MATH. SCAND. 21 (1967), 301–314

NOTE ON WHITEHEAD PRODUCTS IN SPHERES

LEIF KRISTENSEN and IB MADSEN

1. Introduction.

The purpose of the present paper is to show that certain Whitehead products of the form $[\alpha_n, \iota_n]$ are different from zero; here $\alpha_n \in \pi_{n+i}(S^n)$, and ι_n is the generator in $\pi_n(S^n)$ represented by the identity mapping $1: S^n \to S^n$. The main results are contained in Theorems 1.3 and 1.8 below.

Many results in this direction have been obtained earlier (Whitehead, Hilton, Toda, Adams, Barratt, Mahowald, etc.). Notably, $[\iota_n, \iota_n] \neq 0$ for $n \neq 1,3,7$ and $[\iota_n, \iota_n] = 0$ for $n = 1,3,7$ (see Adams [1]).

For $n = 1$, 3 and 7 we have mappings

$$S^n \times S^n \to S^n \tag{1}$$

of type (ι_n, ι_n). The Hopf construction applied to (1) gives a mapping

$$S^{2n+1} = S^n * S^n \to S(S^n) ,$$

and hence an element in $\pi_{2n+1}(S^{n+1})$. These elements are denoted η_2, ν_4 and σ_8 respectively. The suspensions of these elements are denoted

$$\eta_n, \nu_n, \sigma_n \in \pi_{n+i}(S^n), \quad i = 1,3,7 .$$

Let $\alpha_n = \eta_n$, ν_n or σ_n. Then the mapping cone C_{α_n} is a two-cell space

$$C_{\alpha_n} = S^n \cup_{\alpha_n} e^{n+i+1}, \quad i = 1,3,7 .$$

In mod 2 cohomology of this space the Steenrod operation

$$Sq^{i+1} : H^n(C_{\alpha_n}) \to H^{n+i+1}(C_{\alpha_n})$$

is non-zero (see Steenrod [8]). We say that α_n is detected by Sq^{i+1}.

The determination of $[\alpha_n, \iota_n]$ in case $\alpha_n = \eta_n, \nu_n, \sigma_n$ has in most cases been carried out by Mahowald [5]. Some cases still remain unsolved (see (13) and Theorem 1.8 below). The method used here goes as follows:

Assume that $[\alpha_n, \iota_n] = 0$. Then there exists a mapping

Received January 31, 1967.

$$(2) \qquad S^{n+i} \times S^n \to S^n$$

of type (α_n, ι_n). The Hopf construction applied to (2) gives a mapping

$$(3) \qquad f: S^{2n+i+1} \to S^{n+1}$$

detected by a secondary operation

$$Qu(r): H^{n+1}(C_f) \to H^{2n+i+2}(C_f),$$

where $r = R(i+1, n+1)$ is the Adem relation

$$(4) \qquad R(i+1, n+1): Sq^{i+1}Sq^{n+1} + \sum \tbinom{n+j}{i+1-2j} Sq^{n+i+2-j}Sq^j = 0 .$$

This is an immediate consequence of the following two theorems.

THEOREM 1.1. *Let $\alpha_n \in \pi_{n+i}(S^n)$, $i > 0$, and let $[\alpha_n, \iota_n] = 0$. Let*

$$\beta \in \pi_{2n+i+1}(S^{n+1})$$

be the Hopf construction on some map $S^{n+i} \times S^n \to S^n$ of type (α_n, ι_n). Then there is a CW-complex E of the form

$$E = (S^n \cup_\alpha e^{n+i+1}) \cup e^{2n+i+1}$$

with cup product pairing

$$H^n(E) \otimes H^{n+i+1}(E) \to H^{2n+i+1}(E)$$

an isomorphism. Also,

$$SE = C_{S\alpha \vee \beta},$$

where $S\alpha \vee \beta$ is the mapping

$$S\alpha \vee \beta: S^{n+i+1} \vee S^{2n+i+1} \to S^{n+1} .$$

Hence there is a map $C_\beta \to SE$ inducing isomorphisms on cohomology in dimensions different from $n+i+1$.

This theorem, we believe, is well known. A proof is given in Section 2 below. Let

$$(5) \qquad r: \hat{a}Sq^{n+1} + \sum Sq^{n+1+\deg \hat{a}_0} \hat{a}_0 + \sum \hat{\alpha}_\nu \hat{a}_\nu + \hat{b} = 0 ,$$

be a relation in the Steenrod algebra, with excess $\hat{\alpha}_\nu \hat{a}_\nu > n+1$ and excess $\hat{b} > n+1$. The element $\hat{a}_0 \in \mathcal{a}$ appears as the middle term in the Cartan formula for $\hat{a}$:

$$\Delta(\hat{a}) = \sum \hat{a}' \otimes \hat{a}'' + \sum \hat{a}_0 \otimes \hat{a}_0 + \sum \hat{a}'' \otimes \hat{a}' .$$

THEOREM 1.2. *There is a secondary cohomology operation $Qu(r)$ associated with the relation r in (5) taking the values*

$$Qu(r)(\hat{x}) = \begin{cases} 0 & \text{if} \quad \deg\hat{x} < n, \\ \sum \hat{a}''(\hat{x})\,\hat{a}'(\hat{x}) & \text{if} \quad \deg\hat{x} = n. \end{cases}$$

This theorem is proved in [2] and in [3].

