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COHOMOLOGY AND HOMOLOGY OF PAIRS OF PRESHEAVES

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We define the cohomology of pairs of presheaves. As a particular case, we obtain the Grothendieck cohomology of a topological space with coefficients in a presheaf of abelian groups.

In the final section, we introduce the resolution of a presheaf by spreads, which gives rise to a homology of pairs of presheaves; in particular is defined the homology of a topological space with coefficients in a given presheaf of abelian groups.

1.

Let $F=\{F_U,\varphi_V{}^U\}$ be a presheaf of abelian groups (or modules over a fixed ring) over a topological space X.

For every open set U of X, let $F\,|\,U$ denote the presheaf restriction of F to U; the sheaf of germs of $F\,|\,U$ may be identified with the restriction to U of the sheaf of germs $\mathscr{F}$ of F, which we denote by $\mathscr{F}\,|\,U$.

Let $F=\{F_U,\varphi_V{}^U\}$, $C=\{C_U,\gamma_V{}^U\}$ be presheaves of abelian groups over X, and let $\mathscr{C}$ be the sheaf of germs of C.

For every open set $U\subset X$, let M_U be the abelian group of all continuous sheaf-homomorphisms of $\mathscr{F}\,|\,U$ into $\mathscr{C}$.

If $V\subset U$ are open sets, there is a natural homomorphism

$$\mu_V{}^U:\;\; M_U\to M_V\,,$$

and it follows immediately that $M=\{M_U,\mu_V{}^U\}$ is a presheaf, and even a sheaf over X (see [1, p. 185]).

As it is well known, for every presheaf F over X, $C \rightsquigarrow M(F,C)$ is a left-exact covariant functor.

2.

In order to define the n^{th} cohomology presheaf of a pair of presheaves, let us recall the concept of canonical complex of a presheaf G.

Let $\mathscr{G}$ be the sheaf of germs of G, let $C^0=C^0(X,G)$ be the presheaf of

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(not necessarily continuous) sections of $\mathscr{G}$, with the natural restriction mappings $\gamma_V{}^U$ (for $V \subset U$ open sets). There is a natural X-morphism $j^0 \colon G \to C^0$; let $Z^1 = C^0/j^0(G)$ be the quotient presheaf and $p^0 \colon C^0 \to Z^1$ the canonical X-epimorphism; hence $p^0 \circ j^0 = 0$. By induction, we define the

$$X\text{-homomorphism} \qquad j^n : Z^n \to C^n$$

and

$$X\text{-epimorphism} \qquad p^n : C^n \to Z^{n+1} ,$$

where

$$Z^n = C^{n-1}/j^{n-1}(Z^{n-1}), \qquad C^n = C^0(X, Z^n) .$$

Putting $\delta^n = j^n \circ p^{n-1}$, then $\delta^n \circ \delta^{n-1} = 0$ for every $n \geqq 1$ and we obtain the complex of presheaves:

$$(1) \qquad C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} C^2 \xrightarrow{\delta^2} \dots$$

Since M is a left-exact covariant functor, we have a complex of presheaves:

$$(2) \qquad O \xrightarrow{\Delta^{-1}} M(F, C^0) \xrightarrow{\Delta^0} M(F, C^1) \xrightarrow{\Delta^1} M(F, C^2) \xrightarrow{\Delta^2} \dots ,$$

because

$$\delta^n \circ \delta^{n-1} = 0 \quad \text{implies} \quad \Delta^n \circ \Delta^{n-1} = 0$$

for every $n \geqq 1$.

DEFINITION 1. For every $n \geqq 0$ the presheaf

$$H^n(F, G) = \operatorname{Ker} \Delta^n / \operatorname{Im} \Delta^{n-1}$$

is called *the* n^{th} *cohomology presheaf* of the pair (F, G).

