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A COVERING PROPERTY OF SIMPLEXES

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1.

A classical result of Sperner [4] (see also [3]) states that if an n-simplex $S=v_0v_1\ldots v_n$ is covered by $n+1$ closed sets A_i, $0\leqq i\leqq n$, such that for each i, A_i is disjoint from $v_0v_1\ldots v_{i-1}v_{i+1}\ldots v_n$, then $\bigcap_{i=0}^n A_i \neq \emptyset$. This can be formulated as follows: If an n-simplex $S=v_0v_1\ldots v_n$ is covered by $n+1$ closed sets A_i, $0\leqq i\leqq n$, such that $v_0v_1\ldots v_{i-1}v_{i+1}\ldots v_n \subset A_i$ for each i, then $\bigcap_{i=0}^n A_i \neq \emptyset$. Each of these two statements can be easily derived from the other. More precise than the second statement is the following result.

THEOREM 1. *Let $A_0,A_1,\ldots,A_n$ be $n+1$ closed subsets of an n-simple $S=v_0v_1\ldots v_n$ such that $v_0v_1\ldots v_{i-1}v_{i+1}\ldots v_n \subset A_i$ for $0\leqq i\leqq n$, an $\bigcap_{i=0}^n A_i = \emptyset$. If B is a closed set such that $B\cup(\bigcup_{i=0}^n A_i)=S$, then for ever non-empty subset I of $\{0,1,\ldots,n\}$, we have*

$$B\cap\Big(\bigcap_{i\notin I} A_i\Big)\cap\Big(\bigcap_{i\in I} A_i'\Big) \neq \emptyset\,, \tag{1}$$

where $A_i' = S\setminus A_i$.

The assertion amounts to say that

$$\Big(\bigcap_{i\notin I} A_i\Big)\cap\Big(\bigcap_{i\in I} A_i'\Big) \not\subset \operatorname{Int}\bigcup_{i=0}^{n} A_i \tag{2}$$

for every non-empty subset I of $\{0,1,\ldots,n\}$. Here Int denotes the interior relative to S. A closely related result is the following one.

THEOREM 2. *Let $T=v_0v_1\ldots v_p$ be a p-simplex, and let S denote the n-face $v_0v_1\ldots v_n$ of T, where $0\leqq n<p$. Let f be a continuous mapping from S into the n-skeleton of T and having the following two properties:*

(3) *For each $i=0,1,\ldots,n$ and for every point x in the $(n-1)$-face $v_0v_1\ldots v_{i-1}v_{i+1}\ldots v_n$ of S, $f(x)$ is in a face of T not containing the vertex v_i.*

(4) *$f(S)$ is disjoint from the face $v_{n+1}v_{n+2}\ldots v_p$ of T.*

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Then for every continuous mapping g from S into S, there exists a point $x \in S$ such that $f(x)=g(x)$. In particular: $S \subset f(S)$, and f has a fixed point.

We recall that the n-skeleton of T is the union of all n-faces of T. Theorem 2 specializes to the Brouwer fixed point theorem when we take the identity mapping of S as f.

2.

We first prove Theorem 2, which will be used in the proof of Theorem 1.

Proof of Theorem 2. For $x \in S$, write

$$f(x) = \sum_{j=0}^{p} \alpha_j(x) v_j, \qquad g(x) = \sum_{i=0}^{n} \beta_i(x) v_i ,$$

where $\alpha_j(x) \geqq 0$, $0 \leqq j \leqq p$, $\sum_{j=0}^{p} \alpha_j(x) = 1$ and $\beta_i(x) \geqq 0$, $0 \leqq i \leqq n$, $\sum_{i=0}^{n} \beta_i(x) = 1$. For $0 \leqq i \leqq n$, let A_i denote the set of all $x \in S$ satisfying

$$\alpha_i(x) \left[1 + \sum_{j=n+1}^{p} \alpha_j(x)\right] \leqq \beta_i(x) \left[1 - \sum_{j=n+1}^{p} \alpha_j(x)\right]. \tag{5}$$

