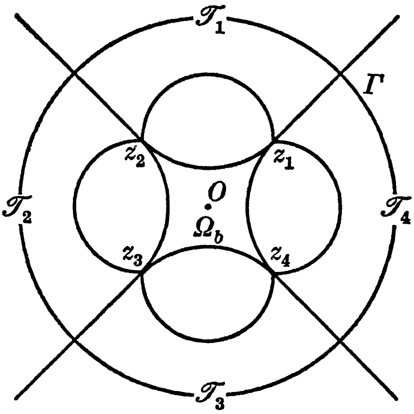

Fig. 5b.

a) $O \in \mathsf{Z}(i)$, hence we have the situation in fig. 5a. The 8 points $z_1, z_2, \ldots, z_8$ indicated in this figure obviously satisfy the conditions (i) and (ii), but also (iii) since $\alpha_j = \frac{1}{4}\pi$, $1 \leqq j \leqq 8$.

b) Let $O \in \Omega_b$ as shown in fig. 5b, where Ω_b is bounded by circular arcs with radius $1/2^{\frac{1}{2}}$. The 4 points z_1, z_2, z_3, z_4 indicated in this figure obviously satisfy the conditions (i) and (ii), but also (iii) since $\frac{1}{4}\pi \leqq \alpha_j \leqq \frac{3}{4}\pi$, $1 \leqq j \leqq 4$.

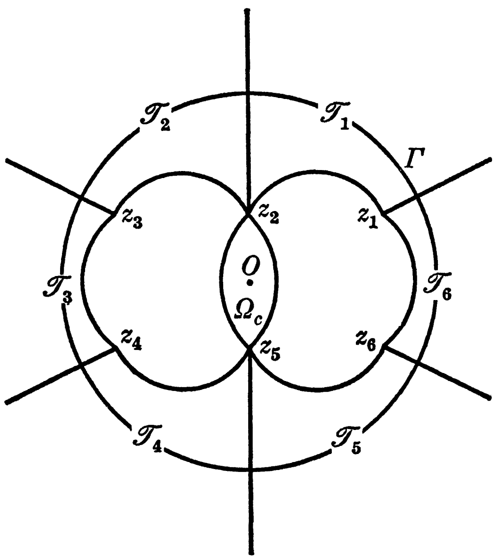

Fig. 5c.

c) Let $O \in \Omega_c$ as shown in fig. 5c, where Ω_c is bounded by circular arcs with radius $1/2^{\frac{1}{2}}$. The 6 points $z_1, z_2, \ldots, z_6$ indicated in this figure obviously satisfy the condition (i) and (ii), but also (iii) since $\frac{1}{4}\pi \leqq \alpha_j \leqq \frac{3}{4}\pi$, $1 \leqq j \leqq 6$.

For reasons of symmetry this proves lemma 1 for any position of O in the lattice $\mathsf{Z}(i)$.

Proof of theorem 3, $m = 1$. In this case the theorem states that the closed disc bounded by the circle C with centre at p_0/q_0 and radius $1/(2^{\frac{1}{2}}|q_0|^2)$ is covered by the set of all Farey triangles having p_0/q_0 as a vertex. In fact, it will be shown that 4, 6 or 8 such Farey triangles will suffice to cover the disc bounded by C (cf. figs. 6a, 6b, 6c).

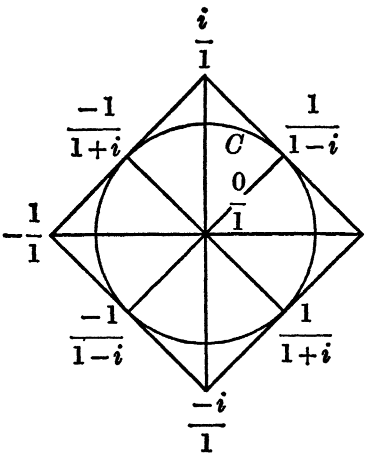

Fig. 6a.

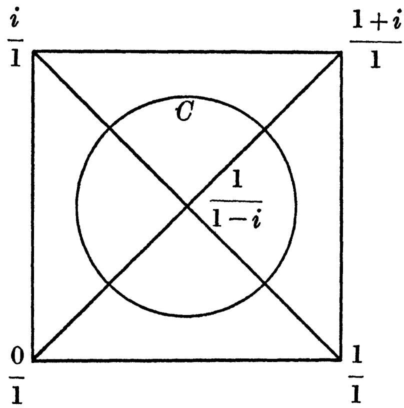

Fig. 6b.

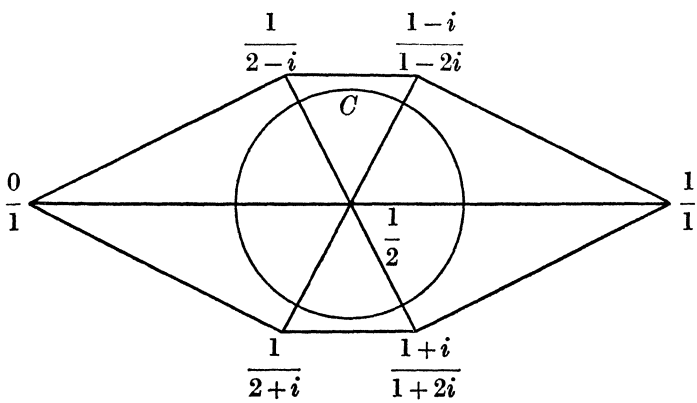

Fig. 6c.

Since $p_0, q_0 \in \mathsf{Z}(i)$ are relatively prime and $\mathsf{Z}(i)$ is a principal ideal ring, there are $P, Q \in \mathsf{Z}(i)$ such that

$$p_0 Q - P q_0 = 1 , \tag{31}$$

and we consider now the unimodular homographic map

$$\varphi \colon w = (p_0 z + P)(q_0 z + Q)^{-1} .$$

By (28)

$$|q_0 w - p_0| = 1/|q_0 z + Q| , \tag{32}$$

and hence

$$\varphi^{-1}(C) = \Gamma ,$$

where Γ is the circle in the z-plane with radius $2^{\frac{1}{2}}$ and centre at $-Q/q_0$. Also by (32) the interior of C corresponds to the exterior of Γ.

By lemma 1, the n closed regions $\mathcal{T}_j$, $1 \leqq j \leqq n$ ($n = 4$, 6 or 8) indicated in figs. 5a, 5b, 5c, cover the part of the complex z-plane outside or on Γ. Hence the n triangles $T_j = \varphi(\mathcal{T}_j)$, $1 \leqq j \leqq n$, cover the closed disc in the w-plane bounded by C.

Finally since

$$\Phi(1,0) = (p_0, q_0), \quad \Phi(z_j, 1) = (p_j, q_j), \ 1 \leqq j \leqq n ,$$

where Φ is the unimodular linear map corresponding to the unimodular homographic map φ, it follows from condition (i) of lemma 1 and (18) that the triangles $T_j = T_j(p_0/q_0, p_j/q_j, p_{j+1}/q_{j+1})$, $1 \leqq j \leqq n$, are Farey triangles.

That the constant $a_1 = 2^{\frac{1}{2}}$ is smallest possible is illustrated by fig. 6a.

In view of theorem 3 it is important to describe of the set of FT's and FQ's of type $(2^{\frac{1}{2}}, 2^{\frac{1}{2}})$ containing a fixed complex number ξ. Incidentally we know that this set is non-empty, since the subdivisions in figs. 1–4 generate tessellations of the whole complex plane into FT's, $m = 1, 3$, and into FT's and FQ's of type $(2^{\frac{1}{2}}, 2^{\frac{1}{2}})$, $m = 2, 7$. The central idea in our description is that of *subdivision* of a given FT (FQ) into a finite number of FT's (FT's + FQ's). Here subdivision is to be taken in a combinatorial sense rather than a geometric one. In fact, the subdivisions of a FT or FQ we are going to consider give in general only a covering of the given FT or FQ.

THEOREM 4. *Every Farey triangle in* $\mathsf{Q}(im^{\frac{1}{2}})$, *except the* FT's *in the tessellations mentioned above, is in two different ways subdivisible into* FT's *and* FQ's *as follows:*

$m = 1$: 7 FT's (*cf. fig.* 8),
$m = 2$: 7 FT's + 6 FQ's *of type* $(2^{\frac{1}{2}}, 2^{\frac{1}{2}})$ (*cf. fig.* 10),

$m=3$: 3 FT's (*cf. fig.* 12),
$m=7$: 1 FT + 3 FQ's *of type* $(2^{\frac{1}{2}}, 2^{\frac{1}{2}})$ (*cf. fig.* 13).

Every Farey quadrangle of type $(2^{\frac{1}{2}}, 2^{\frac{1}{2}})$ *in* $Q(im^{\frac{1}{2}})$, *except the* FQ's *in the tessellations mentioned above, is in two different ways subdivisible into* FT's *and* FQ's *as follows*:

$m=1$: 4 FT's (*cf. fig.* 9),
$m=2$: 8 FT's + 5 FQ's *of type* $(2^{\frac{1}{2}}, 2^{\frac{1}{2}})$ (*cf. fig.* 11),
$m=7$: 2 FT's + 2 FQ's *of type* $(2^{\frac{1}{2}}, 2^{\frac{1}{2}})$ (*cf. figs.* 14a *and* 14b).

The vertices of each subdivision all lie on one side of the circumscribed circle (line) of FT *or* FQ. *Further the vertices of the two different subdivisions of a* FT *or* FQ *of type* $(2^{\frac{1}{2}}, 2^{\frac{1}{2}})$ *lie on either side of the circumscribed circle (line) of* FT *or* FQ, *in fact they are inverse (symmetric) with respect to this circle (line) in all cases except the last one.*

For the exceptional FT's *and* FQ's *of type* $(2^{\frac{1}{2}}, 2^{\frac{1}{2}})$ *occurring in the tessellations of the complex plane generated by the subdivisions in figs.* 1–4 *there is only one subdivision of the kind described above. The vertices of this subdivision all lie inside the circumscribed circle of the* FT *or* FQ.

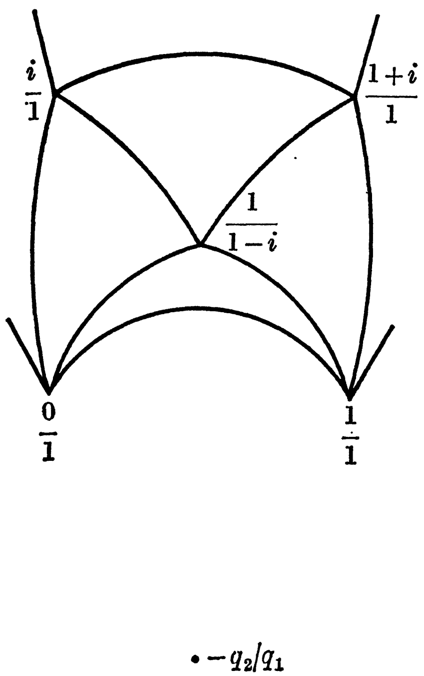

Fig. 7.

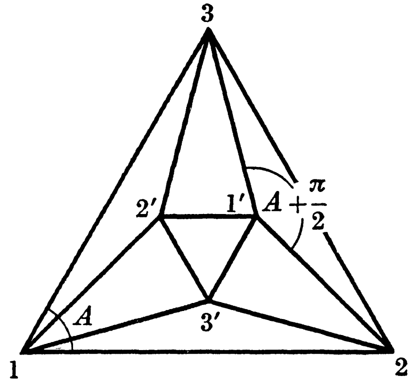

Fig. 8.

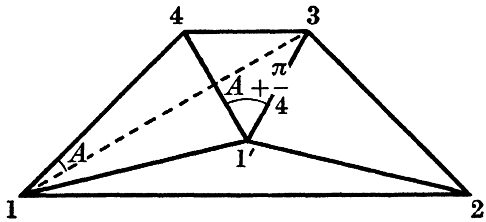

Fig. 9.

PROOF. The proofs of the seven cases in theorem 4 are all based on the subdivisions of the fundamental parallelograms in figs. 1–4, and since the proofs are quite similar, we shall consider only the subdivisions of a FT in the case $m=1$.

By theorem 1 we may suppose that the Farey triangle is of the form $\mathrm{FT}(p_1/q_1, p_2/q_2, p_3/q_3)$ with

$$\begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{pmatrix} = \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

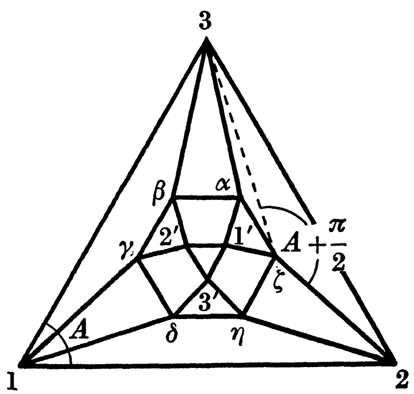

Fig. 10.

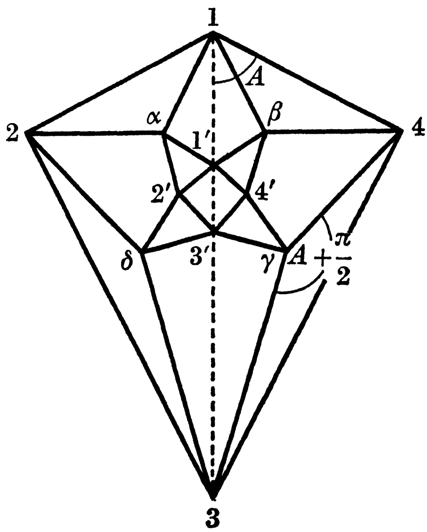

Fig. 11.

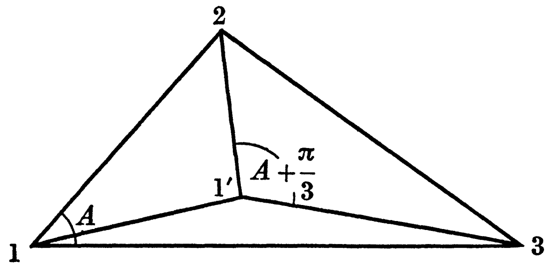

Fig. 12.

Using the properties of the unimodular homographic map

$$\varphi:\ w = (p_1z+p_2)(q_1z+q_2)^{-1}$$

and the corresponding unimodular linear map Φ deduced in section 2, the proof follows from fig. 7, which shows the image of the inner subdivision of the Farey triangle by the map φ^{-1} in case $\operatorname{Im}(-q_2/q_1) < 0$.

Note that by theorem 4 we may distinguish between inner and outer subdivision of a non-degenerate FT or FQ of type $(2^{\frac{1}{2}}, 2^{\frac{1}{2}})$, however in the degenerate case we shall consider any of the two subdivisions as being both inner and outer subdivisions. In figs. 8–14 only the inner subdivisions are shown.

DEFINITION 4. *A chain of* FT's *and* FQ's *is an infinite sequence*

$$\mathrm{FP}^{(0)}, \mathrm{FP}^{(1)}, \ldots, \mathrm{FP}^{(n)}, \ldots \tag{33}$$

of different FT's *and* FQ's *of type* $(2^{\frac{1}{2}}, 2^{\frac{1}{2}})$, *such that*

(i) $\mathrm{FP}^{(n+1)}$ *is one of the* FT's *or* FQ's *in the inner subdivision of* $\mathrm{FP}^{(n)}$, $n \geqq 0$.

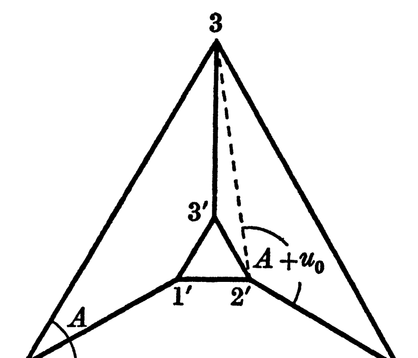

Fig. 13.

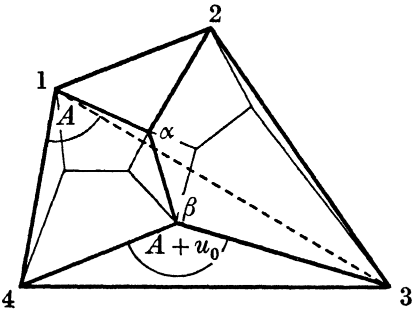

Fig. 14a.