If $\mathscr{F}$, $\mathscr{G}$ are sheaves of abelian groups over X, we define $H^n(\mathscr{F}, \mathscr{G})$ to be the sheaf of germs of the presheaf $H^n(\Gamma(\mathscr{F}), \Gamma(\mathscr{G}))$, where $\Gamma(\mathscr{F})$, $\Gamma(\mathscr{G})$ denote respectively the presheaves of sections of $\mathscr{F}$, $\mathscr{G}$. We call $H^n(\mathscr{F}, \mathscr{G})$ *the* n^{th} *cohomology sheaf* of the pair $(\mathscr{F}, \mathscr{G})$.

We emphasize that we have used the complex (1) (instead of an injective resolution of G); hence, there is no reason why $H^n(F, G)$ should be the same as $\operatorname{Ext}^n(F, G)$, defined in [1, p. 187]. As a matter of fact, if X is a space consisting of only one point, then F, G are identified with abelian groups, and

$$H^0(F, G) = \operatorname{Hom}(F, G) ,$$
$$H^n(F, G) = 0 \quad \text{for every } n \geqq 1 ,$$

for all choices of F, G. However, $\operatorname{Ext}^n(F, G) = 0$ for all $n \geqq 1$ and all groups F, if and only if G is a divisible group.

The following results may be proved in straightforward manner:

PROPOSITION 1. *For every presheaf F over X, the functors*

$$G \rightsquigarrow M(F,G) \quad \text{and} \quad G \rightsquigarrow H^0(F,G)$$

are isomorphic.

PROPOSITION 2. *Let F be a locally free presheaf of abelian groups over X. To each short exact sequence $\mathscr{S}$ of presheaves over X,*

$$\mathscr{S}:\ 0 \to G' \xrightarrow{\pi'} G \xrightarrow{\pi} G'' \to 0\ ,$$

there corresponds the long exact sequence $\mathscr{L}$ of cohomology presheaves,

$$\begin{aligned} \mathscr{L}:\ 0 \longrightarrow M(F,G') \xrightarrow{\Pi'} M(F,G) \xrightarrow{\Pi} M(F,G'') \xrightarrow{\Delta^0} \\ \xrightarrow{\Delta^0} H^1(F,G') \xrightarrow{\Pi'^1} H^1(F,G) \xrightarrow{\Pi^1} H^1(F,G'') \xrightarrow{\Delta^1} \\ \xrightarrow{\Delta^1} H^2(F,G') \xrightarrow{\Pi'^2} H^2(F,G) \xrightarrow{\Pi^2} H^2(F,G'') \xrightarrow{\Delta^2} \cdots . \end{aligned}$$

The functor $\mathscr{S} \rightsquigarrow \mathscr{L}$ is natural.

By means of standard arguments, we are reduced to establish:

LEMMA 1. *Let F be a locally free presheaf of abelian groups over X. For every $n \geqq 0$ the sequence*

$$0 \longrightarrow M(F,C'^n) \xrightarrow{\Pi'^n} M(F,C^n) \xrightarrow{\Pi^n} M(F,C''^n) \longrightarrow 0$$

is exact.

PROOF. By the inductive definition of the presheaves C'^n, C^n, C''^n, it is enough to prove the lemma for the case $n=0$. Since $M(F,\cdot)$ is a left-exact functor, we need only to show that Π^0 is an X-epimorphism, that is, for every open set $U \subset X$, and for every $\lambda'' \in [M(F,C''^0)]_U$ there exists $\lambda \in [M(F,C^0)]_U$ such that $\Pi_U{}^0(\lambda)=\lambda''$.

Let V be an open set (perhaps empty), $V \subset U$, such that there exists λ_V, a continuous sheaf-homomorphism from $\mathscr{F}\,|\,V$ into $\mathscr{C}^0$, such that $\Pi_V{}^0(\lambda_V)=\lambda'' \circ i_V{}^U$, where $i_V{}^U$ is the natural continuous sheaf-monomorphism from $\mathscr{F}\,|\,V$ into $\mathscr{F}\,|\,U$.