The relation r in (4) sometimes contains an unfactored term Sq^{n+i+2}. In these cases the operation $Qu(r)$ is unstable, which means that it only is defined in dimensions less than $n+i+2$. In this range, however, it commutes with suspension (for more details see [2]). In some of these cases there are Adem relations r_j of excess larger than $n+1$ such that $r' = r + \sum_j r_j$ has no unfactored term. Then

$$Qu(r') = Qu(r + \sum r_j)$$

is a stable secondary operation, and $[f] \in \pi_{2n+i+1}(S^{n+1})$, defined in (3), is detected by this operation.

If $Qu(r')$ can be factored,

$$(6) \qquad Qu(r') = \sum \hat{a}_i \psi_i ,$$

where $a_i \in \mathcal{a}$ (Steenrod's algebra), ψ_i is a secondary cohomology operation, and $\deg a_i > 0$, $\deg \psi_i > 0$, then $Qu(r')$ is zero in a two-cell space. Hence we get a contradiction to the fact that $Qu(r')$ detects $[f]$. Our assumption $[\alpha_n, \iota_n] = 0$ is consequently false, and we have proved that $[\alpha_n, \iota_n] \neq 0$.

Information about the possibility of a factorization (6) can be obtained from the cohomology of the Steenrod algebra $\mathrm{Ext}_{\mathcal{a}}^{**}(Z_2, Z_2)$. It is a consequence of results in Adams [1] that a factorization (6) exists if

$$\mathrm{Ext}_{\mathcal{a}}^{2,\,n+i+2}(Z_2, Z_2) = 0 .$$

This is the case when

$$n+i+2 \neq 2^s + 2^t, \quad s, t \in Z .$$

Some new results in this direction are contained in Theorem 1.8 below.

In the present paper we are concerned with Whitehead products $[\alpha_n, \iota_n]$, where α_n is detected by a secondary operation. The elements we consider are (notation as in Toda [9])

$$(7) \qquad \begin{aligned} \eta_n{}^2 &= \eta_{n+1}\eta_n \in \pi_{n+2}(S^n), && n \geqq 2, \\ \sigma\eta_n &= \sigma_{n+1}\eta_n \in \pi_{n+8}(S^n), && n \geqq 7, \\ \nu_n{}^2 &= \nu_{n+3}\nu_n \in \pi_{n+6}(S^n), && n \geqq 4, \\ \sigma_n{}^2 &= \sigma_{n+7}\sigma_n \in \pi_{n+14}(S^n), && n \geqq 8, \\ \bar{\nu}_n &\in \pi_{n+8}(S^n), && n \geqq 6, \\ \omega_n &\in \pi_{n+16}(S^n), && n \geqq 14, \\ \xi_n &\in \pi_{n+18}(S^n), && n \geqq 12 . \end{aligned}$$

The elements $\bar{\nu}_6$, ω_{14}, ξ_{12} are Hopf constructions (see (3)) obtained from

$$[\nu_5, \iota_5] = 0, \quad [\nu_{13}, \iota_{13}] = 0, \quad [\sigma_{11}, \iota_{11}] = 0 ,$$

respectively.

The elements (7) are detected by stable secondary operations associated with relations

$$\begin{aligned} &R(2,2) ,\\ &R(2,8) + R(4,6) ,\\ &R(4,4) ,\\ (8) \qquad &R(8,8) ,\\ &R(4,6) + R(2,8) ,\\ &R(4,14) + R(2,16) ,\\ &R(8,12) + R(4,16) . \end{aligned}$$

In the four composition cases this is well known. In the three Hopf construction cases this follows from Theorem 1.1 and Theorem 1.2 as explained above.

We shall show that $[\alpha_n, \iota_n]$ is non-zero in a number of cases when α_n is one of the elements (7).