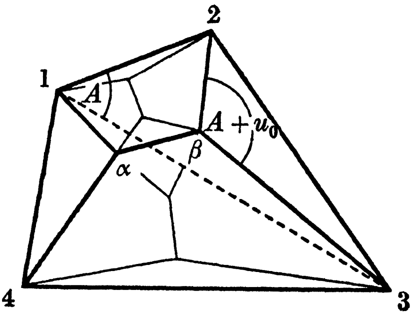

Fig. 14b.

If in addition for a fixed complex number ξ

(ii) $\xi \in \mathrm{FP}^{(n)}$ *for all* $n \geqq 0$,

we say that (33) *is a chain of* FT's *and* FQ's *containing* ξ.

THEOREM 5. *Every complex number* ξ *is contained in a chain of* FT's *in cases* $\mathrm{Q}(im^{1/2})$, $m=1,3$.

Every complex number ξ *is contained in a chain of* FT's *and* FQ's *in cases* $\mathrm{Q}(im^{1/2})$, $m=2,7$.

For any chain (33) *containing a complex number* ξ

$$N^{(n)} = N(\mathrm{FP}^{(n)}) \to \infty \quad \text{as } n \to \infty ,$$

where

$$N^{(n)} = N(\mathrm{FT}^{(n)}) = \sum_{j=1}^{3} N(q_j^{(n)}) \quad \text{or} \quad N^{(n)} = N(\mathrm{FQ}^{(n)}) = \sum_{j=1}^{4} N(q_j^{(n)})$$

according as $\mathrm{FP}^{(n)}$ *is a* FT *or a* FQ.

For any chain (33) *containing a complex number* ξ,

$$\lim_{n\to\infty} p_j^{(n)}/q_j^{(n)} = \xi, \quad j=1,2,3,(4) .$$

PROOF. For the existence proofs we shall restrict our attention to the case $m=1$, in which case we shall prove, by induction, the existence of a chain $\mathrm{FT}^{(n)}$, $n \geqq 0$, containing ξ and having the additional property

$$(34) \qquad N(\mathrm{FT}^{(n+1)}) > N(\mathrm{FT}^{(n)}) \quad \text{for } n \geqq 0 .$$

Indeed, as $\mathrm{FT}^{(0)}$ we may take one of the FT's in the tessellation of the complex plane generated by the subdivision in fig. 2. For the inductive step assume without restriction that

$$N(q_1^{(n)}) \geqq N(q_2^{(n)}) \geqq N(q_3^{(n)}) .$$

Then from fig. 7, fig. 8 and (26), using a map φ as before,

$$N(q_1'^{(n)}) \geqq \max\{N(q_2^{(n)}), N(q_3^{(n)})\} ,$$
$$N(q_j'^{(n)}) > \max\{N(q_1^{(n)}), N(q_2^{(n)}), N(q_3^{(n)})\}, \quad j=2,3 ,$$

and

$$N(q_1'^{(n)}) > N(q_1^{(n)}) \iff A < \tfrac{3}{4}\pi .$$

Further by fig. 7 and fig. 8

$$p_1'^{(n)}/q_1'^{(n)} \in \mathrm{FT}^{(n)} \iff A \leqq \tfrac{2}{3}\pi .$$

It follows from these inequalities that all 7 FT's in the inner subdivision of $\mathrm{FT}^{(n)}$ have $N(\mathrm{FT}) > N(\mathrm{FT}^{(n)})$, except for

$$\mathrm{FT} = \mathrm{FT}(p_1'^{(n)}/q_1'^{(n)},\ p_2^{(n)}/q_2^{(n)},\ p_3^{(n)}/q_3^{(n)})$$

in case $A \geqq \frac{3}{4}\pi$. However, if $A \geqq \frac{3}{4}\pi > \frac{2}{3}\pi$, then $p_1'^{(n)}/q_1'^{(n)} \notin \mathrm{FT}^{(n)}$, and consequently in any case (cf. fig. 8) a $\mathrm{FT} = \mathrm{FT}^{(n+1)}$ belonging to the inner subdivision of $\mathrm{FT}^{(n)}$, satisfying $N(\mathrm{FT}) > N(\mathrm{FT}^{(n)})$ and $\xi \in \mathrm{FT}$, may be found.

Obviously, for any positive integer n there are only finitely many FP's having $N(\mathrm{FP}) \leqq n$ and $\xi \in \mathrm{FP}$, and hence, the FP's in a chain (33) being different by definition, $N(\mathrm{FP}^{(n)}) \to \infty$ for $n \to \infty$.

Finally, a simple calculation shows that $N(\mathrm{FP}^{(n)}) \to \infty$ implies that $\mathrm{diam}(\mathrm{FP}^{(n)}) \to 0$, thus proving the last assertion in theorem 5.

4. Linear norm relations.

In this section we shall continue the investigation of the subdivisions of Farey triangles and Farey quadrangles of type $(2^{\frac{1}{2}}, 2^{\frac{1}{2}})$ in $\mathsf{Q}(im^{\frac{1}{2}})$, $m = 1,2,3,7$, described in the preceding paragraph. The linear norm relations connected with these subdivisions will be deduced by means of the following

LEMMA 2. *Let*

$$\Phi(\pi_j, \varrho_j) = (p_j, q_j), \qquad 1 \leqq j \leqq n ,$$

where Φ is a unimodular linear map of the form (15). *Suppose the following linear relations hold*

$$\sum_{j=1}^{n} b_j N(\pi_j) = \sum_{j=1}^{n} b_j N(\varrho_j) = \sum_{j=1}^{n} b_j \pi_j \bar{\varrho}_j = 0, \qquad b_j \in \mathsf{R},\ 1 \leqq j \leqq n .$$

Then the corresponding linear relations

$$\sum_{j=1}^{n} b_j N(p_j) = \sum_{j=1}^{n} b_j N(q_j) = \sum_{j=1}^{n} b_j p_j \bar{q}_j = 0$$

are also valid.

PROOF. It follows from the assumptions of the lemma that

$$\begin{pmatrix}\sum b_j N(p_j) & \sum b_j p_j \bar{q}_j \\ \sum b_j q_j \bar{p}_j & \sum b_j N(q_j)\end{pmatrix} = \begin{pmatrix} b_1 p_1 \dots b_n p_n \\ b_1 q_1 \dots b_n q_n \end{pmatrix} \begin{pmatrix} \bar{p}_1 & \bar{q}_1 \\ \vdots & \vdots \\ \bar{p}_n & \bar{q}_n \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} b_1 \pi_1 \dots b_n \pi_n \\ b_1 \varrho_1 \dots b_n \varrho_n \end{pmatrix} \begin{pmatrix} \bar{\pi}_1 & \bar{\varrho}_1 \\ \vdots & \vdots \\ \bar{\pi}_n & \bar{\varrho}_n \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix}\sum b_j N(\pi_j) & \sum b_j \pi_j \bar{\varrho}_j \\ \sum b_j \varrho_j \bar{\pi}_j & \sum b_j N(\varrho_j)\end{pmatrix} \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

which proves lemma 2.

Now we consider any Farey triangle or Farey quadrangle of type $(2^{\frac{1}{2}}, 2^{\frac{1}{2}})$ in $\mathsf{Q}(im^{\frac{1}{2}})$. $m = 1,2,3,7$. Then at least one of the corresponding Farey matrices has the form announced in theorem 1. Hence the points p_j/q_j in the subdivisions of the Farey triangle or the Farey quadrangle have the form

$$(p_j, q_j) = \Phi(\pi_j, \varrho_j),$$

where Φ is a unimodular linear map of the form (15) and the (π_j, ϱ_j) of interest are listed in the tables 1–7 below, the indices being in agreement with figs. 8–14.

$m = 1$ (FT):

j	π_j	ϱ_j	$N(\pi_j)$	$N(\varrho_j)$	$\pi_j\bar{\varrho}_j$
1	1	0	1	0	0
2	0	1	0	1	0
3	1	1	1	1	1
1′	1	$1-i$	1	2	$1+i$
2′	$1+i$	1	2	1	$1+i$
3′	i	1	1	1	i
1*	1	$1+i$	1	2	$1-i$
2*	$1-i$	1	2	1	$1-i$
3*	$-i$	1	1	1	$-i$

Table 1

$m = 1$ (FQ):

j	π_j	ϱ_j	$N(\pi_j)$	$N(\varrho_j)$	$\pi_j\bar{\varrho}_j$
1	1	0	1	0	0
2	0	1	0	1	0
3	1	$1-i$	1	2	$1+i$
4	$1+i$	1	2	1	$1+i$
1′	1	1	1	1	1
1*	i	1	1	1	i

Table 2

$m = 2$ (FT):

j	π_j	ϱ_j	$N(\pi_j)$	$N(\varrho_j)$	$\pi_j\bar{\varrho}_j$
1	1	0	1	0	0
2	0	1	0	1	0
3	1	1	1	1	1
$1'$	$1+\omega$	2	3	4	$2+2\omega$
$2'$	2	$1-\omega$	4	3	$2+2\omega$
$3'$	$-1+\omega$	$1+\omega$	3	3	$1+2\omega$
1^*	$1-\omega$	2	3	4	$2-2\omega$
2^*	2	$1+\omega$	4	3	$2-2\omega$
3^*	$-1-\omega$	$1-\omega$	3	3	$1-2\omega$
α	ω	$1+\omega$	2	3	$2+\omega$
β	$1-\omega$	$-\omega$	3	2	$2+\omega$
γ	$1+\omega$	1	3	1	$1+\omega$
δ	ω	1	2	1	ω
η	1	$-\omega$	1	2	ω
ζ	1	$1-\omega$	1	3	$1+\omega$

Table 3

$m = 2$ (FQ):

j	π_j	ϱ_j	$N(\pi_j)$	$N(\varrho_j)$	$\pi_j\bar{\varrho}_j$
1	1	0	1	0	0
2	0	1	0	1	0
3	1	$-\omega$	1	2	ω
4	ω	1	2	1	ω
$1'$	$1-\omega$	$-\omega$	3	2	$2+\omega$
$2'$	ω	$1+\omega$	2	3	$2+\omega$
$3'$	$1+\omega$	2	3	4	$2+2\omega$
$4'$	2	$1-\omega$	4	3	$2+2\omega$
1^*	$1+\omega$	$-\omega$	3	2	$-2+\omega$
2^*	ω	$1-\omega$	2	3	$-2+\omega$
3^*	$-1+\omega$	2	3	4	$-2+2\omega$
4^*	-2	$1+\omega$	4	3	$-2+2\omega$
α	1	1	1	1	1
β	$1+\omega$	1	3	1	$1+\omega$

Table 4

$m = 3$ (FT):

j	π_j	ϱ_j	$N(\pi_j)$	$N(\varrho_j)$	$\pi_j\bar{\varrho}_j$
1	1	0	1	0	0
2	0	1	0	1	0
3	1	1	1	1	1
$1'$	ω	1	1	1	ω
1^*	$1-\omega$	1	1	1	$1-\omega$

Table 5

$m = 7$ (FT):

j	π_j	ϱ_j	$N(\pi_j)$	$N(\varrho_j)$	$\pi_j\bar{\varrho}_j$
1	1	0	1	0	0
2	0	1	0	1	0
3	1	1	1	1	1
1′	ω	1	2	1	ω
2′	1	$\bar{\omega}$	1	2	ω
3′	$-\bar{\omega}$	ω	2	2	$1+\omega$
1*	$\bar{\omega}$	1	2	1	$\bar{\omega}$
2*	1	ω	1	2	$\bar{\omega}$
3*	$-\omega$	$\bar{\omega}$	2	2	$1+\bar{\omega}$

Table 6

$m = 7$ (FQ):

j	π_j	ϱ_j	$N(\pi_j)$	$N(\varrho_j)$	$\pi_j\bar{\varrho}_j$
1	1	0	1	0	0
2	0	1	0	1	0
3	1	$\bar{\omega}$	1	2	ω
4	ω	1	2	1	ω
α	1	1	1	1	1
β	$-\bar{\omega}$	ω	2	2	$1+\omega$
γ	$-\bar{\omega}$	1	2	1	$-\bar{\omega}$
δ	-1	ω	1	2	$-\bar{\omega}$

Table 7a

j	π_j	ϱ_j	$N(\pi_j)$	$N(\varrho_j)$	$\pi_j\bar{\varrho}_j$
1	1	0	1	0	0
2	0	1	0	1	0
3	1	$\bar{\omega}$	1	2	ω
4	ω	1	2	1	ω
α	$-\bar{\omega}$	1	2	1	$-\bar{\omega}$
β	-1	ω	1	2	$-\bar{\omega}$
γ	1	1	1	1	1
δ	$-\bar{\omega}$	ω	2	2	$1+\omega$

Table 7b

Alternatively the primes and asterisks in tables 1–6 should be interchanged, the last six lines in table 3 replaced by their complex conjugates, and the last two lines in table 4 replaced by the following

j	π_j	ϱ_j	$N(\pi_j)$	$N(\varrho_j)$	$\pi_j\bar{\varrho}_j$
α	-1	1	1	1	-1
β	$-1+\omega$	1	3	1	$-1+\omega$

From tables 1–7 or the alternative ones the following linear norm relations connected with the subdivisions in figs. 8–14 are deduced by means of lemma 2

$m = 1$ (FT):

(35) $N' + N^* = 4N$,

(36) $N_1 + N_1' = N_2 + N_2' = N_3 + N_3'$,

(37) $N_1 + N_1^* = N_2 + N_2^* = N_3 + N_3^*$,

$m = 1$ (FQ):

(38) $N_1' + N_1^* = \frac{1}{2}N$,

(39) $N_1 + N_3 = N_2 + N_4$,

$m = 2$ (FT):

(40) $N' + N^* = 10N$,

(41) $N_1 + N_1' = N_2 + N_2' = N_3 + N_3'$,

(42) $N_1 + N_1^* = N_2 + N_2^* = N_3 + N_3^*$,

(43) $N_\alpha + N_\delta = N_\beta + N_\eta = N_\gamma + N_\zeta = N_1 + N_1'$,

(44) $N_\alpha - N_\beta = N_2 - N_1$,

(45) $N_\gamma - N_\delta = N_3 - N_2$,

(46) $N_\beta - N_\gamma = N_3 - N_1$,

$m = 2$ (FQ):

(47) $N' + N^* = 6N$,

(48) $N_1' - N_1 = N_2' - N_2 = N_3' - N_3 = N_4' - N_4$,

(49) $N_1^* - N_1 = N_2^* - N_2 = N_3^* - N_3 = N_4^* - N_4$,

(50) $N_1 + N_3 = N_2 + N_4$,

(51) $N_1' + N_3' = N_2' + N_4'$,

(52) $N_1^* + N_3^* = N_2^* + N_4^*$,

(53) $N_\alpha - N_\beta = N_2 - N_4$,

(54) $N_\alpha + N_\beta = N_1 + N_1'$,

$m = 3$ (FT):

(55) $N_1' + N_1^* = N$,

$m = 7$ (FT):

(56) $N' + N^* = 5N$,

(57) $N_1' - N_1 = N_2' - N_2 = N_3' - N_3$,

(58) $N_1^* - N_1 = N_2^* - N_2 = N_3^* - N_3$,

$m = 7$ (FQ):

$$
\begin{array}{lllll}
(59) & N_\alpha + N_\beta & + N_\gamma + N_\delta & = \tfrac{3}{2}N\,, \\
(60) & N_1 + N_3 & = N_2 + N_4\,, \\
(61a) & N_\alpha - N_\beta & = N_1 - N_4\,, \\
(61b) & N_\alpha - N_\beta & = N_1 - N_2\,, \\
(62a) & N_\gamma - N_\delta & = N_1 - N_2\,, \\
(62b) & N_\gamma - N_\delta & = N_1 - N_4\,.
\end{array}
$$

In these relations N_j means $N(q_j)$, while

$$(63) \qquad N = \sum_{j=1}^{3} N_j \quad \text{and} \quad N = \sum_{j=1}^{4} N_j$$

for Farey triangles and Farey quadrangles respectively.

Of course, by lemma 2 all the norm relations above are valid with $N_j = N(p_j)$ as well.