Let $x \in U$, $x \notin V$. We shall show that there exists an open set W, $x \in W \subset U$, and a continuous sheaf-homomorphism

$$\mu_W :\ \mathscr{F}\,|\,W \to \mathscr{C}^0$$

such that

$$\Pi_W{}^0(\mu_W)\ =\ \lambda'' \circ i_W{}^U\ .$$

By the hypothesis on F, for every $x \in X$ there exists an open set W, $x \in W \subset U$, and there exists a family S of continuous sections of $\mathscr{F}$

over W, such that if $y \in W$, $\alpha \in \mathscr{F}_y$, then α is, in a unique way, a linear combination with integral coefficients

$$\alpha = \sum_{k=1}^{n} m_k \, s_k(y), \qquad \text{where } s_k \in S .$$

In order to define μ_W, let $\lambda'' \circ i_W{}^U(s_k(y))$ be the germ of the presheaf C''^0 which is represented by the triple

$$(t_k'', y, W_k'') ,$$

where $t_k'' \in C''^0_{W_k''}$, $y \in W_k'' \subset W$. Since π is epic, given the section $t_k''\colon W_k'' \to \mathscr{G}''$, there exists some section $t_k\colon W_k'' \to \mathscr{G}$ such that $\bar{\pi} \circ t_k = t_k''$.

With above notations, we define $\mu_W(\alpha)$ to be the germ of C° represented by the triple

$$\left(\sum_{k=1}^{n} m_k \, t_k, y, \bigcap_{k=1}^{n} W_k'' \right).$$

Clearly, μ_W is fiber-preserving and on each fiber it is a group-homomorphism. By its definition, we have

$$\Pi_W{}^0(\mu_W) = \bar{\pi} \circ \mu_W = \lambda'' \circ i_W{}^U .$$

We show now that μ_W is continuous. Let be given the neighbourhood O'' of $\mu_W(\alpha)$, defined by

$$\sum_{k=1}^{n} m_k \, t_k \in C_W{}^0, \qquad y \in W'' \subset \bigcap_{k=1}^{n} W_k'' .$$

Since each section s_k is continuous, given the neighborhood $O_{f_k, V''}$ (where $f_k \in F_{V''}$) of $s_k(y)$, there exists a sufficiently small neighborhood V_0'' of y,

$$V_0'' \subset V'' \cap W'' ,$$

such that for every $z \in V_0''$, and for every $k = 1, \ldots, n$, the germ $s_k(z)$ is represented by the triple (f_k, z, V''). Let O be the neighborhood of α defined by

$$\left(\varphi^{V''}_{V_0''} \left(\sum_{k=1}^{n} m_k f_k \right) \in F_{V_0''} \right).$$

Then

$$\mu_W(0) \subset O'' .$$

Indeed, if $\beta \in O$, it is represented by the triple

$$\left(\varphi^{V''}_{V_0''} \left(\sum_{k=1}^{n} m_k f_k \right), z, V_0'' \right),$$

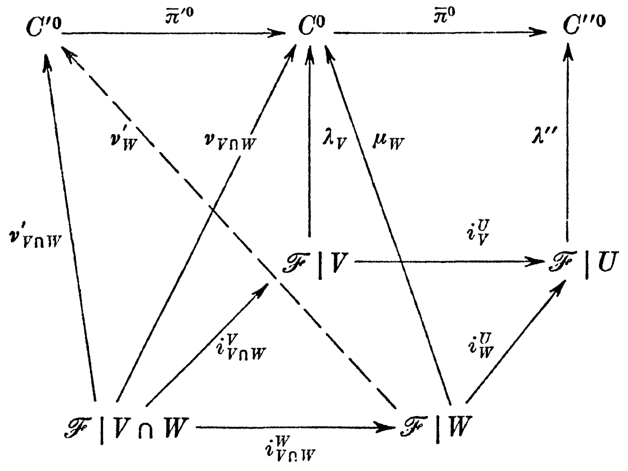
where $z \in V_0''$, so

$$\beta = \sum_{k=1}^{n} m_k s_k(z)$$

and by definition of μ_W it follows that $\mu_W(\beta)$ is the germ represented by

$$\left(\sum_{k=1}^{n} m_k t_k,\ z,\ W''\right)$$

and therefore $\mu_W(\beta) \in O''$.