By property (3) of f, we have $\alpha_i(x)=0$ for $x \in v_0 v_1 \ldots v_{i-1} v_{i+1} \ldots v_n$, $0 \leqq i \leqq n$. Thus $v_0 v_1 \ldots v_{i-1} v_{i+1} \ldots v_n \subset A_i$ for $0 \leqq i \leqq n$. For each $x \in S$, $f(x)$ is in the n-skeleton of T, so at least $p-n$ of $\alpha_j(x)$, $0 \leqq j \leqq p$, are 0. If $\alpha_{n+1}(x), \alpha_{n+2}(x), \ldots, \alpha_p(x)$ are not all 0, then there is an index h such that $0 \leqq h \leqq n$ and $\alpha_h(x)=0$, so $x \in A_h$. On the other hand, if $\alpha_{n+1}(x) = \alpha_{n+2}(x) = \ldots = \alpha_p(x) = 0$, then $\sum_{i=0}^{n} \alpha_i(x) = \sum_{i=0}^{n} \beta_i(x) = 1$, so there is an index k such that $0 \leqq k \leqq n$ and $\alpha_k(x) \leqq \beta_k(x)$. Then $x \in A_k$. This shows that $\bigcup_{i=0}^{n} A_i = S$. By Sperner's result, it follows that $\bigcap_{i=0}^{n} A_i \neq \emptyset$.

Consider a point $x_0 \in \bigcap_{i=0}^{n} A_i$. Summing (5) over $i=0,1,\ldots,n$, we obtain

$$\left[1 - \sum_{j=n+1}^{p} \alpha_j(x_0)\right] \left[1 + \sum_{j=n+1}^{p} \alpha_j(x_0)\right] \leqq 1 - \sum_{j=n+1}^{p} \alpha_j(x_0) . \tag{6}$$

Since $f(x_0) \notin v_{n+1} v_{n+2} \ldots v_p$ by (4), we have $\sum_{j=n+1}^{p} \alpha_j(x_0) < 1$. Therefore (6) implies $\alpha_j(x_0) = 0$ for $n+1 \leqq j \leqq p$. Then $\sum_{i=0}^{n} \alpha_i(x_0) = \sum_{i=0}^{n} \beta_i(x_0) = 1$, and (5) becomes $\alpha_i(x_0) \leqq \beta_i(x_0)$, $0 \leqq i \leqq n$. Hence $\alpha_i(x_0) = \beta_i(x_0)$, $0 \leqq i \leqq n$, and $f(x_0) = g(x_0)$.

By considering those g which map S to a single point of S, we conclude that $S \subset f(S)$. On the other hand, if we take the identity mapping of S as g, then the existence of a fixed point of f follows.

Proof of Theorem 1. We regard $S = v_0 v_1 \ldots v_n$ as a face of an $(n+1)$-simplex $T = v_0 v_1 \ldots v_n v_{n+1}$. For $x \in S$, let $\delta_i(x)$ denote the distance from

x to A_i, $0 \leqq i \leqq n$, and let $\delta_{n+1}(x)$ denote the distance from x to B. Since $\bigcap_{i=0}^n A_i = \emptyset$, we have $\sum_{i=0}^n \delta_i(x) > 0$ for all $x \in S$. Let

$$f(x) = \left(\sum_{j=0}^{n+1} \delta_j(x)\right)^{-1} \sum_{i=0}^{n+1} \delta_i(x) v_i , \qquad x \in S .$$

The relation $B \cup (\bigcup_{i=0}^n A_i) = S$ means that for every $x \in S$, at least one of $\delta_0(x), \delta_1(x), \ldots, \delta_{n+1}(x)$ is 0. Thus f is a continuous mapping from S into the n-skeleton of T. If $0 \leqq i \leqq n$ and $x \in v_0 v_1 \ldots v_{i-1} v_{i+1} \ldots v_n$, then $\delta_i(x) = 0$ and therefore $f(x) \in v_0 v_1 \ldots v_{i-1} v_{i+1} \ldots v_n v_{n+1}$. As $\bigcap_{i=0}^n A_i = \emptyset$, v_{n+1} is not in $f(S)$.

By Theorem 2, we have $S \subset f(S)$. Therefore for any $n+1$ non-negative numbers $\gamma_i \geqq 0$, $0 \leqq i \leqq n$, with $\sum_{i=0}^n \gamma_i = 1$, there is a point $x \in S$ for which

$$\delta_i(x) = \gamma_i \sum_{j=0}^{n} \delta_j(x) \quad \text{for} \quad 0 \leqq i \leqq n, \qquad \delta_{n+1}(x) = 0 . \tag{7}$$

In particular, for any non-empty subset I of $\{0, 1, \ldots, n\}$, if the γ's are so chosen that $\gamma_i > 0$ for $i \in I$ and $\gamma_i = 0$ for $i \notin I$, then every point x satisfying (7) is in the intersection

$$B \cap \left(\bigcap_{i \notin I} A_i\right) \cap \left(\bigcap_{i \in I} A_i'\right).$$

3.