It should be noticed that a great number of the norm relations listed above are just the norm relation (23) in theorem 2, since this relation has the particularly simple form

$$(64) \qquad N_1 + N_3 = N_2 + N_4$$

for a Farey quadrangle of type $(2^{\frac{1}{2}}, 2^{\frac{1}{2}})$.

Finally the number of linearly independant norm relations listed above in each case equals the number of points involved minus four, which is the maximal number of linearly independant relations obtainable by means of lemma 2, since $\mathsf{R} \times \mathsf{R} \times \mathsf{C}$ is a 4-dimensional vector space over R.

Now, given a Farey triangle or a Farey quadrangle of type $(2^{\frac{1}{2}}, 2^{\frac{1}{2}})$ in $\mathsf{Q}(im^{\frac{1}{2}})$, $m = 1, 2, 3, 7$, only three independant norms are given by (64), and hence there is one norm relation missing in each case in order that the N_j in the subdivisions should be determined. It follows from the definition of a Farey triangle and a Farey quadrangle of type $(2^{\frac{1}{2}}, 2^{\frac{1}{2}})$ that once all the N_j in the subdivisions are known the points p_j/q_j themselves are determined geometrically.

5. Angular relations. Non-linear norm relations.

It was pointed out in the preceding section that given a Farey triangle or a Farey quadrangle of type $(2^{\frac{1}{2}}, 2^{\frac{1}{2}})$ in $\mathsf{Q}(im^{\frac{1}{2}})$, $m = 1, 2, 3, 7$, the inner and outer subdivisions are not completely determined by the linear norm relations found in that section, but that there is one norm relation missing in each case. It was also motivated that this norm relation must be non-linear.

In this section we shall show that the missing non-linear norm relations originate from the angular relations already indicated in figs. 8–14, where the angle u_0 in figs. 13 and 14 is given by $0 < u_0 < \pi$, $\cos u_0 = 1/8^{\frac{1}{2}}$.

The angular relations in figs. 8–14 all follow from the conformal property of the unimodular homographic maps φ considered in section 3, e.g. the angular relation in fig. 12 is an immediate consequence of fig. 15 below. In this particular case, however, the angular relation is well-known, since the points p_1'/q_1', p_1^*/q_1^* are just the inner and outer isodynamic points of the Farey triangle.

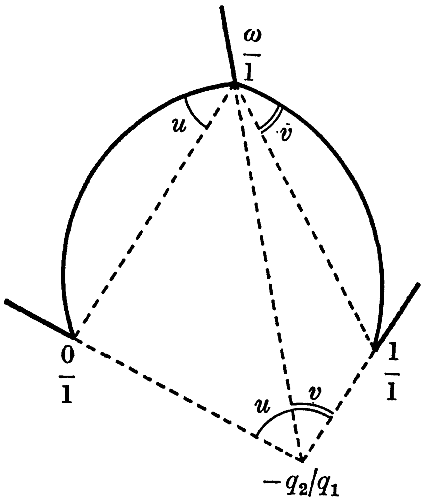

Fig. 15.

From these angular relations the following non-linear norm relations may be deduced

$m = 1$ (FT):

$$N' = 2N + 3(N^2 - 2N^{(2)})^{\frac{1}{2}}, \tag{65}$$

$m = 1$ (FQ):

$$N_1' = \tfrac{1}{4}N + (\tfrac{1}{8})^{\frac{1}{2}}(2N^2 - 5N^{(2)} - 2(N_1N_3 + N_2N_4))^{\frac{1}{2}}, \tag{66}$$

$m = 2$ (FT):

$$N' = 5N + 6(2)^{\frac{1}{2}}(N^2 - 2N^{(2)})^{\frac{1}{2}}, \tag{67}$$

$m = 2$ (FQ):

$$N' = 3N + 4(2N^2 - 5N^{(2)} - 2(N_1N_3 + N_2N_4))^{\frac{1}{2}}, \tag{68}$$

$m = 3$ (FT):

$$N_1' = \tfrac{1}{2}N + \tfrac{1}{2}3^{\frac{1}{2}}(N^2 - 2N^{(2)})^{\frac{1}{2}}, \tag{69}$$

$m = 7$ (FT):

$$N' = \tfrac{5}{2}N + \tfrac{3}{2}7^{\frac{1}{2}}(N^2 - 2N^{(2)})^{\frac{1}{2}}, \tag{70}$$

$m = 7$ (FQ):

$$N_\alpha + N_\beta = \tfrac{3}{4}N + (\tfrac{7}{8})^{\frac{1}{2}}(2N^2 - 5N^{(2)} - 2(N_1N_3 + N_2N_4))^{\frac{1}{2}}. \tag{71}$$

In these formulas we have used the notation

$$N^{(2)} = \sum_{j=1}^{3} N_j^2 \quad \text{and} \quad N^{(2)} = \sum_{j=1}^{4} N_j^2 \tag{72}$$

for Farey triangles and Farey quadrangles respectively.

The non-linear norm relations (65)–(71) all follow from the angular relations in figs. 8–14 in the same manner, so we shall only give the deduction of, say, (69) from the angular relation in fig. 12. In fact, it follows from fig. 12 that

$$\cos(A + \tfrac{1}{3}\pi) = \frac{N_2 + N_3 - N_1'}{2(N_2N_3)^{\frac{1}{2}}}$$

and

$$\cos A = \frac{N_2 + N_3 - N_1}{2(N_2N_3)^{\frac{1}{2}}},$$

whence by the relation

$$\cos(A + \tfrac{1}{3}\pi) = \tfrac{1}{2}\cos A - \tfrac{1}{2}3^{\frac{1}{2}}\sin A,$$

$$\frac{N_2 + N_3 - N_1'}{2(N_2N_3)^{\frac{1}{2}}} = \frac{1}{2}\,\frac{N_2 + N_3 - N_1}{2(N_2N_3)^{\frac{1}{2}}} - \frac{1}{2}3^{\frac{1}{2}}\,\frac{(N^2 - 2N^{(2)})^{\frac{1}{2}}}{2(N_2N_3)^{\frac{1}{2}}}.$$

Now formula (69) follows by reduction.

6. Approximation lemmas.

In this section we shall deduce a number of important approximation lemmas of a purely geometric nature, however formulated by means of complex numbers. The degree of approximation of a quotient p/q of complex numbers to a complex number $\xi \neq p/q$ will in these lemmas be measured by means of the real number c defined by

$$c = (|q|\,|q\xi - p|)^{-1}. \tag{73}$$

LEMMA 3. *Let p', p'', q', q'' be complex numbers, $q', q'' \neq 0$, such that*

(74) $$\Delta = |p'q'' - p''q'| > 0 ,$$

and

(75) $$f = |q''/q'| \geqq 1 .$$

Further let ξ be any complex number different from p'/q' and p''/q''. The real numbers c' and c'' are given by (73), *and the angle u, $0 \leqq u \leqq \pi$, is the angle $p'/q'\ \xi\ p''/q''$.*

Suppose $f \geqq f_0 \geqq 1$, $u \geqq u_0 \geqq \frac{1}{2}\pi$, then

(76) $$\max(c', c'') \geqq (f_0^2 + 1/f_0^2 - 2\cos u_0)^{\frac{1}{2}}/\Delta ,$$

where the equality sign occurs if and only if simultaneously

$$f = f_0, \quad u = u_0, \quad |q'\xi - p'|/|q''\xi - p''| = f_0 ,$$

in which case

$$c' = c'' = (f_0^2 + 1/f_0^2 - 2\cos u_0)^{\frac{1}{2}}/\Delta .$$

PROOF. Let ξ be restricted to the two circular arcs γ and γ' from which the segment from p'/q' to p''/q'' is seen under the angle $u \geqq \frac{1}{2}\pi$ (fig. 16). By symmetry we need only consider $\xi \in \gamma$. As ξ moves continuously from p'/q' to p''/q'' along γ, c' decreases strictly and continuously from $+\infty$ to f/Δ by (73), (74) and (75) and the assumption $u \geqq \frac{1}{2}\pi$, while c'' increases strictly and continuously from $1/(f\Delta)$ to $+\infty$.

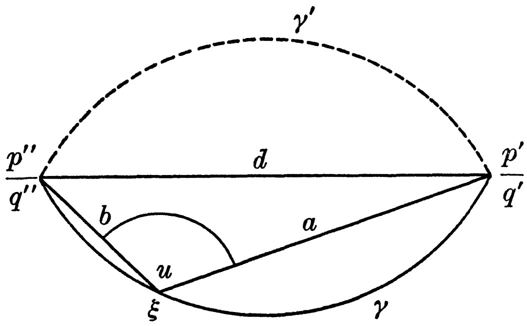

Fig. 16.

Consequently $\max(c', c'')$ attains its minimum in exactly one point of γ, determined by $c' = c''$, that is, $|q'\xi - p'|/|q''\xi - p''| = f$, or with the notation of fig. 16

(77) $$a/b = f^2 .$$

Further by (74) and fig. 16

(78) $$d = \Delta/|q'q''| .$$

From (78) and (73) we obtain in this extreme case

$$\frac{d^2}{ab} = \frac{\Delta^2}{|q'|^2 a |q''|^2 b} = \Delta^2 c'c'' = (\Delta c')^2 = (\Delta c'')^2 . \tag{79}$$

Now since

$$2\cos u = \frac{a^2+b^2-d^2}{ab} = \frac{a}{b}+\frac{b}{a}-\frac{d^2}{ab},$$

we get by applying (77) and (79)

$$2\cos u = f^2+1/f^2-(\Delta c')^2 = f^2+1/f^2-(\Delta c'')^2 ,$$

whence

$$c' = c'' = (f^2+1/f^2-2\cos u)^{\frac{1}{2}}/\Delta .$$

So far we have proved that

$$\max_{\xi\in\gamma\cup\gamma'}(c',c'') \geqq (f^2+1/f^2-2\cos u)^{\frac{1}{2}}/\Delta = F(f,u) ,$$

with equality if and only if $c'=c''$ or equivalently $|q'\xi-p'|/|q''\xi-p''|=f$. However, the function $F(f,u)$, $f\geqq f_0\geqq 1$, $u\geqq u_0\geqq \frac{1}{2}\pi$, is a strictly increasing function of each of the two variables, the other being kept fixed.

This proves lemma 3.

Lemma 4. *Let p', p'', q', q'' be complex numbers, $q',q''\neq 0$, such that*

$$\Delta = |p'q''-p''q'| > 0 ,$$

and

$$f = |q''/q'| \geqq 1 .$$

Further let ξ be any complex number different from p'/q' and p''/q'' and lying on the segment from p'/q' to p''/q''.

Suppose $f\geqq f_0\geqq 1$. Then

$$\max(c',c'') \geqq (f_0+1/f_0)/\Delta , \tag{80}$$

where the equality sign occurs if and only if simultaneously

$$f=f_0 \quad \textit{and} \quad |q'\xi-p'|/|q''\xi-p''| = f_0 ,$$

in which case

$$c' = c'' = (f_0+1/f_0)/\Delta .$$

Proof. Lemma 3 with $u=u_0=\pi$.

Lemma 5. *Let p_1, p_2, p_3, q_1, q_2, q_3 be complex numbers, $q_1,q_2,q_3\neq 0$, such that*

$$|p_1q_2-p_2q_1| = |p_1q_3-p_3q_1| = |p_2q_3-p_3q_2| = 1 . \tag{81}$$

Further let ξ *be any complex number different from* p_1/q_1, p_2/q_2, p_3/q_3 *and lying in the closed triangle* $T=T(p_1/q_1, p_2/q_2, p_3/q_3)$ *with vertices* p_1/q_1, p_2/q_2, p_3/q_3. *Then*

$$\max(c_1,c_2,c_3) \geqq 3^{\frac{1}{2}} \tag{82}$$

with strict inequality unless T *is equilateral and* ξ *is its centre, in which case* $c_1=c_2=c_3=3^{\frac{1}{2}}$.

PROOF. Let u_j, $0<u_j<\pi$, be the angle $p_k/q_k\,\xi\,p_l/q_l$, where (j,k,l) is any permutation of $(1,2,3)$. If $u_j>\frac{2}{3}\pi$, then $\max(c_k,c_l)>3^{\frac{1}{2}}$ by lemma 3 with

$$\Delta = 1, \qquad f = \max(|q_k/q_l|,|q_l/q_k|) \geqq f_0 = 1, \qquad u_j > u_0 = \tfrac{2}{3}\pi\,.$$

If $u_1=u_2=u_3=\frac{2}{3}\pi$ and $f=\max(|q_k/q_l|,|q_l/q_k|)>1$, then $\max(c_k,c_l)>3^{\frac{1}{2}}$ by lemma 3 with

$$\Delta = 1, \qquad f > f_0 = 1, \qquad u_j = u_0 = \tfrac{2}{3}\pi\,.$$

In the remaining case $|q_1|=|q_2|=|q_3|$, that is, T is equilateral by (81), and $u_1=u_2=u_3=\frac{2}{3}\pi$, that is, ξ is the centre of T. Evidently $c_1=c_2=c_3=3^{\frac{1}{2}}$ in this case.

LEMMA 6. *Let* p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4 *be complex numbers,* $q_1,q_2,q_3,q_4 \neq 0$, *such that*

$$|p_1q_2-p_2q_1| = |p_2q_3-p_3q_2| = |p_3q_4-p_4q_3| = |p_4q_1-p_1q_4| = 1\,, \tag{83}$$

and such that the closed quadrangle $Q=Q(p_1/q_1,p_2/q_2,p_3/q_3,p_4/q_4)$ *with vertices* p_1/q_1, p_2/q_2, p_3/q_3, p_4/q_4 *is convex. Further let* ξ *be any complex number different from* p_1/q_1, p_2/q_2, p_3/q_3, p_4/q_4 *and lying in* Q. *Then*

$$\max(c_1,c_2,c_3,c_4) \geqq 2^{\frac{1}{2}} \tag{84}$$

with strict inequality unless Q *is a square and* ξ *is its centre, in which case* $c_1=c_2=c_3=c_4=2^{\frac{1}{2}}$.

PROOF. Let $u_{j,k}$, $0<u_{j,k}<\pi$, be the angle $p_j/q_j\,\xi\,p_k/q_k$, where (j,k) is $(1,2)$, $(2,3)$, $(3,4)$ or $(4,1)$. If $u_{j,k}>\frac{1}{2}\pi$, then $\max(c_j,c_k)>2^{\frac{1}{2}}$ by lemma 3 with

$$\Delta = 1, \qquad f = \max(|q_j/q_k|,\ |q_k/q_j|) \geqq f_0 = 1, \qquad u_{j,k} > u_0 = \tfrac{1}{2}\pi\,.$$

If $u_{1,2}=u_{2,3}=u_{3,4}=u_{4,1}=\frac{1}{2}\pi$ and $f=\max(|q_j/q_k|,\ |q_k/q_j|)>1$, then

$$\max(c_j,c_k) > 2^{\frac{1}{2}}$$

by lemma 3 with

$$\Delta = 1, \qquad f > f_0 = 1, \qquad u_{j,k} = u_0 = \tfrac{1}{2}\pi\,.$$

In the remaining cases $|q_1|=|q_2|=|q_3|=|q_4|$, that is, Q is a rhombus by (83), and $u_{1,2}=u_{2,3}=u_{3,4}=u_{4,1}=\frac{1}{2}\pi$, that is, ξ is the centre of Q. From lemma 3 it follows that $\max(c_1,c_2) > 2^{\frac{1}{2}}$, unless $|q_1\xi - p_1| = |q_2\xi - p_2|$, that is, Q is a square. Evidently $c_1 = c_2 = c_3 = c_4 = 2^{\frac{1}{2}}$ in this particular case.

LEMMA 7. *Let p_1, p_2, p_3, q_1, q_2, q_3 be complex numbers, $q_1,q_2,q_3 \neq 0$, such that*

$$(85)\qquad |p_1q_2 - p_2q_1| = |p_2q_3 - p_3q_2| = 1, \qquad |p_1q_3 - p_3q_1| = \Delta, \quad 0 < \Delta \leqq 2^{\frac{1}{2}},$$

and let

$$f = \max(|q_1/q_3|, |q_3/q_1|).$$

Further let ξ be any complex number different from p_1/q_1, p_2/q_2, p_3/q_3 and lying in the closed triangle $T = T(p_1/q_1, p_2/q_2, p_3/q_3)$ with vertices p_1/q_1, p_2/q_2, p_3/q_3.