We now consider the continuous sheaf homomorphism

$$\gamma_{V\cap W} = \lambda_V \circ i^V_{V\cap W} - \mu_W \circ i^W_{V\cap W}\,,$$

which followed by $\bar{\pi}^0$ is $\bar{\pi}^0 \circ \nu_{V\cap W} = 0$. So

$$\nu_{V\cap W} \in \operatorname{Ker} \Pi^0 = \operatorname{Im} \Pi'^0\,,$$

hence there exists a continuous sheaf-homomorphism

$$\nu'_{V\cap W}:\ \mathscr{F}\,|\,V\cap W \to \mathscr{C}'^0$$

such that

$$\bar{\pi}'^0 \circ \nu'_{V\cap W} = \nu_{V\cap W}\,.$$

For every continuous section $s \in S$, the restriction of $\nu'_{V\cap W} \circ s$ to $V\cap W$ is a continuous section of the sheaf $\mathscr{C}'^0$ over $V\cap W$. Since $\mathscr{C}'^0$ is a flabby sheaf, there exists a continuous section σ of $\mathscr{C}'^0$ over W, whose restriction to $V\cap W$ coincides with $\nu'_{V\cap W} \circ s$. We now define a continuous sheaf-homomorphism

$$\nu_W' : \mathscr{F} \mid W \to \mathscr{C}'^0$$

as follows. If $\alpha \in \mathscr{F} \mid W$ has center $y \in W$, then

$$\alpha = \sum_{k=1}^{n} m_k s_k(y) ,$$

where $s_k \in S$, $m_k \in \mathsf{Z}$ are uniquely defined; we put

$$\nu_W'(\alpha) = \sum_{k=1}^{n} m_k \sigma_k(y) .$$

It is obvious that ν_W' is a continuous sheaf-homomorphism. Moreover,

$$\nu'_W \circ i^W_{V \cap W} = \nu'_{V \cap W} ,$$

since the restriction of σ_k to $V \cap W$ is equal to s_k. It follows that

$$\begin{aligned}(\bar{\pi}'^0 \nu'_W + \mu_W) \circ i^W_{V \cap W} &= \bar{\pi}'^0 \nu'_{V \cap W} + \mu_W \, i^W_{V \cap W} \\ &= \nu_{V \cap W} + \mu_W \, i^W_{V \cap W} = \lambda_V \, i^V_{V \cap W} .\end{aligned}$$

From this, it follows that λ_V and $\bar{\pi}'^0 \nu_W' + \mu_W$ have a common extension to a continuous sheaf-homomorphism

$$\lambda_{V \cup W} : \mathscr{F} \mid V \cup W \to \mathscr{C}^0 ,$$

which is defined in the obvious way. Moreover

$$\Pi^0_{V \cup W}(\lambda_{V \cup W}) = \lambda'' \circ i^U_{V \cap W} .$$

The preceding considerations are the essential part in the proof of the lemma. Indeed, let us consider the family F of all couples (V, λ_V) where V is an open set, $V \subset U$, and

$$\lambda_V : \mathscr{F} \mid V \to \mathscr{C}^0$$

is a continuous sheaf-homomorphism such that

$$\Pi_V{}^0(\lambda_V) = \lambda'' \circ i_V{}^U .$$

This family is not empty, by taking $V = \emptyset$. We order F in natural way, by letting

$$(V, \lambda_V) \leq (V', \lambda_{V'})$$

whenever

$$V \subset V' \quad \text{and} \quad \lambda_V = \lambda_{V'} \circ i_V{}^{V'} .$$

Since F is clearly inductive, by Zorn's lemma F has a maximal element, which in virtue of our proof must be (U, λ_U). Thus $\Pi_U{}^0(\lambda_U) = \lambda''$, and this proves the lemma.

3.

