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A NOTE ON THE HOMOLOGY OF LOCAL RINGS

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In [5] Tate showed how his technique of adjoining variables to R-algebras could be used to define a canonical algebra structure on $\mathrm{Tor}^R(R/\mathfrak{a}, R/\mathfrak{c})$, $\mathfrak{a}$ and $\mathfrak{c}$ being ideals in a Noetherian ring R.

The present note contains a theorem concerning the algebra $\mathrm{Tor}^R(k,k)$, R being a local ring with residue class field $k = R/\mathfrak{m}$. As a corollary we may conclude that the sequence of Betti-numbers of a non-regular local ring R is non-decreasing. Moreover we can prove that the Betti-numbers are bounded if and only if

$$\operatorname{codim} R \geqq \dim_k(\mathfrak{m}/\mathfrak{m}^2) - 1 .$$

We also extend a result due to Scheja [3] concerning the change in the homology when R is divided by a non-zerodivisor not contained in $\mathfrak{m}^2$.

The present note is part of my cand. real. thesis [1]. I wish to thank professor O. A. Laudal for his help and advice during the preparation of the thesis.

Notations. Throughout this note R denotes a local, Noetherian ring with residue class field $k = R/\mathfrak{m}$. The vectorspace dimensions $\dim_k \mathrm{Tor}_p{}^R(k,k)$ are called the Betti-numbers of R. They are denoted by $b_p(R)$. The Betti-series is defined to be the formal power series

$$B(R) = \sum_{p=0}^{\infty} b_p(R) Z^p .$$

The term "R-algebra" will be used in the sense of [5] i.e. graded, differential, strictly skew-commutative, connected algebra over R, such that the homogeneous components are finitely generated modules over R, trivial in negative degrees. We shall use (cf. [5]) the symbol

$$X\langle T \rangle ;\, dT = t$$

to denote the R-algebra obtained from an R-algebra X by "adjoining a variable" T of degree w killing a cycle t of degree $w-1$. We recall that the adjunction of a variable T leads to an exact homology sequence.

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It should be noted that $\mathrm{Tor}^R(R/\mathfrak{a}, R/\mathfrak{c})$ may be regarded as an R-algebra with a trivial differential.

THEOREM 1. *Let R be a local ring. Let $t_1,\ldots,t_n$ be a minimal generating system for $\mathfrak{m}$. Assume that $n>0$ and let $\mathfrak{a}=(t_1,\ldots,t_{n-1})$. Then we have one of the following canonical isomorphisms of graded algebras*

(i) *If $\mathfrak{a}$ is a prime ideal, then*

$$\mathrm{Tor}^R(k,k) \approx \mathrm{Tor}^R(R/\mathfrak{a},k)\langle T\rangle;\ dT=0 \tag{1}$$

with T of degree 1.

(ii) *If $\mathfrak{a}$ is non-prime, then*

$$\mathrm{Tor}^R(k,k) \approx \mathrm{Tor}^R(R/\mathfrak{a},k)\langle T,S\rangle;\ dT=0,\ dS=0 \tag{2}$$

with T of degree 1, S of degree 2. In particular $\mathrm{Tor}^R(k,k)$ is canonically isomorphic as a graded k-module to the graded ring of polynomials $\mathrm{Tor}^R(R/\mathfrak{a},k)[Z]$ in one variable Z of degree 1.

PROOF. Consider the R-algebra

$$E = R\langle T_1,\ldots,T_{n-1}\rangle;\ dT_i=t_i\,. \tag{3}$$

If $n=1$, E is taken to be R considered as an R-algebra. Adjoin to E sufficiently many variables of degree $\geqq 2$ to kill the cycles of positive degrees. In that way one obtains an acyclic R-algebra X such that $H_0(X)=R/\mathfrak{a}$. Now adjoin a variable T of degree 1 to kill t_n. Put

$$Y = X\langle T\rangle;\ dT=t_n\,.$$

Consider the exact homology sequence associated to the adjunction of T

$$\ldots \to H_{i+1}(X) \to H_{i+1}(Y) \to H_i(X) \to \ldots\,, \tag{4}$$

$$\begin{array}{ccccccccccc} \ldots \to & H_1(X) & \to H_1(Y) \overset{j_{*1}}{\to} & H_0(X) & \overset{d_{*0}}{\to} & H_0(X) & \to & H_0(Y) & \to 0\,. \\ & \| & & \| & & \| & & \| & \\ & 0 & & R/\mathfrak{a} & & R/\mathfrak{a} & & k & \end{array} \tag{5}$$