Suppose $f \geqq f_0 \geqq 1$, then

$$(86)\qquad \max(c_1,c_2,c_3) \geqq \min\left((2(f_0 + 1/f_0) - \Delta^2)^{\frac{1}{2}}, 2\right).$$

REMARK. It is possible though complicated to formulate a more general result than lemma 7, in which there are no restrictions on Δ and the constant 2 under the min sign does not occur. But since we shall have occasion only to apply lemma 7 for $\Delta = 1$ or $2^{\frac{1}{2}}$ and for $\max(c_1,c_2,c_3) < 2$, we shall stick to the present formulation of the lemma and thereby profit from a considerable simplification of the proof.

PROOF. Since $\max(c_1,c_2,c_3)$ is a continuous function of ξ on the set $T \setminus \{p_1/q_1, p_2/q_2, p_3/q_3\}$ and $c_j \to +\infty$ as $\xi \to p_j/q_j$, $1 \leqq j \leqq 3$, there exists in this set a complex number ξ_0 where $\max(c_1,c_2,c_3)$ attains its least value. Hence in order to prove the lemma it suffices to establish the inequality

$$(87)\qquad \max(c_1,c_2,c_3) \geqq \min\left((2(f + 1/f) - \Delta^2)^{\frac{1}{2}}, 2\right),$$

when ξ_0 has this property.

Suppose first that ξ_0 is an interior point of T. Then $c_1 = c_2 = c_3$, since otherwise there would exist a complex number $\xi \in T \setminus \{p_1/q_1, p_2/q_2, p_3/q_3\}$ near ξ_0 with a smaller value of $\max(c_1,c_2,c_3)$.

An easy calculation shows that u_1, u_2, u_3 and $c = c_1 = c_2 = c_3$ satisfy the system of equations

$$(88)\qquad \begin{cases} \cos u_1 = \frac{1}{2}\{N_2/N_3 + N_3/N_2 - c^2\}, \\ \cos u_2 = \frac{1}{2}\{N_1/N_3 + N_3/N_1 - (\Delta c)^2\}, \\ \cos u_3 = \frac{1}{2}\{N_1/N_2 + N_2/N_1 - c^2\}, \\ u_1 + u_2 + u_3 = 2\pi, \end{cases}$$

where as usual $N_j = |q_j|^2$ and $u_j = p_k/q_k\, \xi_0\, p_l/q_l$. Hence c is determined by the equation

$$\text{(89)} \quad \operatorname{Arccos} \tfrac{1}{2}\{N_2/N_3 + N_3/N_2 - c^2\} + \operatorname{Arccos} \tfrac{1}{2}\{N_1/N_3 + N_3/N_1 - (\Delta c)^2\} + \\ + \operatorname{Arccos} \tfrac{1}{2}\{N_1/N_2 + N_2/N_1 - c^2\} = 2\pi .$$

If $c \geqq 2$, (87) is trivially satisfied; consequently we may suppose in the following that $c < 2$. By (88)

$$\tfrac{1}{2}\{N_1/N_3 + N_3/N_1 - (\Delta c)^2\} = \tfrac{1}{2}\{f^2 + 1/f^2 - (\Delta c)^2\} \leqq 1 ,$$

and since $c < 2$ this implies

$$(f + 1/f)^2 \leqq 4 + (\Delta c)^2 < 4 + 4\Delta^2 < (2 + \Delta^2)^2 ,$$

whence

$$\text{(90)} \qquad f + 1/f - \Delta^2 < 2 .$$

We consider now the function

$$G(x,y) = \operatorname{Arccos} \tfrac{1}{2}\{x/N_3 + N_3/x - y\} + \operatorname{Arccos} \tfrac{1}{2}\{N_1/N_3 + N_3/N_1 - \Delta^2 y\} + \\ + \operatorname{Arccos} \tfrac{1}{2}\{x/N_1 + N_1/x - y\} .$$

We know from (89) that

$$\text{(91)} \qquad G(N_2, c^2) = 2\pi ,$$

but also

$$\text{(92)} \qquad G((N_1N_3)^{\frac{1}{2}},\ 2(f + 1/f) - \Delta^2) = 2\pi ,$$

since by (90) and the inequalities $\Delta \leqq 2^{\frac{1}{2}}$ and $f \geqq 1$

$$-2 < \Delta^2 - (f + 1/f) \leqq 0 ,$$

and hence

$$2 \operatorname{Arccos} \tfrac{1}{2}\{\Delta^2 - (f + 1/f)\} + \operatorname{Arccos} \tfrac{1}{2}\{(f + 1/f - \Delta^2)^2 - 2\} \\ = 2A + (2\pi - 2A) = 2\pi .$$

Since $G(x,y)$ is defined at the points $((N_1N_3)^{\frac{1}{2}},\ 2(f+1/f) - \Delta^2)$ and (N_2, c^2), $G(x,y)$ is obviously defined on the broken line joining the points

$$((N_1N_3)^{\frac{1}{2}},\ 2(f+1/f) - \Delta^2), \quad ((N_1N_3)^{\frac{1}{2}},\ c^2), \quad (N_2, c^2) .$$

Now

$$G_x'(x,y) = -\tfrac{1}{2}(1 - \tfrac{1}{4}(x/N_3 + N_3/x - y)^2)^{-\frac{1}{2}}(1/N_3 - N_3/x^2) - \\ - \tfrac{1}{2}(1 - \tfrac{1}{4}(x/N_1 + N_1/x - y)^2)^{-\frac{1}{2}}(1/N_1 - N_1/x^2) ,$$

and hence an easy calculation shows that

$$G_x'(x,c^2) > 0 \quad \text{for } N_2 < x < (N_1N_3)^{\frac{1}{2}} \quad \text{in case } N_2 < (N_1N_3)^{\frac{1}{2}} ,$$
$$G_x'(x,c^2) < 0 \quad \text{for } (N_1N_3)^{\frac{1}{2}} < x < N_2 \quad \text{in case } N_2 > (N_1N_3)^{\frac{1}{2}} .$$

Evidently $G_y'(x,y) > 0$ wherever defined. Thus, considering the function $G(x,y)$ along the broken line defined above and using (91) and (92), it results that

$$c^2 \geqq 2(f+1/f)-\Delta^2\,,$$

which proves (87) when ξ_0 is an interior point of T.

If ξ_0 belongs to one of the segments from p_1/q_1 to p_2/q_2 or from p_2/q_2 to p_3/q_3, $\max(c_1,c_2) \geqq 2$ or $\max(c_2,c_3) \geqq 2$ by lemma 4 with $\Delta = 1$ and $f_0 = 1$.

If finally ξ_0 belongs to the segment from p_1/q_1 to p_3/q_3,

$$\max(c_1,c_3) \geqq (f+1/f)/\Delta \geqq (2(f+1/f)-\Delta^2)^{\frac{1}{2}}$$

by lemma 4 with $f_0 = f$ and the inequality

$$((f+1/f)/\Delta)^2-2(f+1/f)+\Delta^2 = ((f+1/f)/\Delta-\Delta)^2 \geqq 0\,.$$

Thus (87) has been established in all cases. This proves lemma 7.

7. Evaluation of some important approximation constants.

Let $\xi \notin \mathsf{Q}(im^{\frac{1}{2}})$, $m = 1,2,3,7$, be contained in a chain of Farey triangles $\mathrm{FT}(p_1^{(n)}/q_1^{(n)}, p_2^{(n)}/q_2^{(n)}, p_3^{(n)}/q_3^{(n)})$, $n \geqq 0$. By theorem 1 we may suppose that the corresponding 2×3 Farey matrices are of the form

$$\mathfrak{T}^{(n)} = \begin{pmatrix} p_1^{(n)} & p_2^{(n)} & p_3^{(n)} \\ q_1^{(n)} & q_2^{(n)} & q_3^{(n)} \end{pmatrix} = \mathfrak{M}^{(n)} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad n \geqq 0\,. \tag{93}$$

Further suppose that

$$\mathfrak{T}^{(n+1)} = \mathfrak{M}^{(n)}\mathfrak{S}^{(n)} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad n \geqq 0\,, \tag{94}$$

or equivalently by (93)

$$\mathfrak{M}^{(n+1)} = \mathfrak{M}^{(n)}\mathfrak{S}^{(n)}, \quad n \geqq 0\,, \tag{95}$$

where $\mathfrak{S}^{(n)}$, $n \geqq 0$, is a unimodular matrix over $\mathsf{Z}(im^{\frac{1}{2}})$ obtainable from one of the tables 1, 3, 5, 6 or the alternative ones when $\mathfrak{T}^{(n+1)}$ is given in terms of $\mathfrak{T}^{(n)}$. By (95)

$$\mathfrak{M}^{(n+1)} = \mathfrak{M}^{(0)}\mathfrak{S}^{(0)}\mathfrak{S}^{(1)}\ldots\mathfrak{S}^{(n)}, \quad n \geqq 0\,. \tag{96}$$

We consider now the important special case, where the sequence $\mathfrak{S}^{(0)}, \mathfrak{S}^{(1)}\ldots$ is periodic with period λ. Then by (94) and (96)

$$\mathfrak{T}^{(n_0+\lambda k)} = \mathfrak{M}^{(n_0)}\mathfrak{S}^{k} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad k \geqq 0\,, \tag{97}$$

with

$$\mathfrak{S} = \mathfrak{S}^{(n_0)}\mathfrak{S}^{(n_0+1)}\ldots\mathfrak{S}^{(n_0+\lambda-1)}\,, \tag{98}$$

and

$$\mathfrak{M}^{(n_0)} = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}, \quad \mathfrak{S} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{99}$$

By theorem 5

(100) $$\lim_{n\to\infty} p_j^{(n)}/q_j^{(n)} = \xi, \quad j=1,2,3\,.$$

Hence we find by (93), (97), (99) and (100), that

(101) $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^k = \begin{pmatrix} r_k\eta & s_k\eta \\ r_k(1+\delta_k') & s_k(1+\delta_k'') \end{pmatrix}, \quad k \geqq 0\,,$$

where

(102) $$\lim_{k\to\infty} \delta_k' = \lim_{k\to\infty} \delta_k'' = 0\,,$$

and

(103) $$\xi = (a_0\eta+b_0)(c_0\eta+d_0)^{-1}\,.$$

From (101)

$$\begin{pmatrix} r_{k+1}\eta & s_{k+1}\eta \\ r_{k+1}(1+\delta_{k+1}') & s_{k+1}(1+\delta_{k+1}'') \end{pmatrix} = \begin{pmatrix} r_k(a\eta+b(1+\delta_k')) & s_k(a\eta+b(1+\delta_k'')) \\ r_k(c\eta+d(1+\delta_k')) & s_k(c\eta+d(1+\delta_k'')) \end{pmatrix},$$

and hence by (102)

(104) $$\eta = (a\eta+b)(c\eta+d)^{-1}\,,$$

so that η is one of the roots of the quadratic form

(105) $$f(x,y) = cx^2+(d-a)xy-by^2\,.$$

By a well-known argument of Perron [9], [10], [11] the approximation constant (cf. (1))

(106) $$C(\eta) \leqq \sqrt{|D|}/\mu\,,$$

where

(107) $$D = (d-a)^2+4bc$$

and

(108) $$\mu = \inf|f(x,y)|\,,$$

the infimum being taken over all $(x,y) \neq (0,0)$, $x,y \in \mathrm{Z}(im^{\frac{1}{2}})$. Further, by (99) and (103)

(109) $$\xi \sim \eta\,,$$

and consequently by a simple and well-known argument

(110) $$C(\xi) = C(\eta)\,.$$

We shall give four particularly simple examples of chains of Farey triangles, one in each of the cases $\mathrm{Q}(im^{\frac{1}{2}})$, $m=1,2,3,7$. Each of these examples plays an important role in the investigation of the respective approximation spectra to be presented in the next section.

Example 1. $\mathrm{Q}(i)$. Let $\xi \notin \mathrm{Q}(i)$ be contained in a chain of Farey triangles $\mathrm{FT}(p_1^{(n)}/q_1^{(n)}, p_2^{(n)}/q_2^{(n)}, p_3^{(n)}/q_3^{(n)})$, $n \geqq 0$, such that (cf. fig. 8)

$$\mathrm{FT}(p_1^{(n+1)}/q_1^{(n+1)}, p_2^{(n+1)}/q_2^{(n+1)}, p_3^{(n+1)}/q_3^{(n+1)}) = \mathrm{FT}(p_1'^{(n)}/q_1'^{(n)}, p_2'^{(n)}/q_2'^{(n)}, p_3'^{(n)}/q_3'^{(n)})$$

for all $n \geqq n_0$.

Then by table 1 or the alternative one the corresponding chain of 2×3 Farey matrices is of the form (or associated with) (94), where

$$\mathfrak{S}^{(n)} = \begin{pmatrix} 1 & -1+i \\ 1-i & i \end{pmatrix} = \mathfrak{S}' \quad \text{or} \quad \mathfrak{S}^{(n)} = \begin{pmatrix} 1 & -1-i \\ 1+i & -i \end{pmatrix} = \mathfrak{S}^*$$

for $n \geqq n_0$. Since

$$\mathfrak{S}'\mathfrak{S}^* = \mathfrak{S}^*\mathfrak{S}' = -\mathfrak{E},$$

$\mathfrak{E}$ being the unity matrix, $\mathfrak{T}^{(n)}$ is of the form (97) with $\lambda = 1$ and $\mathfrak{S} = \mathfrak{S}'$ or $\mathfrak{S} = \mathfrak{S}^*$. In both cases ξ is equivalent with one of the roots of the quadratic form

(111) $$f(x,y) = x^2 - xy + y^2$$

by (104), (105) and (109), and since the two roots of the form (111) are equivalent, we have proved that

(112) $$\xi \sim \tfrac{1}{2}(1+i3^{\frac{1}{2}}).$$

By (111), (107) and (108), $D = -3$ and $\mu = 1$, hence it follows from (106) and (110) that

(113) $$C(\xi) \leqq 3^{\frac{1}{2}}.$$

An easy calculation based on the norm relations (36) and (65) shows that (cf. fig. 17)

(114) $$\lim_{n\to\infty} (|q_j^{(n)}|\,|q_j^{(n)}\xi - p_j^{(n)}|)^{-1} = 3^{\frac{1}{2}}, \quad j = 1,2,3,$$

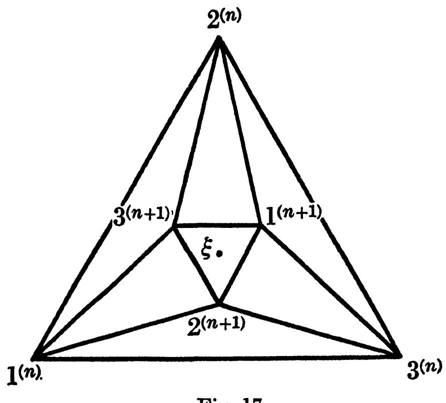

Fig. 17.

which together with (113) shows that

$$C(\xi) = 3^{\frac{1}{2}} . \tag{115}$$

In addition it follows by lemma 5, applied to the Farey triangles in the chain, that the inequality

$$(|q_j^{(n)}|\,|q_j^{(n)}\xi - p_j^{(n)}|)^{-1} > 3^{\frac{1}{2}}, \quad j=1,\,2 \text{ or } 3\,, \tag{116}$$

has infinitely many solutions $(p_j^{(n)}, q_j^{(n)})$.