We shall show that the Grothendieck cohomology of a topological space X, with coefficients in a presheaf G over X, may be considered as a particular case.

For every open set $U \subset X$, let A_U be the ring of continuous functions of U into Z. If $V \subset U$ let $\varrho_V{}^U$ be the restriction mapping. Thus $A(X) = \{A_U, \varrho_V{}^U\}$ is a presheaf of rings over X; let $\mathscr{A}$ be the sheaf of germs of $A(X)$.

It is clear that for every topological space X the presheaf $A(X)$ is locally free.

Theorem 1. *Let X be a topological space, let $A = A(X)$ be the presheaf of abelian groups associated with X (as it was defined above). Let G be any presheaf of abelian groups over X. Then, for every integer $n \geqq 0$, there is a natural isomorphism*

$$h^n(X,G) \cong [H^n(A(X),G)]_X,$$

where $h^n(X,G)$ denotes the n^{th} Grothendieck cohomology group of X with coefficients in G.

Proof. Let

$$0 \to C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} C^2 \xrightarrow{\delta^2} \dots$$

be the canonical complex of G. Let

$$0 \to \Gamma(X,C^0) \xrightarrow{\gamma^0} \Gamma(X,C^1) \xrightarrow{\gamma^1} \Gamma(X,C^2) \xrightarrow{\gamma^2} \dots$$

be the complex of abelian groups of global continuous sections over X, which is induced by the canonical complex of G. By definition,

$$h^n(X,G) = \operatorname{Ker}\gamma^n/\operatorname{Im}\gamma^{n-1}.$$

To prove the theorem, we first define a homomorphism

$$\varphi^n : \quad \Gamma(X,C^n) \to [M(A,C^n)]_X$$

in the following way. Let $\alpha \in \mathscr{A}$ be the germ represented by (f,x,U), where U is open in X, $x \in U$, $f \in A_U$. If $c \in \Gamma(X,C^n)$, we define

$$\varphi^n(c) : \quad \mathscr{A} \to \mathscr{C}^n$$

by letting

$$\varphi^n(c)(\alpha) = f(x)\cdot c(x);$$

the mapping $\varphi^n(c)$ does not depend on the triple which represents α. The continuity of $\varphi^n(c)$ is easily established from the definition and the fact that c is a continuous section and f is a continuous mapping. Thus

$$\varphi^n(c) \in [M(A,C^n)]_X .$$

It is also clear that if $c \in \operatorname{Ker}\gamma^n$ then

$$\varphi^n(c) \in \operatorname{Ker}\Delta_X{}^n .$$

Indeed, if $\delta^n: C^n \to C^{n+1}$, if $\bar{\delta}^n: \mathscr{C}^n \to \mathscr{C}^{n+1}$ is the associated sheaf-homomorphism, then $\gamma^n: \Gamma(X,C^n) \to \Gamma(X,C^{n+1})$ is such that

$$\big(\gamma^n(c)\big)(x) = \bar{\delta}^n\big(c(x)\big);$$

hence

$$\begin{aligned}\big(\Delta_X{}^n(\varphi^n(c))\big)(\alpha) &= \bar{\delta}^n\big(\varphi^n(c)(\alpha)\big)\\ &= \bar{\delta}^n\big(f(x)\,c(x)\big)\\ &= f(x)\,\bar{\delta}^n\big(c(x)\big) = 0\end{aligned}$$

for every $\alpha \in \mathscr{A}$, where α is the germ represented by (f,x,U).