THEOREM 1.3. *Let N denote the set of numbers given in Definition* 1.4 *below. Then we have*:

$$\begin{aligned} [\eta_n{}^2, \iota_n] = 0 \quad &\textit{implies} \quad n+5 \in N \quad \textit{or} \quad n \equiv 2,3 \pmod 4,\\ [\sigma\eta_n, \iota_n] = 0 \quad &\textit{implies} \quad n+11 \in N \quad \textit{or} \quad n \equiv -1 \pmod 4\\ &\qquad \textit{or} \quad n \equiv 2 \pmod{16},\\ [\nu_n{}^2, \iota_n] = 0 \quad &\textit{implies} \quad n+9 \in N \quad \textit{or} \quad n \equiv 4,5,7 \pmod 8 ,\\ [\sigma_n{}^2, \iota_n] = 0 \quad &\textit{implies} \quad n+17 \in N \quad \textit{or} \quad n \equiv 9,11,15 \pmod{16} ,\\ [\bar{\nu}_n, \iota_n] = 0 \quad &\textit{implies} \quad n+11 \in N \quad \textit{or} \quad n \equiv -1 \pmod 4\\ &\qquad \textit{or} \quad n \equiv -2 \pmod{16} ,\\ [\omega_n, \iota_n] = 0 \quad &\textit{implies} \quad n+19 \in N \quad \textit{or} \quad n \equiv -1 \pmod 4\\ &\qquad \textit{or} \quad n \equiv -2 \pmod{32} ,\\ [\xi_n, \iota_n] = 0 \quad &\textit{implies} \quad n+21 \in N \quad \textit{or} \quad n \equiv -1 \pmod 8\\ &\qquad \textit{or} \quad n \equiv -3 \pmod{32}. \end{aligned}$$

This theorem can be strengthened by reducing the size of the set N (see Remark 1.11 at the end of this section). Some of the results contained in Theorem 1.3 have also been obtained by M. Barratt (using different methods).

DEFINITION 1.4. By N we denote the set of positive integers of the form $2^i + 2^j + 2^k$ for all triples (i,j,k), $i \leqq j \leqq k$, different from triples of the form

$$(i, i+1, k), \quad k \neq i+3$$
$$(i, j, j+1),$$
$$(i, i+2, i+2).$$

The proof of Theorem 1.3 is analogous to the proof in the case above. We assume that $[\alpha_n, \iota_n] = 0$ for α_n one of the elements in (7). First we shall see that the associated Hopf construction $f \in \pi_{2n+i+1}(S^{n+1})$ is detected by a tertiary cohomology operation.

In Section 3 we introduce for each triple (a, b, c) of integers, with $2b > a$ and $2c > b$, a relation among relations,

$$R(a, b, c),$$

in the Steenrod algebra. These play the same role for tertiary operations as Adem relations (4) play for secondary operations: To each sum $\sum R(a, b, c)$ there is associated a tertiary operation. This operation might be unstable in the sense that it is defined only in dimensions less than a certain integer. It commutes with suspension whenever this makes sense (for more details see L. Kristensen and I. Madsen [3]).

Let n and k be fixed integers and let

$$(9) \qquad R = \sum \lambda(a, b, j)\, R(a, b, n+1+j), \quad \lambda(a, b, j) \in Z_2,$$

where the summation is taken over all triples (a, b, j) of non-negative integers with $a+b+j = k$. Let

$$(10) \qquad r = \sum \lambda(a, b, 0)\, R(a, b), \quad a+b = k,$$

determine a stable secondary operation (i.e. contain no unfactored term). Then we have the following theorem which in a slightly more general form was proved in [3].

THEOREM 1.5. *There is a tertiary operation $Qu(R)$ associated with R in* (9) *taking the following values on classes $\hat{x}$ annihilated by all primary operations of degree i with $0 < i \leqq k$:*

$$Qu(R)(\hat{x}) = \begin{cases} 0 & \text{if } \deg \hat{x} \leqq n-1, \\ Qu(r)(\hat{x}) \cdot \hat{x} & \text{if } \deg \hat{x} = n, \end{cases}$$

where $Qu(r)$ is a secondary operation associated with r in (10).

It follows from Theorem 1.1 and Theorem 1.5 that the Hopf construction $[f] \in \pi_{2n+i+1}(S^{n+1})$ associated with $[\alpha_n, \iota_n] = 0$ is detected by tertiary operations associated with

$$
(11)\qquad
\begin{aligned}
&R(2,2,n+1)\,,\\
&R(2,8,n+1)+R(4,6,n+1)\,,\\
&R(4,4,n+1)\,,\\
&R(8,8,n+1)\,,\\
&R(4,6,n+1)+R(2,8,n+1)\,,\\
&R(4,14,n+1)+R(2,16,n+1)\,,\\
&R(8,12,n+1)+R(4,16,n+1)\,,
\end{aligned}
$$

respectively. We shall prove Theorem 1.3 in one special case; all other cases are similar. Let us show that

$$[\nu_n, \iota_n] \neq 0 \quad \text{for} \quad n \equiv 2 \ (\mathrm{mod}\, 16) \ \text{and} \ n+11 \notin N\,.$$