EXAMPLE 2. $\mathsf{Q}(i2^{\frac{1}{2}})$. Let $\xi \notin \mathsf{Q}(i2^{\frac{1}{2}})$ be contained in a chain of Farey triangles $\mathrm{FT}(p_1^{(n)}/q_1^{(n)}, p_2^{(n)}/q_2^{(n)}, p_3^{(n)}/q_3^{(n)})$, $n \geqq 0$, such that (cf. fig. 10)

$$\begin{aligned}\mathrm{FT}(p_1^{(n+1)}/q_1^{(n+1)}, p_2^{(n+1)}/q_2^{(n+1)}, p_3^{(n+1)}/q_3^{(n+1)}) \\ = \mathrm{FT}(p_1'^{(n)}/q_1'^{(n)}, p_2'^{(n)}/q_2'^{(n)}, p_3'^{(n)}/q_3'^{(n)})\end{aligned}$$

for all $n \geqq n_0$. Then by table 3 or the alternative one the corresponding chain of 2×3 Farey matrices is of the form (or associated with) (94), where

$$\mathfrak{S}^{(n)} = \begin{pmatrix} 1+\omega & -2 \\ 2 & -1+\omega \end{pmatrix} = \mathfrak{S}' \quad \text{or} \quad \mathfrak{S}^{(n)} = \begin{pmatrix} 1-\omega & -2 \\ 2 & -1-\omega \end{pmatrix} = \mathfrak{S}^*$$

for $n \geqq n_0$. Since

$$\mathfrak{S}'\mathfrak{S}^* = \mathfrak{S}^*\mathfrak{S}' = -\mathfrak{E}\,,$$

$\mathfrak{T}^{(n)}$ is of the form (97) with $\lambda=1$ and $\mathfrak{S}=\mathfrak{S}'$ or $\mathfrak{S}=\mathfrak{S}^*$. In both cases ξ is equivalent with one of the roots of the quadratic form

$$f(x,y) = x^2 - xy + y^2 \tag{117}$$

by (104), (105) and (109), and since the two roots of the form (117) are equivalent, we have proved that

$$\xi \sim \tfrac{1}{2}(1+i3^{\frac{1}{2}})\,. \tag{118}$$

By (117), (107) and (108), $D=-3$ and $\mu=1$, hence it follows from (106) and (110) that

$$C(\xi) \leqq 3^{\frac{1}{2}} . \tag{119}$$

An easy calculation based on the norm relations (41) and (67) shows that (cf. fig. 18)

$$\lim_{n\to\infty} (|q_j^{(n)}|\,|q_j^{(n)}\xi - p_j^{(n)}|)^{-1} = 3^{\frac{1}{2}}, \quad j=1,2,3\,, \tag{120}$$

which together with (119) shows that

$$C(\xi) = 3^{\frac{1}{2}} . \tag{121}$$

As in example 1, the inequality (116) has infinitely many solutions.

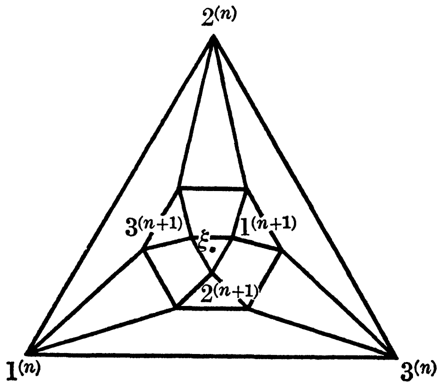

Fig. 18.

EXAMPLE 3. $Q(i7^{\frac{1}{2}})$. Let $\xi \notin Q(i7^{\frac{1}{2}})$ be contained in a chain of Farey triangles $\mathrm{FT}(p_1^{(n)}/q_1^{(n)}, p_2^{(n)}/q_2^{(n)}, p_3^{(n)}/q_3^{(n)})$, $n \geqq 0$, such that (cf. fig. 13)

$$\begin{aligned}\mathrm{FT}(p_1^{(n+1)}/q_1^{(n+1)}, p_2^{(n+1)}/q_2^{(n+1)}, p_3^{(n+1)}/q_3^{(n+1)}) \\ = \mathrm{FT}(p_1'^{(n)}/q_1'^{(n)}, p_2'^{(n)}/q_2'^{(n)}, p_3'^{(n)}/q_3'^{(n)})\end{aligned}$$

for all $n \geqq n_0$. Then by table 6 or the alternative one the corresponding chain of 2×3 Farey matrices is of the form (or associated with) (94), where

$$\mathfrak{S}^{(n)} = \begin{pmatrix} \omega & -1 \\ 1 & -\bar{\omega} \end{pmatrix} = \mathfrak{S}' \quad \text{or} \quad \mathfrak{S}^{(n)} = \begin{pmatrix} \bar{\omega} & -1 \\ 1 & -\omega \end{pmatrix} = \mathfrak{S}^*$$

for $n \geqq n_0$. Since

$$\mathfrak{S}'\mathfrak{S}^* = \mathfrak{S}^*\mathfrak{S}' = \mathfrak{E},$$

$\mathfrak{T}^{(n)}$ is of the form (97) with $\lambda = 1$ and $\mathfrak{S} = \mathfrak{S}'$ or $\mathfrak{S} = \mathfrak{S}^*$. In both cases ξ is equivalent with one of the roots of the quadratic form

$$f(x,y) = x^2 - xy + y^2 \tag{122}$$

by (104), (105) and (109), and since the two roots of the form (122) are equivalent, we have proved that

$$\xi \sim \tfrac{1}{2}(1 + i3^{\frac{1}{2}}). \tag{123}$$

By (122), (107) and (108), $D = -3$ and $\mu = 1$, hence it follows from (106) and (110) that

$$C(\xi) \leqq 3^{\frac{1}{2}}. \tag{124}$$

An easy calculation based on the norm relations (57) and (70) shows that (cf. fig. 19)

$$\lim_{n \to \infty} (|q_j^{(n)}|\,|q_j^{(n)}\xi - p_j^{(n)}|)^{-1} = 3^{\frac{1}{2}}, \quad j = 1,2,3, \tag{125}$$

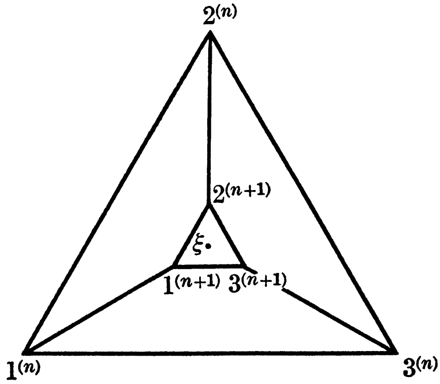

Fig. 19.

which together with (124) shows that

$$C(\xi) = 3^{1/2}\,. \tag{126}$$

As in example 1 the inequality (116) has infinitely many solutions.

Example 4. $\mathbf{Q}(i3^{1/2})$. Let $\xi \notin \mathbf{Q}(i3^{1/2})$ be contained in a chain of Farey triangles $\mathrm{FT}(p_1^{(n)}/q_1^{(n)}, p_2^{(n)}/q_2^{(n)}, p_3^{(n)}/q_3^{(n)})$, $n \geqq 0$, such that (cf. fig. 12)

$$\begin{aligned}\mathrm{FT}(p_1^{(n+1)}/q_1^{(n+1)}, p_2^{(n+1)}/q_2^{(n+1)}, p_3^{(n+1)}/q_3^{(n+1)})\\ = \mathrm{FT}(p_2^{(n)}/q_2^{(n)}, p_3^{(n)}/q_3^{(n)}, p_1'^{(n)}/q_1'^{(n)})\end{aligned}$$

and

$$N_3^{(n)} \geqq N_2^{(n)} \geqq N_1^{(n)}$$

for all $n \geqq n_0$. Then by table 5 or the alternative one the corresponding chain of 2×3 Farey matrices is of the form (or associated with) (94), where

$$\mathfrak{S}^{(n)} = \begin{pmatrix} 0 & \omega \\ 1-\omega & \omega \end{pmatrix} = \mathfrak{S}' \quad \text{or} \quad \mathfrak{S}^{(n)} = \begin{pmatrix} 0 & 1-\omega \\ \omega & 1-\omega \end{pmatrix} = \mathfrak{S}^*$$

for $n \geqq n_0$. Since

$$\mathfrak{S}'\mathfrak{S}^* \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1+\omega & 1 & \omega \\ -1+\omega & 1-\omega & 0 \end{pmatrix},$$

$$\mathfrak{S}^*\mathfrak{S}' \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -\omega & 1 & 1-\omega \\ -\omega & \omega & 0 \end{pmatrix},$$

and $p_1^{(n)}/q_1^{(n)}$ is not a vertex of $\mathrm{FT}(p_1^{(n+2)}/q_1^{(n+2)}, p_2^{(n+2)}/q_2^{(n+2)}, p_3^{(n+2)}/q_3^{(n+2)})$ for $n \geqq n_0$, $\mathfrak{T}^{(n)}$ is of the form (97) with $\lambda = 1$ and $\mathfrak{S} = \mathfrak{S}'$ or $\mathfrak{S} = \mathfrak{S}^*$. Hence ξ is equivalent with a root of one of the two quadratic forms

(127) $$f_1(x,y) = (1-\omega)x^2+\omega xy-\omega y^2\,,$$

(128) $$f_2(x,y) = \omega x^2+(1-\omega)xy-(1-\omega)y^2$$

by (104), (105) and (109), and since the four roots of the forms (127) and (128) are equivalent, we have proved that

(129) $$\xi \sim \frac{-\omega+(3+\omega)^{\frac{1}{2}}}{2-2\omega}.$$

By (127), (107) and (108), $D_1=3+\omega$ and $\mu_1=1$, hence it follows from (106) and (110) that

(130) $$C(\xi) \leqq 13^{\frac{1}{2}}\,.$$

An easy calculation based on the norm relation (69) shows that (cf. fig. 20)

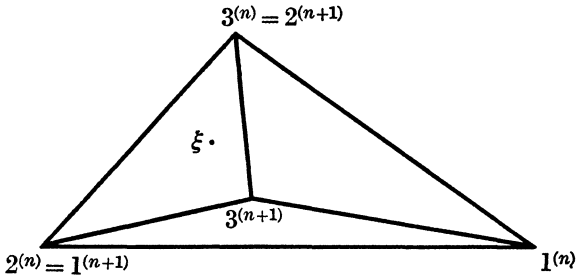

Fig. 20.

(131) $$\lim_{n\to\infty}(|q_j^{(n)}|\,|q_j^{(n)}\xi-p_j^{(n)}|)^{-1} = 13^{\frac{1}{2}}, \quad j=1,2,3\,,$$

which together with (130) shows that

(132) $$C(\xi) = 13^{\frac{1}{2}}\,.$$

It follows by (69) that

(133) $$\lim_{n\to\infty}N_3^{(n)}/N_2^{(n)} = \lim_{n\to\infty}N_2^{(n)}/N_1^{(n)} = F\,,$$

and hence, by (55), $F>1$ satisfies the equation

(134) $$F^4-F^3-F^2-F+1 = 0\,,$$

so that

(135) $$F+1/F = \tfrac{1}{2}(1+13^{\frac{1}{2}})\,.$$

In addition it may be shown that

(136) $$N_3^{(n)}/N_1^{(n)} > F^2$$

for infinitely many n, and hence by (135), (136) and lemma 7 with $\Delta=1$

and $f > f_0 = F$ applied to the corresponding Farey triangles in the chain, the inequality

$$(137) \qquad (|q_j^{(n)}|\,|q_j^{(n)}\xi - p_j^{(n)}|)^{-1} > 13^{\frac{1}{2}}, \quad j = 1,2 \text{ or } 3\,,$$

has infinitely many solutions $(p_j^{(n)}, q_j^{(n)})$.

In the following three examples $\xi \notin \mathsf{Q}(im^{\frac{1}{2}})$, $m = 2,7$, is contained in a chain of Farey quadrangles of type $(2^{\frac{1}{2}}, 2^{\frac{1}{2}})$,

$$\mathrm{FQ}\,(p_1^{(n)}/q_1^{(n)}, p_2^{(n)}/q_2^{(n)}, p_3^{(n)}/q_3^{(n)}, p_4^{(n)}/q_4^{(n)}), \quad n \geqq 0\,.$$

By theorem 1 we may suppose, that the corresponding 2×4 Farey matrices are of the form

$$(138) \quad \mathfrak{Q}^{(n)} = \begin{pmatrix} p_1^{(n)} & p_2^{(n)} & p_3^{(n)} & p_4^{(n)} \\ q_1^{(n)} & q_2^{(n)} & q_3^{(n)} & q_4^{(n)} \end{pmatrix} = \mathfrak{M}^{(n)} \begin{pmatrix} 1 & 0 & 1 & \omega \\ 0 & 1 & \bar{\omega} & 1 \end{pmatrix}, \quad n \geqq 0\,,$$

and hence the apparatus used in describing chains of Farey triangles can be applied to chains of Farey quadrangles as well.

Example 5. $\mathsf{Q}(i2^{\frac{1}{2}})$. Let $\xi \notin \mathsf{Q}(i2^{\frac{1}{2}})$ be contained in a chain of Farey quadrangles of type $(2^{\frac{1}{2}}, 2^{\frac{1}{2}})$, $\mathrm{FQ}\,(p_1^{(n)}/q_1^{(n)}, p_2^{(n)}/q_2^{(n)}, p_3^{(n)}/q_3^{(n)}, p_4^{(n)}/q_4^{(n)})$, $n \geqq 0$, such that (cf. fig. 11)

$$\begin{aligned} &\mathrm{FQ}\,(p_1^{(n+1)}/q_1^{(n+1)}, p_2^{(n+1)}/q_2^{(n+1)}, p_3^{(n+1)}/q_3^{(n+1)}, p_4^{(n+1)}/q_4^{(n+1)}) \\ &\qquad = \mathrm{FQ}\,(p_1'^{(n)}/q_1'^{(n)}, p_2'^{(n)}/q_2'^{(n)}, p_3'^{(n)}/q_3'^{(n)}, p_4'^{(n)}/q_4'^{(n)}) \end{aligned}$$

for all $n \geqq n_0$. Then by table 4 or the alternative one the corresponding chain of 2×4 Farey matrices is of the form (or associated with) (138), where (95) holds and

$$\mathfrak{S}^{(n)} = \begin{pmatrix} -1+\omega & \omega \\ \omega & 1+\omega \end{pmatrix} = \mathfrak{S}' \quad \text{or} \quad \mathfrak{S}^{(n)} = \begin{pmatrix} -1-\omega & \omega \\ \omega & 1-\omega \end{pmatrix} = \mathfrak{S}^*$$

for $n \geqq n_0$. Since

$$\mathfrak{S}'\mathfrak{S}^* = \mathfrak{S}^*\mathfrak{S}' = \mathfrak{E}\,,$$

$\mathfrak{Q}^{(n)}$ is of the form (97) with $\lambda = 1$ and $\mathfrak{S} = \mathfrak{S}'$ or $\mathfrak{S} = \mathfrak{S}^*$, formula (97) being modified so as to become applicable to 2×4 Farey matrices. In both cases ξ is equivalent with one of the roots of the quadratic form

$$(139) \qquad f(x,y) = x^2 - \omega xy - y^2$$

by (104), (105) and (109), and since the two roots of the form (139) are equivalent, we have proved that

$$(140) \qquad \xi \sim \tfrac{1}{2}2^{\frac{1}{2}} + \tfrac{1}{2}\omega\,.$$

By (139), (107) and (108), $D = 2$ and $\mu = 1$, hence (106) and (110) imply

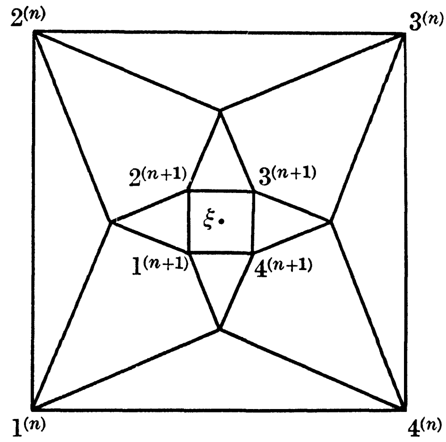

Fig. 21.

$$C(\xi) \leqq 2^{\frac{1}{2}}\,. \tag{141}$$

An easy calculation based on the norm relations (48) and (68) shows that (cf. fig. 21)

$$\lim_{n\to\infty}\left(|q_j^{(n)}|\,|q_j^{(n)}\xi-p_j^{(n)}|\right)^{-1} = 2^{\frac{1}{2}}, \quad j=1,2,3,4\,, \tag{142}$$

which together with (141) shows that

$$C(\xi) = 2^{\frac{1}{2}}\,. \tag{143}$$

In addition it follows by lemma 6 applied to the Farey quadrangles in the chain, that the inequality

$$\left(|q_j^{(n)}|\,|q_j^{(n)}\xi-p_j^{(n)}|\right)^{-1} > 2^{\frac{1}{2}}, \quad j=1,2,3,4\,, \tag{144}$$

has infinitely many solutions $(p_j^{(n)},q_j^{(n)})$.