It is immediate that if $c \in \operatorname{Im}\gamma^{n-1}$ then $\varphi^n(c) \in \operatorname{Im}\Delta_X{}^{n-1}$. Moreover, $\varphi^n(c)$ sends fibers into fibers and it is additive. Thus, φ^n is a group-homomorphism and induces a group-homomorphism

$$\Phi^n: \; h^n(X,G) \to [H^n(A,G)]_X$$

defined by

$$\Phi^n(c+\operatorname{Im}\gamma^{n-1}) = \varphi^n(c) + \operatorname{Im}\Delta_X{}^{n-1} .$$

We shall now show that Φ^n is epic. Let

$$b+\operatorname{Im}\Delta_X{}^{n-1} \in [H^n(A,G)]_X ,$$

where

$$b \in \operatorname{Ker}\Delta_X{}^n \subset [M(A,C^n)]_X .$$

Let $x \in X$ and let $1_x \in A$ be the germ represented by (e,x,X) where $e \in A$ with $e(y)=1$ for every $y \in X$. The mapping $c: X \to \mathscr{C}^n$, defined by $c(x)=b(1_x)$, is a section and it is continuous. Indeed, consider $O_{h,U'}$, U' open, $h \in C_U{}^n$, which is a neighborhood of $c(\alpha)$ for $x \in U'$. Since b is continuous, there exists a neighborhood $O_{g,U''}$ of 1_x, with $x \in U''$, $g \in A_{U''}$, such that $b(O_{g,U''}) \subset O_{h,U'}$. Thus, since $1_x \in O_{g,U''}$ then (e,x,X) and (g,x,U'') determine the germ 1_x and hence $g(x)=1$. Since $g \in A_{U''}$ by continuity there exists a neighborhood V of x, $V \subset U' \cap U''$, such that $g(y)=g(x)$ for every $y \in V$. Now, if $y \in V$ then $1_x \in O_{g,U''}$ since (e,y,V) and (g,y,U'') define the same germ, for $g(y)=g(x)=1=e(y)$ for every $y \in V$. Thus, if $y \in V$ then

$$c(y) = b(1_x) \in b(O_{g,U''}) \subset O_{h,U'} .$$

Moreover,

$$c \in \operatorname{Ker}\gamma^n .$$

Indeed,

$$\gamma^n(c) = \bar{\delta}^n \circ c(x) = \bar{\delta}^n \circ b(1_x) = \Delta_X{}^n(b)(1_x) = 0$$

since $b \in \operatorname{Ker} \Delta_X{}^n$. To show that Φ^n is monic it is enough, by the above considerations to prove that φ^n is monic, and this is immediate.

Finally,

$$\Phi^n(c + \operatorname{Im}\gamma^{n-1}) = b + \operatorname{Im}\Delta_X{}^{n-1} .$$

Indeed, if α is the germ defined by (f,x,U), then

$$\varphi^n(c)(\alpha) = f(x)\, c(x) = f(x)\, b(1_x) = b(f(x)\, 1_x) .$$

Now, since f is continuous there exists a neighborhood V of x such that $f(y) = f(x)$ for every $y \in V$. Thus (f,x,U) and $(f(x)\,e,x,X)$ define the germ α, since $\varrho_V{}^U(f) = f(x)\, e$. Thus $\varphi^n(c)(\alpha) = b(\alpha)$, hence also

$$\Phi^n(c + \operatorname{Im}\gamma^{n-1}) = b + \operatorname{Im}\Delta_X{}^{n-1} .$$

So, we have proved that Φ^n is an isomorphism for every n, and the theorem is proved.

The *cohomological dimension* of a presheaf F is defined in the usual manner. Thus, the cohomological dimension of the topological space X coincides with the cohomological dimension of the presheaf $A(X)$ canonically associated with X.

4.

Let $G = \{G_U, \varrho_V{}^U\}$ be a presheaf over X, let $\mathcal{O}(X)$ be the collection of open sets of X. We shall consider subsets $\mathcal{U} \subset \mathcal{O}(X)$ satisfying:

(*) if $V \subset U \in \mathcal{U}$, then $V \in \mathcal{U}$.

For every $\mathcal{U}$, we consider the families $(g_U)_{U \in \mathcal{U}}$ where each $g_U \in G_U$ and if $V \subset U \in \mathcal{U}$ then $\varrho_V{}^U(g_U) = g_V$.