Since X is acyclic, it follows from (4) that

$$H_i(Y) = 0 \quad \text{for } i\geqq 2\,. \tag{6}$$

We observe that d_{*0} is multiplication by t_n. If therefore $\mathfrak{a}$ is a prime ideal, then d_{*0} is injective. From (5) then follows $H_1(Y)=0$, so Y is acyclic. Tensoring with k commutes with the adjunction of T. Therefore we have canonical isomorphisms of graded algebras

$$\mathrm{Tor}^R(k,k) \approx H(Y\otimes_R k) \approx H(X\otimes_R k\langle T\rangle;\ dT=t_n\otimes 1)\,.$$

Since $t_n \otimes 1 = 0$ this is further isomorphic to

$$H(X \otimes_R k)\langle T\rangle; dT = 0 \approx \operatorname{Tor}^R(R/\mathfrak{a}, k)\langle T\rangle; dT = 0 .$$

We conclude (i).

Now suppose that $\mathfrak{a}$ is non-prime. Then $R/\mathfrak{a}$ is a non-regular ring, so $\dim(R/\mathfrak{a}) = 0$. Since $R/\mathfrak{a}$ is artinian, it follows from (5), considering lengths, that $H_1(Y) \approx k$. Let σ be a homology class generating $H_1(Y)$. Let s be a cycle representing σ. Put

$$L = Y\langle S\rangle; dS = s .$$

From (6) it follows that σ is a skew non-zerodivisor in $H(Y)$, so by [5, theorem 2] L is acyclic. Therefore

$$\operatorname{Tor}^R(k,k) \approx H(L \otimes_R k) \approx H(X \otimes_R k\langle T, S\rangle; dT = t_n \otimes 1, dS = s \otimes 1) .$$

Again $t_n \otimes 1 = 0$. Since $t_1, \ldots, t_n$ are linearly independent modulo $\mathfrak{m}^2$, we have

$$Z_1(Y) \subseteq \mathfrak{m} Y_1, \quad \text{so} \quad s \otimes 1 = 0 .$$

Hence the adjunction of T and S commutes with H. It follows that

$$\begin{aligned} \operatorname{Tor}^R(k,k) &\approx H(X \otimes_R k)\langle T, S\rangle; dT = 0, dS = 0 \\ &\approx \operatorname{Tor}^R(R/\mathfrak{a}, k)\langle T, S\rangle; dT = 0, dS = 0 . \end{aligned}$$

The rest of (ii) now follows easily from (2).

Corollary 2. *Let $\mathfrak{a}$ be as in theorem* 1. *Put $c_i = \dim \operatorname{Tor}_i^R(R/\mathfrak{a}, k)$*

(i) *If $\mathfrak{a}$ is a prime ideal then $b_p(R) = c_p + c_{p-1}$.*
(ii) *If $\mathfrak{a}$ is non-prime then $b_p(R) = \sum_{i=0}^{p} c_i$.*

Corollary 3. *Let R be a non-regular local ring. Put $n = \dim(\mathfrak{m}/\mathfrak{m}^2)$. Then*

(i) $B(R) \geqslant \dfrac{(1+Z)^n}{1-Z^2}$ *(Tate)*

(ii) *The sequence $\{b_p(R)\}$ is non-decreasing.*

Proof. Let d denote the dimension of R. From a theorem of Murthy's [2] it follows that there exists a minimal generating system $t_1, \ldots, t_n$ for $\mathfrak{m}$ such that rank $(t_1, \ldots, t_d) = d$. Put

(7) $$\mathfrak{a} = (t_1, \ldots, t_{n-1}) .$$

Since R is non-regular, we have

$$d \leqq n-1 .$$

It follows that $\mathfrak{a}$ is non-prime. Now (ii) follows from (ii) of corollary 2.

The R-algebra E in (3) can be regarded as an exterior algebra with a differential. It follows from [4, appendice I] that E can be imbedded (as a graded R-module) as a direct factor of a minimal resolution of $R/\mathfrak{a}$. It follows that

$$\dim \mathrm{Tor}_i^R(R/\mathfrak{a},k) \geqq \dim E_i \otimes_R k = \binom{n-1}{i}.$$

Now (i) follows from (ii) of corollary 2.

Lemma 4. *Let $t_1,\ldots,t_n$ be a minimal generating system for $\mathfrak{m}$. Let $n>0$ and assume that t_n is a non-zerodivisor. Put $\mathfrak{a}=(t_1,\ldots,t_{n-1})$ and $\bar{R}=R/(t_n)$. Then we have canonical isomorphisms of graded algebras:*

(i) *If $\mathfrak{a}$ is a prime ideal*

$$\mathrm{Tor}^{\bar{R}}(k,k) \approx \mathrm{Tor}^R(R/\mathfrak{a},k) .$$

(ii) *If $\mathfrak{a}$ is non-prime*

$$\mathrm{Tor}^{\bar{R}}(k,k) \approx \mathrm{Tor}^R(R/\mathfrak{a},k)\langle S\rangle ;\ dS=0$$

with S of degree 2.