Example 6. $Q(i2^{\frac{1}{2}})$. Let $\xi \notin Q(i2^{\frac{1}{2}})$ be contained in a chain of Farey quadrangles of type $(2^{\frac{1}{2}},2^{\frac{1}{2}})$, $\mathrm{FQ}(p_1^{(n)}/q_1^{(n)}, p_2^{(n)}/q_2^{(n)}, p_3^{(n)}/q_3^{(n)}, p_4^{(n)}/q_4^{(n)})$, $n \geqq 0$, such that (cf. figs. 11 and 12)

$$\begin{aligned}\mathrm{FQ}&(p_1^{(n+1)}/q_1^{(n+1)}, p_2^{(n+1)}/q_2^{(n+1)}, p_3^{(n+1)}/q_3^{(n+1)}, p_4^{(n+1)}/q_4^{(n+1)})\\ &= \mathrm{FQ}(p_1'^{(n)}/q_1'^{(n)}, p_\alpha^{(n)}/q_\alpha^{(n)}, p_1^{(n)}/q_1^{(n)}, p_\beta^{(n)}/q_\beta^{(n)})\end{aligned}$$

for all $n \geqq n_0$. Then by table 4 or the alternative one the corresponding chain of 2×4 Farey matrices is of the form (or associated with) (138), where (95) holds and

$$\mathfrak{S}^{(n)} = \begin{pmatrix} 1-\omega & -1 \\ -\omega & -1 \end{pmatrix} = \mathfrak{S}' \quad \text{or} \quad \mathfrak{S}^{(n)} = \begin{pmatrix} 1+\omega & 1 \\ -\omega & -1 \end{pmatrix} = \mathfrak{S}^*$$

for $n \geqq n_0$. Since

$$\mathfrak{S}'^2 = \begin{pmatrix} -1-\omega & \omega \\ -2 & 1+\omega \end{pmatrix} \quad \text{and} \quad \mathfrak{S}^{*2} = \begin{pmatrix} -1+\omega & \omega \\ 2 & 1-\omega \end{pmatrix},$$

and, by fig. 11, $p_3'^{(n)}/q_3'^{(n)}$ is not a vertex of

$$\mathrm{FQ}(p_1^{(n+2)}/q_1^{(n+2)}, p_2^{(n+2)}/q_2^{(n+2)}, p_3^{(n+2)}/q_3^{(n+2)}, p_4^{(n+2)}/q_4^{(n+2)})$$

for $n \geqq n_0$, $\mathfrak{Q}^{(n)}$ is of the form (97) (modified) with $\lambda = 2$ and

$$\mathfrak{S} = \mathfrak{S}'\mathfrak{S}^* = \begin{pmatrix} 3+\omega & 2-\omega \\ 2 & 1-\omega \end{pmatrix}.$$

Hence ξ is equivalent with one of the roots of the quadratic form

$$f(x,y) = \omega x^2 + (2-\omega)xy - (1+\omega)y^2 \tag{145}$$

by (104), (105) and (109), and since the two roots of the form (145) are equivalent, we have proved that

$$\xi \sim \tfrac{1}{2}(1+3^{\frac{1}{2}}) + \tfrac{1}{2}\omega\,. \tag{146}$$

By (145), (107) and (108), $D = -6$ and $\mu = 2^{\frac{1}{2}}$. Indeed

$$f(x,y) = \omega(x^2 - xy - y^2) + 2xy - y^2$$

does not represent ± 1, for otherwise in turn $f(x,y) \equiv 1 \pmod 2$, $f(x,y) \equiv 1 \pmod{\omega}$, $y \equiv 1 \pmod{\omega}$, $y^2 \equiv 1 \pmod 2$ and finally $x^2 - x - 1 \equiv 0 \pmod{\omega}$, which is impossible. Consequently $\mu = 2^{\frac{1}{2}}$. Hence it follows from (106) and (110) that

$$C(\xi) \leqq 3^{\frac{1}{2}}\,. \tag{147}$$

An easy calculation based on the norm relations (48), (53), (54) and (68) shows that (cf. fig. 22)

$$\lim_{n\to\infty} (|q_j^{(n)}|\,|q_j^{(n)}\xi - p_j^{(n)}|)^{-1} = 3^{\frac{1}{2}}, \quad j = 1,3\,, \tag{148}$$

$$\lim_{n\to\infty} (|q_j^{(n)}|\,|q_j^{(n)}\xi - p_j^{(n)}|)^{-1} = 2^{\frac{1}{2}}, \quad j = 2,4\,, \tag{149}$$

which together with (147) shows that

$$C(\xi) = 3^{\frac{1}{2}}\,. \tag{150}$$

EXAMPLE 7. $\mathsf{Q}(i7^{\frac{1}{2}})$. Let $\xi \notin \mathsf{Q}(i7^{\frac{1}{2}})$ be contained in a chain of Farey quadrangles of type $(2^{\frac{1}{2}}, 2^{\frac{1}{2}})$, $\mathrm{FQ}(p_1^{(n)}/q_1^{(n)}, p_2^{(n)}/q_2^{(n)}, p_3^{(n)}/q_3^{(n)}, p_4^{(n)}/q_4^{(n)})$, $n \geqq 0$, such that (cf. figs. 14a and 23)

$$\begin{aligned} &\mathrm{FQ}(p_1^{(n+1)}/q_1^{(n+1)}, p_2^{(n+1)}/q_2^{(n+1)}, p_3^{(n+1)}/q_3^{(n+1)}, p_4^{(n+1)}/q_4^{(n+1)}) \\ &\qquad = \mathrm{FQ}(p_\alpha^{(n)}/q_\alpha^{(n)}, p_\beta^{(n)}/q_\beta^{(n)}, p_4^{(n)}/q_4^{(n)}, p_1^{(n)}/q_1^{(n)}) \end{aligned}$$

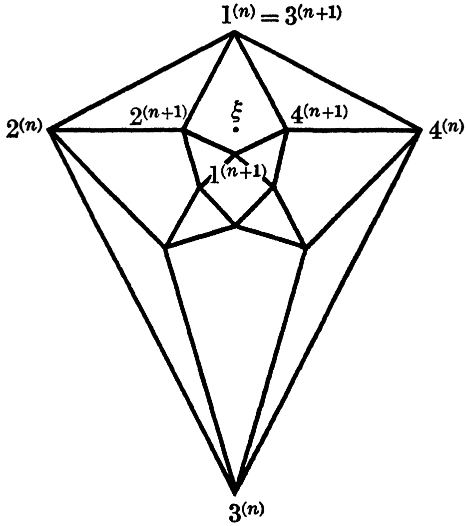

Fig. 22.

for all $n \geqq n_0$. Then by table 7a the corresponding chain of 2×4 Farey matrices is of the form (or associated with) (138), where (95) holds and

$$\mathfrak{S}^{(n)} = \begin{pmatrix} 1 & \bar{\omega} \\ 1 & -\omega \end{pmatrix}$$

for all $n \geqq n_0$. Consequently $\mathfrak{Q}^{(n)}$ is of the form (97) (modified) with $\lambda = 1$ and

$$\mathfrak{S} = \begin{pmatrix} 1 & \bar{\omega} \\ 1 & -\omega \end{pmatrix}.$$

Hence ξ is equivalent with one of the roots of the quadratic form

$$f(x,y) = x^2 - (1+\omega)xy - \bar{\omega}y^2 \tag{151}$$

by (104), (105) and (109), and since the two roots of the form (151) are equivalent, we have proved that

$$\xi \sim \tfrac{1}{2}(1+\omega+(3-\omega)^{\frac{1}{2}}). \tag{152}$$

By (151), (107) and (108), $D = 3-\omega$ and $\mu = 1$, hence it follows from (106) and (110) that

$$C(\xi) \leqq 8^{\frac{1}{2}}. \tag{153}$$

An easy calculation based on the norm relations (61a), (60) and (71) yields (cf. fig. 23)

$$\lim_{n\to\infty} (|q_j^{(n)}|\,|q_j^{(n)}\xi - p_j^{(n)}|)^{-1} = 8^{\frac{1}{2}}, \quad j = 1,3,4, \tag{154}$$

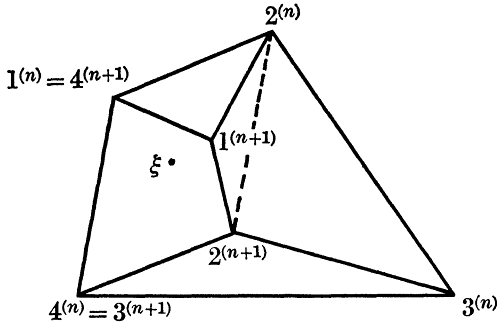

Fig. 23.

$$\lim_{n\to\infty} (|q_2^{(n)}|\,|q_2^{(n)}\xi - p_2^{(n)}|)^{-1} = 2^{\frac{1}{2}}, \tag{155}$$

which together with (153) shows that

$$C(\xi) = 8^{\frac{1}{2}}. \tag{156}$$

It follows from (61a), (60) and (71) that

$$\lim_{n\to\infty} N_1^{(n)}/N_3^{(n)} = F^2, \tag{157}$$

$$\lim_{n\to\infty} N_2^{(n)}/N_4^{(n)} = F + 1/F - 1, \tag{158}$$

where $F > 1$ is determined by

$$F + 1/F = 1 + 2^{\frac{1}{2}}. \tag{159}$$

In addition it may be shown that

$$N_1^{(n)}/N_3^{(n)} > F^2 \tag{160}$$

for infinitely many n, and hence, by (159), (160) and lemma 7 with $\Delta = 2^{\frac{1}{2}}$ and $f > f_0 = F$ applied to the corresponding Farey quadrangles in the chain, the inequality

$$(|q_j^{(n)}|\,|q_j^{(n)}\xi - p_j^{(n)}|)^{-1} > 8^{\frac{1}{2}}, \quad j = 1,3,4, \tag{161}$$

has infinitely many solutions $(p_j^{(n)}, q_j^{(n)})$.

8. Approximation spectra in cases $\mathrm{Q}(im^{\frac{1}{2}})$, $m = 1, 2, 3, 7$.

In this section we shall give a detailed investigation of the approximation spectra in the cases $\mathrm{Q}(im^{\frac{1}{2}})$, $m = 1,2,3,7$, by means of the theory of Farey triangles and Farey quadrangles developed.

$m=1$.

THEOREM 6. *Let $\xi \notin \mathrm{Q}(i)$ have the approximation constant*

$$C(\xi) < (2(2^{\frac{1}{2}}+2^{-\frac{1}{2}})-1)^{\frac{1}{2}} = 1.8007\ldots.$$

Then ξ has the approximation properties described in example 1, *especially*

$$\xi \sim \tfrac{1}{2}(1+i3^{\frac{1}{2}}) \quad and \quad C(\xi) = 3^{\frac{1}{2}}.$$

PROOF. Let $\mathrm{FT}(p_1^{(n)}/q_1^{(n)}, p_2^{(n)}/q_2^{(n)}, p_3^{(n)}/q_3^{(n)})$, $N_3^{(n)} \geqq N_2^{(n)} \geqq N_1^{(n)}$, $n \geqq 0$, be a chain of Farey triangles containing ξ. Since

$$C(\xi) < (2(2^{\frac{1}{2}}+2^{-\frac{1}{2}})-1)^{\frac{1}{2}},$$

there exists a positive integer n_0, such that

$$N_3^{(n)}/N_1^{(n)} < 2 \quad \text{for all } n \geqq n_0 \tag{162}$$

by lemma 7 with $\Delta=1$ and $f_0=2^{\frac{1}{2}}$.

For a $\mathrm{FT}(p_1/q_1, p_2/q_2, p_3/q_3)$ with $N_3 \geqq N_2 \geqq N_1$ and $N_3/N_1 < 2$ we have $N_3' \leqq N_2' \leqq N_1'$ by (36), and hence by (36) and (65)

$$\begin{aligned} N_j'/N_k \geqq N_3'/N_3 &= (N_3'+N_3)/N_3 - 1 \\ &= \tfrac{1}{3}(N'+N)/N_3 - 1 \\ &= (N+(N^2-2N^{(2)})^{\frac{1}{2}})/N_3 - 1 \\ &> \left(N+(N^2-2(\tfrac{1}{4}+\tfrac{1}{16}+\tfrac{1}{16})N^2)^{\frac{1}{2}}\right)/(\tfrac{1}{2}N) - 1 = 2 \end{aligned}$$

for $j=1,2,3$ and $k=1,2,3$. From this result and (162)

$$N_j'^{(n)}/N_k^{(n)} > 2 \quad \text{for } j = 1,2,3 \text{ and } k = 1,2,3 \text{ and for all } n \geqq n_0. \tag{163}$$

Now recall that by definition 4 and theorem 4, the $p_1^{(n+1)}/q_1^{(n+1)}$ are a selection of the $p_k^{(n)}/q_k^{(n)}$ and the $p_j'^{(n)}/q_j'^{(n)}$. Then (163) and the fact that (162) is valid for $n+1$ instead of n shows (cf. fig. 8) that only one selection is possible, i.e.

$$\begin{aligned} &\mathrm{FT}(p_1^{(n+1)}/q_1^{(n+1)}, p_2^{(n+1)}/q_2^{(n+1)}, p_3^{(n+1)}/q_3^{(n+1)}) \\ &\qquad = \mathrm{FT}(p_3'^{(n)}/q_3'^{(n)}, p_2'^{(n)}/q_2'^{(n)}, p_1'^{(n)}/q_1'^{(n)}) \end{aligned}$$

for all $n \geqq n_0$.

This proves theorem 6.

An examination of the proof of theorem 6 shows that it may be modified so as to prove the same result with a somewhat weaker condition on $C(\xi)$. However, the simple method used here is obviously insufficient to find the second minimum of the approximation spectrum.

$m = 3$.

THEOREM 7. *Let* $\xi \notin \mathrm{Q}(i3^{\frac{1}{2}})$ *have the approximation constant*

$$C(\xi) < (2(3^{\frac{1}{2}}+3^{-\frac{1}{2}})-1)^{\frac{1}{2}} = 1.9023\ldots .$$

Then ξ *has the approximation properties described in example* 4, *especially*

$$\xi \sim \frac{-\omega+(3+\omega)^{\frac{1}{2}}}{2-2\omega} \quad \textit{and} \quad C(\xi) = 13^{\frac{1}{4}} = 1.8988\ldots .$$

PROOF. Consider a $\mathrm{FT}(p_1/q_1, p_2/q_2, p_3/q_3)$ with $N_3 \geqq N_2 \geqq N_1$ and angles $A_3 \geqq A_2 \geqq A_1$. Suppose $A_3 \geqq \frac{2}{3}\pi$, then $N_3/N_1 \geqq 3$, since

$$\left(\frac{N_3}{N_1}\right)^{\frac{1}{2}} = \frac{\sin A_3}{\sin A_1} \geqq \frac{\sin 2A_1}{\sin A_1} = 2\cos A_1 \geqq 2\cos\tfrac{1}{6}\pi = 3^{\frac{1}{2}} .$$

Now let $\mathrm{FT}(p_1^{(n)}/q_1^{(n)}, p_2^{(n)}/q_2^{(n)}, p_3^{(n)}/q_3^{(n)})$, $N_3^{(n)} \geqq N_2^{(n)} \geqq N_1^{(n)}$, $n \geqq 0$, be a chain of Farey triangles containing ξ. Since

$$C(\xi) < (2(3^{\frac{1}{2}}+3^{-\frac{1}{2}})-1)^{\frac{1}{2}} ,$$

there exists a positive integer n_0 such that

$$(164) \qquad N_3^{(n)}/N_1^{(n)} < 3 \quad \text{for all } n \geqq n_0$$

by lemma 7 with $\Delta = 1$ and $f_0 = 3^{\frac{1}{2}}$. Hence by the argument above

$$(165) \qquad \tfrac{1}{3}\pi \leqq A_3^{(n)} < \tfrac{2}{3}\pi \quad \text{for all } n \geqq n_0 .$$

Next we apply the angular relation of the subdivision of $\mathrm{FT}(p_1^{(n)}/q_1^{(n)}, p_2^{(n)}/q_2^{(n)}, p_3^{(n)}/q_3^{(n)})$ (fig. 12):

$$A^{(n)} = A_3^{(n)} + \tfrac{1}{3}\pi, \quad n \geqq 0 ,$$

which together with (165) yields

$$(166) \qquad \tfrac{2}{3}\pi \leqq A^{(n)} < \pi \quad \text{for all } n \geqq n_0 .$$

By (166) and the argument above

$$(167) \qquad N_1'^{(n)}/N_1^{(n)} \geqq 3 \quad \text{for all } n \geqq n_0 .$$

By definition 4 and theorem 4 (cf. fig. 12), the $p_j^{(n+1)}/q_j^{(n+1)}$, $1 \leqq j \leqq 3$, are $p_1'^{(n)}/q_1'^{(n)}$ together with two of the $p_j^{(n)}/q_j^{(n)}$, $1 \leqq j \leqq 3$. Hence by (167) and by (164) with $n+1$ instead of n only one possibility remains:

$$\mathrm{FT}(p_1^{(n+1)}/q_1^{(n+1)}, p_2^{(n+1)}/q_2^{(n+1)}, p_3^{(n+1)}/q_3^{(n+1)}) = \mathrm{FT}(p_2^{(n)}/q_2^{(n)}, p_3^{(n)}/q_3^{(n)}, p_1'^{(n)}/q_1'^{(n)})$$

for all $n \geqq n_0$. This proves theorem 7.