We assume $[\bar{\nu}_n, \iota_n] = 0$. The Hopf construction $[f] \in \pi_{2n+9}(S^{n+1})$ is, by (11), detected by a tertiary operation associated with

$$(12)\qquad R' = R(4,6,n+1) + R(2,8,n+1)\,.$$

This operation is not stable. The reason for this is that terms of the form $Sq^{\alpha} Sq^{\beta} Sq^{0}$ are involved in (12). However,

$$R = R' + R(2,7,n+2) + R(1,8,n+2), \quad n \equiv 2 \ (\mathrm{mod}\, 16)\,,$$

determines a stable tertiary operation $Qu(R)$, which also detects $[f] \in \pi_{2n+9}(S^{n+1})$. That R is stable is shown in Section 3. To complete the proof we need only show that $Qu(R)$ is zero in a two-cell space. Since $n+11 \notin N$, this is an immediate consequence of the following two theorems.

THEOREM 1.6. *If* $n \notin N$ *(see Definition* 1.4), *then*

$$\mathrm{Ext}_{\mathcal{A}}^{3,n}(Z_2, Z_2) = 0\,.$$

This theorem is contained in Novikov [6].

THEOREM 1.7. *Let*

$$R = \sum \lambda(a,b,c)\, R(a,b,c), \quad a+b+c=n, \ \lambda(a,b,c) \in Z_2\,,$$

be a stable relation among relations, and let

$$\mathrm{Ext}_{\mathcal{A}}^{3,n}(Z_2, Z_2) = 0\,.$$

Then there is a factorization of the form

$$Qu(R) = \sum \hat{a}_i \psi_i\,,$$

with $a_i \in \mathcal{A}$, $\deg a_i > 0$, *and* ψ_i *a tertiary operation with* $\deg \psi_i > 0$, *valid on classes annihilated by all stable primary and secondary operations.*

This is a generalization of a theorem on secondary operations due to Adams [1, Theorem 3.7.1.]. See also [4].

We now return to the case $[\sigma_n, \iota_n]$. Here we have

THEOREM 1.8. *The Hopf mapping $\sigma_n \in \pi_{n+7}(S^n)$ has the property*

$$[\sigma_n, \iota_n] \neq 0 \quad \textit{if} \quad n=2^i-7,\ i \geqq 4 \quad \textit{or if} \quad n=2^i-5,\ i>5\ .$$

The proof is given in Section 2. There is (to the best of our knowledge) still open questions in connection with $[\alpha_n, \iota_n]$, $\alpha_n = \eta_n$, ν_n or σ_n:

For $n=2^i-3$, $i \geqq 5$, is $[\nu_n, \iota_n]=0$ or $\neq 0$? Also, is $[\sigma_{27}, \iota_{27}]=0$ or $\neq 0$? The following is known:

$$(13)\quad \begin{aligned} [\eta_n, \iota_n] &= 0 && \text{for } n=2,6, \text{ and for } n \equiv -1 \pmod 4\ , \\ [\eta_n, \iota_n] &\neq 0 && \text{otherwise}\ , \\ [\nu_n, \iota_n] &= 0 && \text{for } n=5,13, \text{ and for } n \equiv -1 \pmod 8\ , \\ [\nu_n, \iota_n] &\neq 0 && \text{if } n \not\equiv -1 \pmod 8 \text{ provided } n \neq 2^i-3,\ i \geqq 5\ , \\ [\sigma_n, \iota_n] &= 0 && \text{for } n=11 \text{ and for } n \equiv -1 \pmod{16}\ , \\ [\sigma_n, \iota_n] &\neq 0 && \text{if } n \not\equiv -1 \pmod{16} \text{ provided } n \neq 11,27\ . \end{aligned}$$

In the cases

$$\begin{aligned} [\eta_n, \iota_n] &= 0, & n &\equiv -1 \pmod 4\ , \\ [\nu_n, \iota_n] &= 0, & n &\equiv -1 \pmod 8\ , \\ [\sigma_n, \iota_n] &= 0, & n &\equiv -1 \pmod{16}\ , \end{aligned}$$

the Hopf constructions

$$h(\eta_n) \in \pi_{2n+2}(S^{n+1}), \quad h(\nu_n) \in \pi_{2n+4}(S^{n+1}), \quad h(\sigma_n) \in \pi_{2n+8}(S^{n+1})$$

are detected by unstable operations associated with

$$\begin{aligned} &Sq^2 Sq^{n+1} + Sq^{n+2} Sq^1 + Sq^{n+3} = 0\ , \\ &Sq^4 Sq^{n+1} + Sq^{n+3} Sq^2 + Sq^{n+4} Sq^1 + Sq^{n+5} = 0\ , \\ &Sq^8 Sq^{n+1} + Sq^{n+5} Sq^4 + Sq^{n+7} Sq^2 + Sq^{n+8} Sq^1 + Sq^{n+9} = 0\ . \end{aligned}$$

These operations cannot be stabilized in the sense described earlier. The suspensions

$$\begin{aligned} &S^i h(\eta_n), && i \leqq 1\ , \\ &S^i h(\nu_n), && i \leqq 3\ , \\ &S^i h(\sigma_n), && i \leqq 7\ , \end{aligned}$$

are detected by the same operations. Hence, they are different from zero.