A theorem of Ghouila-Houri [2] on a combinatorial property of convex sets extends a result of Berge [1] and has other interesting corollaries. As an application of Theorem 1, the following result sharpens Ghouila-Houri's theorem.

COROLLARY 1. *In a topological vector space E, let $C_1, C_2, \ldots, C_n$ be n closed convex sets such that $\bigcap_{i=1}^n C_i = \emptyset$, and let $u_j \in \bigcap_{i \neq j} C_i$ for each $j = 1, 2, \ldots, n$. If D is a closed set in E such that $D \cup (\bigcup_{i=1}^n C_i)$ contains the convex hull of $\{u_1, u_2, \ldots, u_n\}$, then for every non-empty subset I of $\{1, 2, \ldots, n\}$, we have*

$$D \cap \left(\bigcap_{i \notin I} C_i\right) \cap \left(\bigcap_{i \in I} C_i'\right) \neq \emptyset , \tag{8}$$

where $C_i' = E \setminus C_i$.

This specializes to Ghouila-Houri's theorem when I contains only one index.

PROOF. Consider an $(n-1)$-simplex $S=v_1v_2\dots v_n$ and define a continuous mapping f from S into $D\cup(\bigcup_{i=1}^n C_i)$ by $f(\sum_{j=1}^n \alpha_j v_j)=\sum_{j=1}^n \alpha_j u_j$ for $\alpha_j \geqq 0$, $1\leqq j\leqq n$, with $\sum_{j=1}^n \alpha_j=1$. Let $A_i=f^{-1}(C_i)$, $1\leqq i\leqq n$, and $B=f^{-1}(D)$. Then $B\cup(\bigcup_{i=1}^n A_i)=S$. Since $u_j\in C_i$ for $i\neq j$ and C_i is convex, we have $f(x)\in C_i$ for $x\in v_1v_2\dots v_{i-1}v_{i+1}\dots v_n$. In other words, $v_1v_2\dots v_{i-1}v_{i+1}\dots v_n\subset A_i$ for $1\leqq i\leqq n$. We have also $\bigcap_{i=1}^n A_i=\emptyset$, because $\bigcap_{i=1}^n C_i=\emptyset$. Applying Theorem 1, we obtain (1) and therefore (8) for every non-empty subset I of $\{1,2,\dots,n\}$.

COROLLARY 2. *Let $f_0,f_1,\dots,f_n$ be $n+1$ real-valued continuous functions defined on an n-simplex $S=v_0v_1\dots v_n$ such that for each $i=0,1,\dots,n$, $f_i(x)\leqq 0$ for $x\in v_0v_1\dots v_{i-1}v_{i+1}\dots v_n$. Then either there exists a point $x\in S$ satisfying $f_i(x)\leqq 0$ for all i; or for every non-empty subset I of $\{0,1,\dots,n\}$, there exists an $x\in S$ such that $f_i(x)>0$ for $i\in I$ and $f_i(x)=0$ for $i\notin I$.*

PROOF. This follows immediately from Theorem 1 by considering the sets $A_i=\{x\in S\colon f_i(x)\leqq 0\}$, $0\leqq i\leqq n$, and $B=\{x\in S\colon f_i(x)\geqq 0 \text{ for all } i\}$.

EXAMPLE. Consider a real $n\times n$ matrix $M=(a_{ij})$ such that $a_{ij}\leqq 0$ for $i\neq j$ and $\sum_{i=1}^n a_{ij}\geqq 0$ for $1\leqq j\leqq n$. Define functions f_i, $1\leqq i\leqq n$, by $f_i(x)=\sum_{j=1}^n a_{ij}x_j$ for $x=(x_1,x_2,\dots,x_n)$ with $x_j\geqq 0$, $1\leqq j\leqq n$, and $\sum_{j=1}^n x_j=1$. Then an application of Corollary 2 yields the following well-known fact: If M is singular, there exist n non-negative numbers x_j, $1\leqq j\leqq n$, not all zero, such that $\sum_{j=1}^n a_{ij}x_j=0$ for $1\leqq i\leqq n$. If M is non-singular, then all elements of M^{-1} are non-negative.

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