Proof. Consider the free acyclic R-algebra X in the proof of theorem 1. We have an exact sequence of complexes

$$(8) \qquad 0 \to X \xrightarrow{t_n} X \to \bar{X} \to 0, \quad \text{where} \quad \bar{X} = X/t_nX$$

has a canonical structure as an R-algebra. Consider the homology-sequence associated with (8).

$$(9) \qquad \ldots \to H_{i+1}(X) \to H_{i+1}(\bar{X}) \to H_i(X) \to \ldots ,$$

$$(10) \qquad \ldots \to \underset{\underset{0}{\|}}{H_1(X)} \to H_1(\bar{X}) \to \underset{\underset{R/\mathfrak{a}}{\|}}{H_0(X)} \xrightarrow{t_n} \underset{\underset{R/\mathfrak{a}}{\|}}{H_0(X)} \to \underset{\underset{k}{\|}}{H_0(\bar{X})} \to 0 .$$

We observe that (9) and (10) are analogous to (4) and (5). We now use arguments similar to those used in the proof of theorem 1. If $\mathfrak{a}$ is prime, then $\bar{X}$ is acyclic and (i) follows. If $\mathfrak{a}$ is non-prime one finds that $H_1(\bar{X})$ is generated by a single homology class $\bar{\sigma}$, represented by a cycle $\bar{s} \in \mathfrak{m}\bar{X}_1$. Again according to [5, theorem 2] the R-algebra

$$L = \bar{X}\langle S\rangle ;\ dS=\bar{s}$$

is acyclic and we have

$$\mathrm{Tor}^{\bar{R}}(k,k) \approx H(L\otimes_{\bar{R}}k) \approx H(X\otimes_R k)\langle S\rangle ;\ dS=0 .$$

From lemma 4 and theorem 1 now follows

THEOREM 5. *Let t be a non-zerodivisor in a local ring R. Assume that $t \in \mathfrak{m}$, $t \notin \mathfrak{m}^2$. Put $\bar{R} = R/(t)$. Then we have an isomorphism of graded algebras,*

$$\mathrm{Tor}^R(k,k) \approx \mathrm{Tor}^{\bar{R}}(k,k)\langle T \rangle ; \, dT = 0 \, ,$$

where T is a variable of degree 1. In particular

$$B(R) = (1+Z)B(\bar{R}) \qquad (\textit{Scheja}) \, .$$

PROPOSITION 6. *The following conditions are equivalent*

(i) *The sequence $\{b_p(R)\}$ is bounded.*
(ii) $\operatorname{codim} R \geqq \dim(\mathfrak{m}/\mathfrak{m}^2) - 1$.

PROOF. We observe that if t is a non-zerodivisor in R such that

$$t \in \mathfrak{m}, \quad t \notin \mathfrak{m}^2, \quad \bar{R} = R/(t) \, ,$$

then by theorem 5 R satisfies (i) (resp. (ii)) if and only if $\bar{R}$ does. On the other hand if $\operatorname{codim} R > 0$ there is by [2, corollary 3] such a non-zerodivisor t not contained in $\mathfrak{m}^2$. Therefore, dividing R by a suitable R-sequence, there is no loss of generality to assume that $\operatorname{codim} R = 0$. In this case the implication (ii) $\Rightarrow$ (i) is almost obvious, see [3], so we show that (i) implies (ii). Assume that

$$\operatorname{codim} R < \dim(\mathfrak{m}/\mathfrak{m}^2) - 1, \quad \text{that is,} \quad \dim(\mathfrak{m}/\mathfrak{m}^2) \geqq 2 \, .$$

Let $\mathfrak{a}$ be as in (7). We have $\mathfrak{a} \neq 0$. Since $\operatorname{codim} R = 0$, the only R-modules of finite homological dimension are the free ones, so

$$dh_R R/\mathfrak{a} = \infty, \quad \text{that is,} \quad \dim \mathrm{Tor}_p{}^R(R/\mathfrak{a},k) \neq 0 \quad \text{for} \quad p \geqq 0 \, .$$

Since R is non-regular, $\mathfrak{a}$ is non-prime. It follows from corollary 2 (ii) that $\{b_p(R)\}$ is non-bounded.

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