Note that $Sh(\eta_n)$, $S^3 h(\nu_n)$ and $S^7 h(\sigma_n)$ are detected by the same operations as the Whitehead products $[\iota_{n+2}, \iota_{n+2}]$, $[\iota_{n+4}, \iota_{n+4}]$ and $[\iota_{n+8}, \iota_{n+8}]$. See [2] and [1′].

REMARK 1.9. If $[\nu_{29}, \iota_{29}]=0$, the Hopf construction gives an element

$$\bar{\omega}_n \in \pi_{n+32}(S^n), \quad n \geqq 30\ ,$$

detected by a stable secondary operation associated with the relation

$$R(4,30) + R(2,32)\,.$$

The same methods as those used above show that

$$[\bar{\omega}_n, \iota_n] \neq 0 \quad \text{if} \quad n+35 \notin N,\ n \not\equiv -1 \ (\text{mod}\,4) \text{ and } n \not\equiv -2 \ (\text{mod}\,64)\,.$$

If $[\sigma_{27}, \iota_{27}] = 0$, the Hopf construction gives an element

$$\bar{\sigma}_n \in \pi_{n+34}(S^n), \quad n \geqq 28\,,$$

detected by a stable secondary operation associated with the relation

$$R(8,28) + R(4,32)\,.$$

Here we have

$$[\bar{\sigma}_n, \iota_n] \neq 0 \quad \text{if} \quad n+37 \notin N,\ n \not\equiv -1 \ (\text{mod}\,8) \text{ and } n \not\equiv -3 \ (\text{mod}\,64)\,.$$

We conjecture that

$$\begin{aligned} [\omega_n, \iota_n] &= 0 \quad \text{if} \quad n \equiv -1 \ (\text{mod}\,4) \text{ or } n \equiv -2 \ (\text{mod}\,32)\,, \\ [\bar{\omega}_n, \iota_n] &= 0 \quad \text{if} \quad n \equiv -1 \ (\text{mod}\,4) \text{ or } n \equiv -2 \ (\text{mod}\,64)\,, \\ [\xi_n, \iota_n] &= 0 \quad \text{if} \quad n \equiv -1 \ (\text{mod}\,8) \text{ or } n \equiv -3 \ (\text{mod}\,32)\,, \\ [\bar{\sigma}_n, \iota_n] &= 0 \quad \text{if} \quad n \equiv -1 \ (\text{mod}\,8) \text{ or } n \equiv -3 \ (\text{mod}\,64)\,. \end{aligned}$$

REMARK 1.10. The case of $[\bar{\nu}_n, \iota_n]$ is somewhat exceptional, since there are two elements, $\sigma\eta$ and $\bar{\nu}$, in 8-stem, detected by the same secondary operation. Hence the similar conjecture

$$[\bar{\nu}_n, \iota_n] = 0 \quad \text{if} \quad n \equiv -1 \ (\text{mod}\,4) \text{ or } n \equiv -2 \ (\text{mod}\,16)\,,$$

is more doubtful. In fact, S. Thomeier claims a counterexample in a low dimensional case.

REMARK 1.11. The results of Theorem 1.3 can be strengthened. From the results in [4] it follows that a stable tertiary operation of degree i can be factored in some cases even if $\text{Ext}_{\mathcal{a}}^{3,i+2}(Z_2, Z_2)$ is different from zero (cf. Theorem 1.7). It can be factored if the differential

$$d_2 : \text{Ext}_{\mathcal{a}}^{3,i+2}(Z_2, Z_2) \to \text{Ext}_{\mathcal{a}}^{5,i+3}(Z_2, Z_2)$$

of the Adams spectral sequence is injective. Using Novikov's result [6], we can reduce the set N (Definition 1.4) a good deal.

REMARK 1.12. There is an element $\gamma \in \pi_{61}(S^{31})$ with $2\gamma = [\iota_{31}, \iota_{31}]$, and

$$[\gamma_n, \iota_n] = 0 \quad \text{implies} \quad n+33 \in N \text{ or } n \equiv 19, 23, 31 \ (\text{mod}\,32)\,.$$

Here $\gamma_n \in \pi_{n+30}(S^n)$ is the suspension of γ. It is detected by a secondary operation associated with $R(16,16)$.

The existence of γ follows from the fact that h_4^2 is a permanent cycle in Adams' spectral sequence (h_i^2 is a permanent cycle if and only if $[\iota_{2i-1}, \iota_{2i-1}]$ can be halved).