$m = 2.$

The investigation of the approximation spectrum in this case is deeper and much more complicated than in the previous cases. In order to present the decisive steps of this investigation as simple as possible we shall need two preliminary lemmas.

LEMMA 8. *Let* $\mathrm{FQ}(p_1/q_1, p_2/q_2, p_3/q_3, p_4/q_4)$ *of type* $(2^{\frac12}, 2^{\frac12})$ *have*

$$N_1 \geqq N_2 \geqq N_4 \geqq N_3, \quad N_1/N_3 = \Lambda, \quad N_2/N_4 = \lambda\,, \tag{168}$$

and

$$N_j' - N_j = kN, \quad 1 \leqq j \leqq 4\,. \tag{169}$$

Suppose $\lambda \leqq \lambda_0 < 3 + 8^{\frac12}$ *and* $\Lambda \leqq \Lambda_0 < 3 + 8^{\frac12}$. *Then*

$$k = \tfrac12\{1 + (8\Lambda/(\Lambda+1)^2 + 8\lambda/(\lambda+1)^2 - 2)^{\frac12}\}\,, \tag{170}$$

and

$$k \geqq \tfrac12\{1 + (8\Lambda_0/(\Lambda_0+1)^2 + 8\lambda_0/(\lambda_0+1)^2 - 2)^{\frac12}\}\,. \tag{171}$$

PROOF. By (50), (63) and (168)

$$N_1 = \frac{\Lambda}{\Lambda+1}\frac{N}{2}, \quad N_2 = \frac{\lambda}{\lambda+1}\frac{N}{2}, \quad N_3 = \frac{1}{\Lambda+1}\frac{N}{2}, \quad N_4 = \frac{1}{\lambda+1}\frac{N}{2}, \tag{172}$$

and by (48), (68) and (169)

$$kN = \tfrac12 N + (2N^2 - 5N^{(2)} - 2(N_1N_3 + N_2N_4))^{\frac12}\,,$$

whence (170) by insertion using (72) and (172).

The inequality (171) follows from (170) since $f(x) = 8x/(x+1)^2$ is a decreasing function of x for $x \geqq 1$ and $f(3+8^{\frac12}) = 1$.

LEMMA 9. *Let* $\mathrm{FQ}(p_1/q_1, p_2/q_2, p_3/q_3, p_4/q_4)$ *of type* $(2^{\frac12}, 2^{\frac12})$ *have the notation* (168), (169) *of lemma* 8 *and further*

$$N_1'/N_3' = \Lambda'\,.$$

Then (*with the notation of fig.* 11):

$$N_1'/N_1 = 1 + 2k(1+\Lambda^{-1})\,, \tag{173}$$

$$N_2'/N_2 = 1 + 2k(1+\lambda^{-1})\,, \tag{174}$$

$$N_3'/N_3 = 1 + 2k(1+\Lambda)\,, \tag{175}$$

$$N_4'/N_4 = 1 + 2k(1+\lambda)\,, \tag{176}$$

$$\Lambda' = 1 + \frac{\Lambda-1}{1+2k(\Lambda+1)}\,, \tag{177}$$

$$N_\alpha/N_\beta \leqq N_\alpha/N_\delta = 1 + \frac{(\Lambda-1)/(\Lambda+1)}{\lambda/(\lambda+1) + k - \frac12(\Lambda-1)/(\Lambda+1)}\,. \tag{178}$$

PROOF. By (169)

$$N_j'/N_j = (N_j+kN)/N_j = 1+kN/N_j, \quad 1 \leqq j \leqq 4 ,$$

and hence (173)–(176) follow from (172). Similarly, by (169) and (172)

$$\Lambda' = \frac{N_1'}{N_3'} = \frac{N_1+kN}{N_3+kN} = 1 + \frac{N_1-N_3}{N_3+kN} = 1 + \frac{\Lambda-1}{1+2k(\Lambda+1)} .$$

Finally, by (53), (168) and a norm relation analogous to (53),

$$N_\alpha - N_\beta = N_2 - N_4 \leqq N_1 - N_3 = N_\alpha - N_\delta ,$$

and hence

$$N_\alpha/N_\beta \leqq N_\alpha/N_\delta .$$

Now by norm relations analogous to (53) and (54)

$$N_\alpha - N_\delta = N_1 - N_3 \quad \text{and} \quad N_\alpha + N_\delta = N_2 + N_2' ,$$

whence

$$\frac{N_\alpha}{N_\delta} = \frac{(N_2+N_2')+(N_1-N_3)}{(N_2+N_2')-(N_1-N_3)} = 1+\frac{2(N_1-N_3)}{(N_2+N_2')-(N_1-N_3)},$$

and hence (178) follows from (168) and (169).

This proves lemma 9.

The result of the investigation of the approximation spectrum is now described in the following two theorems.

THEOREM 8. *Let $\xi \notin \mathrm{Q}(i2^{\frac12})$ have the approximation constant $C(\xi)<\frac{7}{4}$, and suppose that ξ is contained only in a finite number of Farey triangles. Then either* (i) *ξ has the approximation properties described in example* 5, *especially*

$$\xi \sim \tfrac12 2^{\frac12}+\tfrac12\omega \quad \textit{and} \quad C(\xi) = 2^{\frac12} ,$$

or (ii) *ξ has the approximation properties described in example* 6, *especially*

$$\xi \sim \tfrac12(1+3^{\frac12})+\tfrac12\omega \quad \textit{and} \quad C(\xi) = 3^{\frac12} .$$

PROOF. By the assumption of the theorem, ξ is contained only in a finite number of Farey triangles, and hence let $FQ(p_1^{(n)}/q_1^{(n)}, p_2^{(n)}/q_2^{(n)}, p_3^{(n)}/q_3^{(n)}, p_4^{(n)}/q_4^{(n)})$, $N_1^{(n)} \geqq N_2^{(n)} \geqq N_4^{(n)} \geqq N_3^{(n)}$, $n \geqq 0$, be a chain of Farey quadrangles of type $(2^{\frac12}, 2^{\frac12})$ containing ξ. Since

$$C(\xi) < \tfrac{7}{4} < (2(2.045+1/2.045)-2)^{\frac12} = 1.751\ldots ,$$

there exists a positive integer n_0, such that

(179) $$\Lambda^{(n)} = N_1^{(n)}/N_3^{(n)} < 2.045^2 < 4.2 \quad \text{for all } n \geqq n_0$$

by lemma 7 with $\Delta = 2^{\frac{1}{2}}$ and $f_0 = 2.045$ applied to the two triangles

$$T(p_1^{(n)}/q_1^{(n)}, p_2^{(n)}/q_2^{(n)}, p_3^{(n)}/q_3^{(n)}) \quad \text{and} \quad T(p_1^{(n)}/q_1^{(n)}, p_4^{(n)}/q_4^{(n)}, p_3^{(n)}/q_3^{(n)}) .$$

By (179) and the inequality (171) of lemma 8

(180) $$k^{(n)} \geqq \tfrac{1}{2}\{1+(2\times 8\times 4.2\times 5.2^{-2}-2)^{\frac{1}{2}}\} > 0.84 \quad \text{for all } n \geqq n_0 ,$$

whence by (175) and (176)

$$\begin{aligned} N_3'^{(n)}/N_3^{(n)} \geqq N_4'^{(n)}/N_4^{(n)} &= 1+2k^{(n)}(1+\lambda^{(n)}) \\ &\geqq 1+4k^{(n)} \\ &> 1+4\times 0.84 = 4.36 > 4.2 \end{aligned}$$

for all $n \geqq n_0$. This together with (179) (cf. fig. 11) shows that

$$\mathrm{FQ}(p_1^{(n+1)}/q_1^{(n+1)}, p_2^{(n+1)}/q_2^{(n+1)}, p_3^{(n+1)}/q_3^{(n+1)}, p_4^{(n+1)}/q_4^{(n+1)})$$

cannot have $p_3^{(n)}/q_3^{(n)}$ or $p_4^{(n)}/q_4^{(n)}$ as vertices for $n \geqq n_0$. Further by (177), (179) and (180)

$$\Lambda'^{(n)} < 1+\frac{3.2}{1+2\times 0.84\times 5.2} < \tfrac{4}{3} \quad \text{for all } n \geqq n_0 ,$$

and hence by (171)

$$k'^{(n)} > \tfrac{1}{2}\{1+\left(2\times 8\times \tfrac{4}{3}\times (\tfrac{7}{3})^{-2}-2\right)^{\frac{1}{2}}\} > 1.19 \quad \text{for all } n \geqq n_0 .$$

Consequently by (173)–(176)

$$N_j''^{(n)}/N_j'^{(n)} > 1+2\times 1.19\times (1+\tfrac{3}{4}) > 5.1 > 4.2, \quad 1 \leqq j \leqq 4 ,$$

for all $n \geqq n_0$, and hence if for some $n \geqq n_0$

$$\begin{aligned} &\mathrm{FQ}(p_1^{(n+1)}/q_1^{(n+1)}, p_2^{(n+1)}/q_2^{(n+1)}, p_3^{(n+1)}/q_3^{(n+1)}, p_4^{(n+1)}/q_4^{(n+1)}) \\ &\qquad = \mathrm{FQ}(p_1'^{(n)}/q_1'^{(n)}, p_2'^{(n)}/q_2'^{(n)}, p_3'^{(n)}/q_3'^{(n)}, p_4'^{(n)}/q_4'^{(n)}) , \end{aligned}$$

we must have case (i) in theorem 8.

In the continuation we shall suppose that case (i) is not present, and hence by the arguments above

(181) $$\mathrm{FT}(p_1^{(n+1)}/q_1^{(n+1)}, p_2^{(n+1)}/q_2^{(n+1)}, p_3^{(n+1)}/q_3^{(n+1)}, p_4^{(n+1)}/q_4^{(n+1)})$$

has $p_1^{(n)}/q_1^{(n)}$ or $p_2^{(n)}/q_2^{(n)}$ as a vertex for all $n \geqq n_0$.

In both cases it follows from (178), (179) and (180) that

(182) $$\lambda^{(n+1)} \leqq \frac{1+\frac{3.2}{5.2}}{\frac{1}{2}+0.84-\frac{1}{2}\,\frac{3.2}{5.2}} < 1.6 \quad \text{for all } n \geqq n_0 ,$$

and hence by (171), (179) and (182)

$$(183)\qquad k^{(n+1)} \geqq \tfrac{1}{2}\{1+(8\times 4.2\times 5.2^{-2}+8\times 1.6\times 2.6^{-2}-2)^{\frac{1}{2}}\} > 1$$

for all $n \geqq n_0$. Now, if $p_2^{(n)}/q_2^{(n)}$ is a vertex of (181),

$$(184)\qquad \Lambda^{(n+1)} = 1+2k^{(n)}(1+\lambda^{(n)^{-1}}) > 1+2(1+1.6^{-1}) > 4.2$$

for all $n \geqq n_0+1$, by (174), (182) and (183). By (179) and (184), $p_2^{(n)}/q_2^{(n)}$ can be a vertex of (181) only for $n=n_0$, and consequently we must have case (ii) in theorem 8.

This proves theorem 8.

THEOREM 9. *Let* $\xi \notin \mathrm{Q}(i\,2^{\frac{1}{2}})$ *have the approximation constant* $C(\xi) <$ 1.733, *and suppose that* ξ *is contained in an infinite number of Farey triangles. Then* ξ *has the approximation properties described in example* 2, *especially*

$$\xi \sim \tfrac{1}{2}(1+i3^{\frac{1}{2}}) \quad and \quad C(\xi) = 3^{\frac{1}{2}}.$$

PROOF. Since

$$C(\xi) < 1.733 < \big(2(1.0414+1/1.0414)-1\big)^{\frac{1}{2}} = 1.7330007\ldots,$$

there exists a positive integer n_0 such that any $\mathrm{FT}(p_1/q_1, p_2/q_2, p_3/q_3)$, $N_3 \geqq N_2 \geqq N_1$, with $N \geqq n_0$ and containing ξ, satisfies the inequalities

$$(185)\qquad c_j = (N_j d(j,\xi))^{-1} < 1.733 \quad \text{for } 1 \leqq j \leqq 3,$$

where $d(j,\xi) = |p_j/q_j - \xi|$, and hence

$$(186)\qquad \Lambda = N_3/N_1 < 1.0414^2 < 1.085$$

by lemma 7 with $\Delta = 1$ and $f_0 = 1.0414$. Further, since ξ is contained in an infinite number of Farey triangles, there exists a $\mathrm{FT}(p_1/q_1, p_2/q_2, p_3/q_3)$, $N_3 \geqq N_2 \geqq N_1$, with $N \geqq n_0$ and containing ξ. For any Farey triangle having those properties

$$\begin{aligned} N' &\geqq 5N+6\big(2N^2-4(\Lambda^2+2)(\Lambda+2)^{-2}N^2\big)^{\frac{1}{2}} \\ &> N\{5+6\big(2-4(1.085^2+2)3.085^{-2}\big)^{\frac{1}{2}}\} > 9.8\,N \end{aligned}$$

by (67), (72) and (186). Hence by (41) and (186)

$$\begin{aligned} N_3'/N_3 = (N_3'+N_3)/N_3-1 &= \tfrac{1}{3}(N'+N)/N_3 - 1 \\ &> 3.6N/(1.085N/3.085) - 1 > 9.2\,, \end{aligned}$$

whence

$$(187)\qquad (3N_3/N_3')^{\frac{1}{2}} < \tfrac{4}{7}\,.$$

Now from (18), table 3, (187), the inequalities $N_3 \geqq N_2 \geqq N_1$ and the corresponding inequalities $N_3' \leqq N_2' \leqq N_1'$ arising from (41) we deduce

$$d(1,2') \leqq d(1,3') = (3/(N_1N_3'))^{\frac{1}{2}} < \tfrac{4}{7}N_1^{-1}\,,$$
$$d(2,1') \leqq d(2,3') = (3/(N_2N_3'))^{\frac{1}{2}} < \tfrac{4}{7}N_2^{-1}\,,$$
$$d(3,1') \leqq d(3,2') = (3/(N_3N_2'))^{\frac{1}{2}} < \tfrac{4}{7}N_3^{-1}\,,$$

where $d(j,k') = |p_j/q_j - p_k'/q_k'|$, $j \neq k$.

It follows from these inequalities and from (185) (cf. fig. 10) that ξ is an interior point of $\mathrm{FT}(p_1'/q_1', p_2'/q_2', p_3'/q_3')$, and since $N' > 9.8N > N \geqq n_0$, it follows by induction that ξ is contained in a chain of Farey triangles

$$\mathrm{FT}(p_1^{(n)}/q_1^{(n)}, p_2^{(n)}/q_2^{(n)}, p_3^{(n)}/q_3^{(n)}), \quad n \geqq 0\,,$$

beginning with $\mathrm{FT}(p_1/q_1, p_2/q_2, p_3/q_3)$ and having

$$\mathrm{FT}(p_1^{(n+1)}/q_1^{(n+1)}, p_2^{(n+1)}/q_2^{(n+1)}, p_3^{(n+1)}/q_3^{(n+1)}) = \mathrm{FT}(p_3'^{(n)}/q_3'^{(n)}, p_2'^{(n)}/q_2'^{(n)}, p_1'^{(n)}/q_1'^{(n)})$$

for all $n \geqq 0$.