Each $(g_U)_{U \in \mathcal{U}}$ is called a *spread on* $\mathcal{U}$ and it is therefore an element of the projective limit

$$\varprojlim_{\mathcal{U}} G_U .$$

Let $\mathcal{S}_{\mathcal{U}}$ be the set of spreads on $\mathcal{U}$, let $\mathcal{S} = \bigcup \mathcal{S}_{\mathcal{U}}$ be the set of spreads. We note that each $\mathcal{S}_{\mathcal{U}} \neq \emptyset$, since it contains at least the zero spread.

Each $\mathcal{S}_{\mathcal{U}}$ is an abelian group, defining the addition as follows:

$$(g_U)_{U \in \mathcal{U}} + (g_U{}')_{U \in \mathcal{U}} = (g_U + g_U{}')_{U \in \mathcal{U}} .$$

For every open set $U \subset X$ let

$$F_U = \bigoplus_{U \in \mathscr{U}} \mathscr{S}_{\mathscr{U}},$$

direct sum of the groups $\mathscr{S}_{\mathscr{U}}$, where $\mathscr{U}$ runs in the set of allowable families of open sets containing the given open set U.

If $V \subset U$, let $\tau_V{}^U\colon F_U \to F_V$ be the natural inclusion (if $U \in \mathscr{U}$, then $V \in \mathscr{U}$). Then $F = \{F_U, \tau_V{}^U\}$ is a presheaf over X.

For every open set $U_0 \subset X$, we define $\varphi_{U_0}\colon F_{U_0} \to G_{U_0}$ as follows:

$$\varphi_{U_0}\left(\sum_{i=1}^{n} (g_U{}^{(i)})_{U \in \mathscr{U}_i}\right) = \sum_{i=1}^{n} g_{U_0}^{(i)};$$

φ_{U_0} is a group-homomorphism and the following diagram is commutative:

$$\begin{array}{ccc} F_{U_0} & \xrightarrow{\varphi_{U_0}} & G_{U_0} \\ \downarrow{\scriptstyle \tau_{V_0}^{U_0}} & & \downarrow{\scriptstyle \varrho_{V_0}^{U_0}} \\ F_{V_0} & \xrightarrow[\varphi_{V_0}]{} & G_{V_0} \end{array}$$

where $V_0 \subset U_0$ are open sets of X.

Now, we prove that *each mapping* φ_{U_0} *is epic*. Indeed, let $g \in G_{U_0}$, let

$$\mathscr{U}_0 = \{V \text{ open set in } X \mid V \subset U_0\};$$

then $\mathscr{U}_0$ satisfies

(*) $$(\varrho_V{}^{U_0}(g))_{V \in \mathscr{U}_0} \in \mathscr{S}_{\mathscr{U}_0}$$

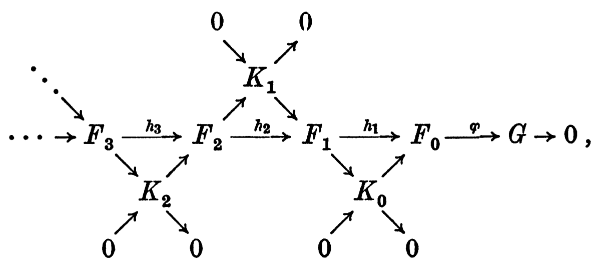
and

$$\varphi_{U_0}[(\varrho_V{}^{U_0}(g))_{V \in \mathscr{U}_0}] = g\,.$$

Thus, we have the presheaf-epimorphism

$$F \xrightarrow[\varphi]{} G \to 0\,,$$

where $\varphi = (\varphi_{U_0})_{U_0 \in \mathscr{O}(X)}$. This gives rise, in the usual manner to a resolution of G:



where $K_0 = \mathrm{Ker}\,\varphi$, F_1 is obtained from K_0 in the same way as $F = F_0$ was defined from G, $K_1 = \mathrm{Ker}\,\varphi_1$, and so on. We have $\mathrm{Im}\,h_1 = \mathrm{Ker}\,\varphi$, $\mathrm{Im}\,h_2 = \mathrm{Ker}\,h_1$, and so on. If K is any presheaf, the resolution gives rise, by means of the tensor product, to the complex

$$\ldots \to F_2 \otimes K \xrightarrow{\bar{h}_2} F_1 \otimes K \xrightarrow{\bar{h}_1} F_0 \otimes K \xrightarrow{\bar{\varphi}} G \otimes K \to 0 .$$