We would like to thank M. Barratt and S. Thomeier for information on the Whitehead product. We would also like to thank H. A. Salomonsen for the great help he gave us in connection with stabilizing tertiary operations (see Section 3). The algebraic computations involved were carried out on a computer. The details are contained in mimeographed notes [7].

2. Proof of Theorems 1.1 and 1.8.

Proof of Theorem 1.1. Let $T = (X, Y; f)$ be a triple consisting of a pair (X, Y) of CW-complexes and a cellular mapping

$$f: \ Y \to Z$$

between CW-complexes. We can construct a CW-complex

$$W(T) = W = Z \cup_f X$$

from $Z \cup X$ by identifying y and $f(y)$ for all $y \in Y$. Mappings

$$Z \xrightarrow{i} W \xrightarrow{j} X/Y$$

are obtained in an obvious way. A mapping between triples

$$g: \ (X, Y; f) \to (X', Y'; f')$$

consists of continuous mappings

$$g_1: \ (X, Y) \to (X', Y'), \qquad g_2: \ Z \to Z'$$

with

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g_2} \\ Y' & \xrightarrow{f'} & Z' \end{array}$$

commutative. A mapping $g: \ (X, Y; f) \to (X', Y'; f')$ induces a mapping

$$g: \ Z \cup_f X \to Z' \cup_{f'} X'$$

such that the diagram

$$\begin{array}{ccccc} Z & \xrightarrow{i} & W & \xrightarrow{j} & Y/Y \\ \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g} & & \downarrow \\ Z' & \xrightarrow{i'} & W' & \xrightarrow{j'} & X'/Y' \end{array}$$

is commutative.

Let

$$f:\ S^{n+i} \times S^n \to S^n$$

be the mapping given in Theorem 1.1. We consider the triples

$$\begin{aligned} T &= (e^{n+i+1} \times S^n, S^{n+i} \times S^n; f)\ , \\ T_1 &= (e^{n+i+1} * S^n, S^{n+i} * S^n; f_1)\ , \\ T_2 &= (S(e^{n+i+1} \times S^n), S(S^{n+i} \times S^n); Sf)\ , \end{aligned}$$

where f_1 is obtained from f by Hopf construction,

$$f_1:\ S^{n+i} * S^n \to SS^n\ .$$

A mapping $h:\ T_1 \to T_2$ is obtained from

$$\begin{aligned} h_1 &:\ e^{n+i+1} * S^n \to S(e^{n+i+1} \times S^n)\ , \\ 1 = h_2 &:\ SS^n \to SS^n\ , \end{aligned}$$

where h_1 is obtained by Hopf construction from the identity

$$1:\ e^{n+i+1} \times S^n \to e^{n+i+1} \times S^n\ .$$

We put $E = W(T)$. It is easy to see that E has the cohomology structure specified in Theorem 1.1. Also $C_\beta = W(T_1)$, and

$$h:\ W(T_1) \to W(T_2) = SW(T)$$

induces an isomorphism on homology in dimensions different from $n+i+1$. The inclusion $C_\alpha \to W(T)$ induces an isomorphism on homology except in dimension $2n+i+1$. Hence there is a mapping $C_{S\alpha \vee \beta} \to SW(T)$ inducing isomorphism on homology in all dimensions. This mapping is, consequently, a homotopy equivalence. This proves Theorem 1.1.

Proof of Theorem 1.8. Let r and s be the following two (stable) relations in Steenrod's algebra

$$\begin{aligned} r = R(8, 2^i-6) &+ R(4, 2^i-2): \\ & Sq(8)\, Sq(2^i-6) + Sq(4)\, Sq(2^i-2) + \\ & \quad + Sq(2^i-1)\, Sq(3) + Sq(2^i-2)\, Sq(4) = 0, \quad i \geqq 4\ , \end{aligned}$$

$s = R(8,2^i-4) + R(4,2^i)$:

$$Sq(4)\,Sq(2^i) + Sq(8)\,Sq(2^i-4) + Sq(2^i)\,Sq(4) + \\ + Sq(2^i+2)\,Sq(2) = 0, \quad i \geqq 5 .$$

According to Section 1 we have to show that secondary operations $Qu(r)$ and $Qu(s)$ are zero in a two cell space. The relation r contains no term $Sq^a Sq^b$ with both a and b a power of 2. Hence there is a formula ([4] Lemma 3.3)

$$Qu(r) = \sum \hat{a}_i Qu(r_i), \quad \hat{a}_i \in \mathfrak{a}, \deg \hat{a}_i \geqq 1 .$$

The operation $Qu(s)$ is nothing but the Adams operation $\Phi_{2,i}$; for $i > 5$ this can be factorized in a sum of products of secondary operations ([4, Theorem B]), and the proof is completed.