This proves theorem 9.

$m = 7$.

The result of the investigation of the approximation spectrum is described in the following two theorems.

THEOREM 10. *Let* $\xi \notin \mathrm{Q}(i\,7^{\frac{1}{2}})$ *have the approximation constant*

$$C(\xi) < (2\cdot 7^{\frac{1}{2}} - 2)^{\frac{1}{2}} = 1.8142\ldots\,,$$

and suppose that ξ *is contained only in a finite number of Farey triangles. Then* ξ *has the approximation properties described in example* 7, *especially*

$$\xi \sim \tfrac{1}{2}(1 + \omega + (3-\omega)^{\frac{1}{2}}) \quad \textit{and} \quad C(\xi) = 8^{\frac{1}{4}}\,.$$

PROOF. By the assumption of the theorem, ξ is contained only in a finite number of Farey triangles, and hence let $\mathrm{FQ}(p_1^{(n)}/q_1^{(n)}, p_2^{(n)}/q_2^{(n)}, p_3^{(n)}/q_3^{(n)}, p_4^{(n)}/q_4^{(n)})$, $N_1^{(n)} \geqq N_2^{(n)} \geqq N_4^{(n)} \geqq N_3^{(n)}$, $n \geqq 0$, be a chain of Farey quadrangles of type $(2^{\frac{1}{2}}, 2^{\frac{1}{2}})$ containing ξ. Since

$$C(\xi) < (2\cdot 7^{\frac{1}{2}} - 2)^{\frac{1}{2}} = \left(2\left(\tfrac{1}{2}(7^{\frac{1}{2}} + 3^{\frac{1}{2}}) + \tfrac{1}{2}(7^{\frac{1}{2}} - 3^{\frac{1}{2}})\right) - 2\right)^{\frac{1}{2}},$$

there exists a positive integer n_0 such that

$$(188) \qquad \Lambda^{(n)} = N_1^{(n)}/N_3^{(n)} < \tfrac{1}{4}(7^{\frac{1}{2}} + 3^{\frac{1}{2}})^2 = \tfrac{1}{2}(5 + 21^{\frac{1}{2}}) = \Lambda_0$$

for all $n \geqq n_0$ by lemma 7 with $\Delta = 2^{\frac{1}{2}}$ and $f_0 = \frac{1}{2}(7^{\frac{1}{2}} + 3^{\frac{1}{2}})$ applied to the two triangles

$$T(p_1^{(n)}/q_1^{(n)}, p_2^{(n)}/q_2^{(n)}, p_3^{(n)}/q_3^{(n)}) \quad \text{and} \quad T(p_1^{(n)}/q_1^{(n)}, p_4^{(n)}/q_4^{(n)}, p_3^{(n)}/q_3^{(n)}) .$$

Now we distinguish between the two cases a) and b) according as the subdivision of $\mathrm{FQ}(p_1^{(n)}/q_1^{(n)}, p_2^{(n)}/q_2^{(n)}, p_3^{(n)}/q_3^{(n)}, p_4^{(n)}/q_4^{(n)})$ is as in fig. 14a or as in fig. 14b. For reasons of simplicity we shall leave out the index (n) in these considerations.

a) Let $N_2/N_4 = \lambda$, then by (71), (72) and (172)

$$(189) \qquad N_\alpha + N_\beta = \tfrac{1}{4}N\{3 + 7^{\frac{1}{2}}(4\Lambda(\Lambda+1)^{-2} + 4\lambda(\lambda+1)^{-2} - 1)^{\frac{1}{2}}\} ,$$

and by (61a) and (172)

$$(190) \qquad N_\alpha - N_\beta = N_1 - N_4 = \left(\frac{\Lambda}{\Lambda+1} - \frac{1}{\lambda+1}\right)\frac{N}{2}.$$

By (189), (190), two concavity inequalities, and $1 \leqq \lambda \leqq \Lambda \leqq \Lambda_0$,

$$\begin{aligned}(191) \quad N_\alpha &= \tfrac{1}{4}N\{\tfrac{3}{2} + \tfrac{1}{2}7^{\frac{1}{2}}(4\Lambda(\Lambda+1)^{-2} + 4\lambda(\lambda+1)^{-2} - 1)^{\frac{1}{2}} + \\ &\qquad + \Lambda/(\Lambda+1) - 1/(\lambda+1)\} \\ &\geqq \tfrac{1}{4}N \min[\{\tfrac{3}{2} + \tfrac{1}{2}7^{\frac{1}{2}}(8\Lambda(\Lambda+1)^{-2} - 1)^{\frac{1}{2}} + (\Lambda-1)/(\Lambda+1)\} , \\ &\qquad \{\tfrac{3}{2} + \tfrac{1}{2}7^{\frac{1}{2}}(4\Lambda(\Lambda+1)^{-2})^{\frac{1}{2}} + \Lambda/(\Lambda+1) - \tfrac{1}{2}\}] \\ &\geqq \tfrac{1}{4}N \min[\tfrac{3}{2} + \tfrac{1}{2}7^{\frac{1}{2}}, \tfrac{5}{2} + \tfrac{1}{14}21^{\frac{1}{2}}, 2 + \tfrac{1}{7}21^{\frac{1}{2}}] \\ &= \tfrac{1}{4}N(2 + \tfrac{1}{7}21^{\frac{1}{2}}) .\end{aligned}$$

Hence by (172) and (188)

$$N_\alpha/N_1 > \tfrac{1}{4}N(2 + \tfrac{1}{7}21^{\frac{1}{2}})/(\tfrac{1}{2}N\Lambda_0/(\Lambda_0+1)) = \tfrac{1}{4}(11 - 21^{\frac{1}{2}}) = 1.60\ldots .$$

Since ξ is contained in one of the two Farey quadrangles (cf. fig. 14a) $\mathrm{FQ}(p_\alpha/q_\alpha, p_\beta/q_\beta, p_4/q_4, p_1/q_1)$ or $\mathrm{FQ}(p_\alpha/q_\alpha, p_\beta/q_\beta, p_3/q_3, p_2/q_2)$, we have $N_\alpha/N_4 < \Lambda_0$ by (188), and hence

$$\begin{aligned}\lambda = N_2/N_4 &= (N_\alpha/N_4)(N_2/N_\alpha) \\ &\leqq (N_\alpha/N_4)(N_1/N_\alpha) \\ &< \tfrac{1}{2}(5 + 21^{\frac{1}{2}})/(\tfrac{1}{4}(11 - 21^{\frac{1}{2}})) \\ &= \tfrac{1}{25}(38 + 8\cdot 21^{\frac{1}{2}}) = 2.98\ldots < 3 .\end{aligned}$$

Consequently, since $\lambda < 3$, we obtain by (191), (188), two concavity inequalities, and $1 \leqq \lambda \leqq 3$, $\lambda \leqq \Lambda \leqq \Lambda_0$,

$$\begin{aligned}N_\alpha &\geqq \tfrac{1}{4}N \min[\{\tfrac{3}{2} + \tfrac{1}{2}7^{\frac{1}{2}}(4\Lambda(\Lambda+1)^{-2})^{\frac{1}{2}} + \Lambda/(\Lambda+1) - \tfrac{1}{2}\} , \\ &\qquad \{\tfrac{3}{2} + \tfrac{1}{2}7^{\frac{1}{2}}(4\Lambda(\Lambda+1)^{-2} - \tfrac{1}{4})^{\frac{1}{2}} + \Lambda/(\Lambda+1) - \tfrac{1}{4}\}] \\ &\geqq \tfrac{1}{4}N \min\{\tfrac{3}{2} + \tfrac{1}{2}7^{\frac{1}{2}}, \tfrac{5}{2} + \tfrac{1}{14}21^{\frac{1}{2}}, 2 + \tfrac{1}{4}14^{\frac{1}{2}}, \tfrac{7}{4} + \tfrac{1}{4}11^{\frac{1}{2}} + \tfrac{1}{14}21^{\frac{1}{2}}\} ,\end{aligned}$$

so that

(192) $$N_\alpha \geqq \tfrac14 N(\tfrac32+\tfrac12 7^{\frac12}) .$$

Also, since $\lambda < 3$, we obtain by (189), (190) and (188)

$$2N_\beta > N\{\tfrac34+\tfrac14 7^{\frac12}(4\Lambda_0(\Lambda_0+1)^{-2}-\tfrac14)^{\frac12}+\tfrac12(\tfrac14-\Lambda_0/(\Lambda_0+1))\} ,$$

so that

(193) $$2N_\beta > N(1-\tfrac{1}{28}21^{\frac12}) = N\times 0.836\ldots .$$

Now by (172) and (188)

(194) $$\begin{aligned} 2N_1 &= N\Lambda/(\Lambda+1) \\ &< N\Lambda_0/(\Lambda_0+1) \\ &= N(\tfrac12+\tfrac{1}{14}21^{\frac12}) = N\times 0.827\ldots , \end{aligned}$$

and hence by (193) and (194)

(195) $$N_\beta > N_1 .$$

b) By (189) and (188)

$$N_\alpha+N_\beta > N\{\tfrac34+\tfrac14 7^{\frac12}(8\Lambda_0(\Lambda_0+1)^{-2}-1)^{\frac12}\} = N ,$$

and by (61b), (172) and (188)

$$\begin{aligned} N_\beta-N_\alpha = N_2-N_1 &= \tfrac12 N(\lambda/(\lambda+1)-\Lambda/(\Lambda+1)) \\ &> \tfrac12 N(\tfrac12-\Lambda_0/(\Lambda_0+1)) , \end{aligned}$$

so that the inequality (193) is valid in this case also. However, since the inequality (194) holds in both cases, the inequality (195) is also valid in this case.

From the validity of (195) in both cases a) and b) we conclude that in any case

$$p_1^{(n+1)}/q_1^{(n+1)} = p_\alpha^{(n)}/q_\alpha^{(n)}, \quad p_2^{(n+1)}/q_2^{(n+1)} = p_\beta^{(n)}/q_\beta^{(n)}$$

for all $n \geqq n_0$, and hence figs. 14a and 14b show that the subdivision of $\mathrm{FQ}(p_1^{(n)}/q_1^{(n)}, p_2^{(n)}/q_2^{(n)}, p_3^{(n)}/q_3^{(n)}, p_4^{(n)}/q_4^{(n)})$ is as in fig. 14a for all $n \geqq n_0+1$. Consequently by (192) and (172)

$$\Lambda^{(n+1)} = N_1^{(n+1)}/N_3^{(n+1)} \geqq N_\alpha^{(n)}/N_4^{(n)} > \tfrac14 N(\tfrac32+\tfrac12 7^{\frac12})/(\tfrac14 N) = \tfrac32+\tfrac12 7^{\frac12}$$

for all $n \geqq n_0+1$. On the other hand, by (192), (172) and (188)

$$N_\alpha^{(n+1)}/N_1^{(n+1)} > \tfrac14 N(\tfrac32+\tfrac12 7^{\frac12})/(\tfrac12 N\Lambda_0/(\Lambda_0+1)) = \tfrac18(3+7^{\frac12})(7-21^{\frac12}) ,$$

for all $n \geqq n_0$, and hence

$$\begin{aligned} N_\alpha^{(n+1)}/N_3^{(n+1)} &= (N_\alpha^{(n+1)}/N_1^{(n+1)})(N_1^{(n+1)}/N_3^{(n+1)}) \\ &> \tfrac{1}{16}(3+7^{\frac12})^2(7-21^{\frac12}) = 4.81\ldots > \Lambda_0 = 4.79\ldots \end{aligned}$$

for all $n \geqq n_0+1$. By (188) this implies (cf. fig. 14a) that

$$\mathrm{FQ}(p_1^{(n+1)}/q_1^{(n+1)}, p_2^{(n+1)}/q_2^{(n+1)}, p_3^{(n+1)}/q_3^{(n+1)}, p_4^{(n+1)}/q_4^{(n+1)}) = \mathrm{FQ}(p_\alpha^{(n)}/q_\alpha^{(n)}, p_\beta^{(n)}/q_\beta^{(n)}, p_4^{(n)}/q_4^{(n)}, p_1^{(n)}/q_1^{(n)})$$

for all $n \geqq n_0 + 2$.

This proves theorem 10.

THEOREM 11. *Let $\xi \notin \mathsf{Q}(i7^{\frac12})$ have the approximation constant $C(\xi) < \frac74$, and suppose that ξ is contained in an infinite number of Farey triangles. Then ξ has the approximation properties described in example* 3, *especially*

$$\xi \sim \tfrac12(1+i3^{\frac12}) \quad \text{and} \quad C(\xi) = 3^{\frac12}.$$

PROOF. Since

$$C(\xi) < \tfrac74 < (2(2.045+1/2.045)-2)^{\frac12} = 1.751\ldots,$$

there exists a positive integer n_0 such that any Farey quadrangle of type $(2^{\frac12}, 2^{\frac12})$, $\mathrm{FQ}(p_1/q_1, p_2/q_2, p_3/q_3, p_4/q_4)$, $N_1 \geqq N_2 \geqq N_4 \geqq N_3$, with $N \geqq n_0$ and containing ξ, satisfies the inequality

$$(196) \qquad \Lambda_Q = N_1/N_3 < 2.045^2 < 4.2$$

by lemma 7 with $\Delta = 2^{\frac12}$ and $f_0 = 2.045$ applied to the two triangles

$$T(p_1/q_1, p_2/q_2, p_3/q_3) \quad \text{and} \quad T(p_1/q_1, p_4/q_4, p_3/q_3).$$

Similarly, since

$$C(\xi) < \tfrac74 < (2(\sqrt{\tfrac32}+\sqrt{\tfrac23})-1)^{\frac12} = 1.755\ldots,$$

n_0 may be chosen, such that also any Farey triangle $\mathrm{FT}(p_1/q_1, p_2/q_2, p_3/q_3)$, $N_3 \geqq N_2 \geqq N_1$, with $N \geqq n_0$ and containing ξ, satisfies the inequality

$$(197) \qquad \Lambda_T = N_3/N_1 < \tfrac32$$

by lemma 7 with $\Delta = 1$ and $f_0 = \sqrt{\tfrac32}$.

Further, since ξ is contained in an infinite number of Farey triangles, there exists a $\mathrm{FT}(p_1/q_1, p_2/q_2, p_3/q_3)$, $N_3 \geqq N_2 \geqq N$, with $N \geqq n_0$ and containing ξ. For any Farey triangle having those properties

$$N' \geqq \tfrac52 N + \tfrac32 7^{\frac12}(N^2 - 2(\Lambda_T^2+2)(\Lambda_T+2)^{-2}N^2)^{\frac12} > N\{\tfrac52 + \tfrac32 7^{\frac12}(1-2((\tfrac32)^2+2)(\tfrac32+2)^{-2})^{\frac12}\} > 4.6N$$

by (70), (72) and (197). Hence by (57) and (197)

$$N_j'/N_j = (N_j'-N_j)/N_j + 1 = \tfrac13(N'-N)/N_j + 1 > 1.2N/(\tfrac32 N/4) + 1 = 4.2$$

for $j = 1, 2$. From this result, the inequalities $N_3 \geqq N_2 \geqq N_1$ and the corresponding inequalities $N_3' \geqq N_2' \geqq N_1'$ arising from (57) we deduce

(198) $N_3'/N_1 > 4.2, \quad N_3'/N_2 > 4.2, \quad N_2'/N_1 > 4.2\,.$

It follows from (196) and (198) (cf. fig. 13) that ξ is an interior point of $\mathrm{FT}(p_1'/q_1', p_2'/q_2', p_3'/q_3')$, and since $N' > 4.6N > n_0$, it follows by induction that ξ is contained in the chain of Farey triangles beginning with $\mathrm{FT}(p_1/q_1, p_2/q_2, p_3/q_3)$.

This proves theorem 11.

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