DEFINITION 2. For every $n \geqq 0$, the presheaf

$$H_n(G,K) = \mathrm{Ker}\,\bar{h}_n / \mathrm{Im}\,\bar{h}_{n+1}$$

is called *the* n^{th} *homology presheaf* of the pair (G,K).

As we did for the cohomology, we may also define *the* n^{th} *homology sheaf* of the pair $(\mathscr{G},\mathscr{K})$ of sheaves.

Let us note that if X is a space consisting of only one point, then G may be identified with an abelian group and the resolution (1) becomes trivial; then

$$H_0(G,K) = G \otimes K ,$$
$$H_n(G,K) = 0 \quad \text{for every } n \geqq 1 ,$$

whatever be the groups G,K. Thus, $H_n(G,K)$ is not equal to $\mathrm{Tor}_n(G,K)$, since $\mathrm{Tor}_n(G,K) = 0$ for every $n \geqq 1$ and for every K, if and only if G is a torsion-free abelian group.

For the particular case where $K = A(X)$, the presheaf canonically associated with the topological space X, we define *the* n^{th} *homology group of* X *with coefficients in* G as being

$$H_n(G,X) = [H_n(G,A(X))]_X .$$

The homology groups $H_n(G,X)$ do not, in general, coincide with the homology groups $H_n(X,G)$ defined in [3], as it is shown by the following example, using an idea in Hilton–Wylie [2, page 360]:

Let X be the subset of the Cartesian plane consisting of the set of points

$$\left(x, \sin\frac{1}{x}\right), \quad 0 < x \leqq 1 ,$$

and the set of points

$$(0,y), \quad -1 \leqq y \leqq 1 .$$

Then it is easy to show that, for any sheaf G over X,

$$H_0(G,x) \cong G \quad \text{while} \quad H_0(X,G) \cong G \otimes G .$$

It is natural to define the *domain of the spread* $(g_U)_{U \in \mathscr{U}}$ as being $\bigcup_{U \in \mathscr{U}} U$; it is an open set of X.

In the collection $\mathcal{O}(X)$ of open sets of X, we may introduce a topology as follows. Given $U \in \mathcal{O}(X)$ and $x_1, \ldots, x_n \in U$, let

$$\mathcal{N}_{x_1,\ldots,x_n}(U) = \{U' \in \mathcal{O}(X) \mid x_1, \ldots, x_n \in U'\}.$$

These sets constitute a fundamental system of neighborhoods of U for a topology in $\mathcal{O}(X)$, which may be said to be of Zariski type, since the closure of $\{U\}$ is equal to $\{U' \in \mathcal{O}(X) \mid U' \supset U\}$.

We may also define a topology on $\mathcal{S}$, in the following manner. Given $(g_U)_{U \in \mathcal{U}} \in \mathcal{S}$ and any elements $x_1, \ldots, x_n$ belonging to the domain of $(g_U)_{U \in \mathcal{U}}$, let

$$\mathcal{N}_{x_1,\ldots,x_n}((g_U)_{U \in \mathcal{U}}) = \{(g_U')_{U \in \mathcal{U}'} \in \mathcal{S} \mid \text{for every } i = 1, \ldots, n, \text{ there exists } U_i \in \mathcal{U} \cap \mathcal{U}' \text{ such that } x_i \in U_i \text{ and } g'_{U_i} = g_{U_i}\}.$$

Again, these sets constitute a fundamental system of neighborhoods of $(g_U)_{U \in \mathcal{U}}$ for a topology on $\mathcal{S}$. It is straightforward to verify that the mapping $\mathcal{S} \to \mathcal{O}(X)$ which associates with every spread its domain is a continuous mapping.

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