3. Steenrod's algebra.

Let us consider symbols of the form

$$(14) \qquad Sq^a R(b,c), \quad R(\alpha,\beta)\,Sq^\gamma ,$$

with a,b,c,α,β and γ non-negative integers satisfying $2c > b$ and $2\beta > \alpha$. We shall say that

$$(15) \qquad \left.\begin{matrix} Sq^a R(b,c) \\ R(\alpha,\beta)\,Sq^\gamma \end{matrix}\right\} \text{ is admissible if } \left\{\begin{matrix} a \geqq 2b , \\ \beta \geqq 2\gamma . \end{matrix}\right.$$

Other elements (14) are called inadmissible. Let V_a (V_i) be the Z_2-vector space generated by admissible (inadmissible) symbols (14). The vector spaces V_a and V_i are graded by

$$\deg(Sq^a R(b,c)) = a + b + c ,$$
$$\deg(R(\alpha,\beta)\,Sq^\gamma) = \alpha + \beta + \gamma .$$

Let F denote the free associative algebra (without unit) generated by symbols Sq^a, $a = 0,1,\ldots$. We define mappings

$$d : V_\nu \to F, \quad \nu = a,i ,$$

by (cf. (4) in Section 1)

$$d(Sq^a R(b,c)) = Sq^a(Sq^b Sq^c + \sum\tbinom{c-1-j}{b-2j} Sq^{b+c-j} Sq^b) ,$$
$$d(R(\alpha,\beta)\,Sq^\gamma) = (Sq^\alpha Sq^\beta + \sum\tbinom{\beta-1-j}{\alpha-2j} Sq^{\alpha+\beta-j} Sq^j)\,Sq^\gamma .$$

Let $I = (i_0, i_1, \ldots, i_r)$ be a sequence of non-negative integers. The excess of I, $\operatorname{exc} I$, is defined by

$$\operatorname{exc} I = \max_j \{i_j - (i_{j+1} + i_{j+2} + \ldots)\}.$$

We put

$$\operatorname{exc}(Sq^a R(b,c)) = \operatorname{exc}(a,b,c),$$
$$\operatorname{exc}(R(\alpha,\beta)\, Sq^\gamma) = \operatorname{exc}(\alpha,\beta,\gamma).$$

LEMMA 3.1. *The kernel af the mapping $d: V_a \to F$ is equal to zero.*

PROOF. Let

(16) $$\sum \lambda(s,t,u)\, Sq^s R(t,u) + \sum \delta(s,t,u)\, R(s,t)\, Sq^u \in V_a$$

be a homogeneous element in the kernel of d. The functions λ and δ are defined on all triples (s,t,u) of non-negative integers, and take values in Z_2. They have to satisfy some obvious conditions in order that (16) belongs to V_a. We order triples (s,t,u) lexicographically from the right. Let (s_0,t_0,u_0) be the largest triple in $\lambda^{-1}(1)$. The term $Sq^{s_0} Sq^{t_0} Sq^{u_0}$ appears in $d(\sum(\text{left}))$ (see (16)). Hence it must appear in a term of the form $d(R(s_0+t_0-y,y)\, Sq^{u_0})$. Thus there is a y such that

$$\delta(s_0+t_0-y,y,u_0) = 1.$$

We have $s_0 \geqq 2t_0$ and $2y > s_0+t_0-y$. Hence $y > t_0$ and

$$(s_0,t_0,u_0) < (s_0+t_0-y,y,u_0) \leqq (s_1,t_1,u_1),$$

where (s_1,t_1,u_1) is the largest triple in $\delta^{-1}(1)$. Similarly one sees that

$$(s_0,t_0,u_0) > (s_1,t_1,u_1).$$

This implies that $\lambda = \delta = 0$, and the lemma is proved.

THEOREM 3.2. *Let a, b, c be non-negative integers with $2b > a$, $2c > b$. There is a unique element $R \in V_a$ such that*

(17) $$R(a,b,c) = Sq^a R(b,c) + R(a,b)\, Sq^c + R,$$

is in the kernel of $d: V = V_a \oplus V_i \to F$. All terms in R have excess larger than or equal to c.

PROOF. Uniqueness is an immediate consequence of Lemma 3.1. The rest of the proof is omitted. One constructs R by a repeated application of Adem relations.

An element

$$S = \sum Sq^s(\sum R(t,u)) + \sum(\sum R(\alpha,\beta))\, Sq^\gamma$$

in V is called stable if each s and γ is larger than zero.

Let a, b, n be positive integers. We say that $R(a,b,n)$ can be stabilized if there is a function λ taking values in Z_2 such that

$$S = R(a,b,n) + \sum\lambda(s,t,u)\, R(s,t,u), \quad (s,t,u) > (a,b,n) ,$$

is stable; the ordering $(s,t,n) > (a,b,n)$ is lexicographical from the right. Use of a computer yields

LEMMA 3.3. *$R(a,b,n)$ can be stabilized in the following cases:*

For $a=4$, $b=4$ and $n \geqq 5$ if

$$n \not\equiv 0,5,6 \pmod 8 ,$$

for $a=4$, $b=6$ and $n \geqq 6$ if

$$n \not\equiv 0 \pmod 4 \text{ and } n \not\equiv -1 \pmod{16} ,$$

for $a=8$, $b=8$ and $n \geqq 9$ if

$$n \not\equiv 0,10,12 \pmod{16} ,$$

for $a=4$, $b=14$ and $n \geqq 10$ if

$$n \not\equiv 0 \pmod 4 \textit{ and } n \not\equiv -1 \pmod{32} ,$$

for $a=8$, $b=12$ and $n \geqq 11$ if

$$n \not\equiv 0 \pmod 8 \text{ and } n \not\equiv -2 \pmod{32} ,$$

for $a=4$, $b=30$ and $n \geqq 15$ if

$$n \not\equiv 0 \pmod 4 \text{ and } n \not\equiv -1 \pmod{64} ,$$

for $a=16$, $b=16$ and $n \geqq 14$ if

$$n \not\equiv 0,20,24 \pmod{32} .$$

As mentioned in Section 1, the details are contained in [7]. As an example, we state the results for $a=4$, $b=6$ in more detail:

LEMMA 3.4. *The following expressions are stable:*

$$
\begin{array}{ll}
R(4,6,n) + R(2,8,n) & \textit{if } n \equiv 1 \pmod{16} , \\
R(4,6,n) + R(3,7,n) + R(2,8,n) & \textit{if } n \equiv 2 \pmod{16} , \\
R(4,6,n) + R(2,8,n) + R(2,7,n+1) + & \\
\quad + R(1,8,n+1) & \textit{if } n \equiv 3 \pmod{16} , \\
R(4,6,n) + R(2,8,n) & \textit{if } n \equiv 5 \pmod{16} , \\
R(4,6,n) + R(3,7,n) + R(2,8,n) & \textit{if } n \equiv 6 \pmod{16} , \\
R(4,6,n) + R(2,8,n) + R(4,5,n+1) + & \\
\quad + R(1,8,n+1) + R(1,6,n+3) & \textit{if } n \equiv 7 \pmod{16} , \\
R(4,6,n) + R(2,8,n) + R(2,2,n+6) & \textit{if } n \equiv 9 \pmod{16} ,
\end{array}
$$

$$\begin{aligned}
&R(4,6,n) + R(3,7,n) + R(2,8,n) + \\
&\quad + R(1,9,n) + R(5,4,n+1) + \\
&\quad + R(4,5,n+1) + R(1,8,n+1) + \\
&\quad + R(3,4,n+3) + \\
&\quad + R(1,6,n+3) + R(2,2,n+6) \qquad \textit{if } n \equiv 10 \pmod{16}, \\
&R(4,6,n) + R(2,8,n) + R(2,7,n+1) + \\
&\quad + R(1,8,n+1) + R(4,4,n+2) + \\
&\quad + R(2,3,n+5) + R(1,4,n+5) \qquad \textit{if } n \equiv 11 \pmod{16}, \\
&R(4,6,n) + R(2,8,n) + R(2,6,n+2) \qquad \textit{if } n \equiv 13 \pmod{16}, \\
&R(4,6,n) + R(2,8,n) + R(4,5,n+1) + \\
&\quad + R(2,6,n+2) \qquad \textit{if } n \equiv 14 \pmod{16},
\end{aligned}$$

REFERENCES

1. J. F. Adams, *On the non-existence of elements of Hopf invariant one*, Ann. of Math. 72 (1960), 20–104.

1'. E. H. Brown Jr. and F. P. Peterson, *Whitehead products and cohomology operations*, Quart. J. Math. Oxford (2) 15 (1964), 116–120.

2. L. Kristensen, *On secondary cohomology operations*, Math. Scand. 12 (1963), 57–82.

3. L. Kristensen and I. Madsen, *On evaluation of higher order cohomology operations*, Math. Scand. 20 (1967), 114–130.

4. L. Kristensen and I. Madsen, *Some relations among cohomology operations*, Math. Scand., (to appear).

5. M. Mahowald, *Some Whitehead products in S^n*, Topology 4 (1965), 17–26.

6. C. P. Novikov, *On the cohomology of Steenrod's algebra*, Dokl. Akad. Nauk. SSSR (Moscow) 128 (1959), 893–895. (In Russian.)

7. H. A. Salomonsen, *Computations in the Steenrod algebra*, Mimeographed notes, Matematisk Institut, Aarhus Universitet, 1966.

8. N. Steenrod, *Products of cocycles and extensions of mappings*, Ann. of Math. 48 (1947), 290–320.

9. H. Toda, *Composition methods in homotopy groups of spheres*, Ann. Math. Studies 49, Princeton, N. J., 1962.

UNIVERSITY OF AARHUS, DENMARK