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MEASURE THEORY FOR C^* ALGEBRAS II

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As the title indicates, this paper is a sequel of [4], to which we refer the reader for motivation and general terminology. The main result is the extension of the notion of C^* integrals, introduced in [4], to cover also non-positive integrals. Before this we establish some auxiliary results about order-related C^* subalgebras, some of which may have independent interest. In section 3 we divert ourselves with the very simple example, already mentioned in [4], of a C^* algebra generated by two projections. Since, however, the set of available examples of C^* algebras is very small, we feel justified in doing so.

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1. Order-related C^* subalgebras.

Let A be a C^* algebra universally represented as operators on the Hilbert space H, and let A'' be the double commutant of A. Then A'' is also the weak closure of A in $B(H)$, and as a vector space it is iso morphic to the second dual of A. We let X denote the set of projections from A'' which can be approximated strongly from below by elements from A.

If S is the set of positive linear functionals on A, and M and N are subsets of S and A^+, respectively, we denote by $M^\perp$ (resp. $N^\perp$) the elements in A^+ (resp. S) vanishing on M (resp. N).

A *subalgebra B of A is called order-related (or hereditary) if B^+ is an order ideal in A^+, and B is the linear span of B^+.

THEOREM 1.1. *There is a one-to-one correspondence between*

(1) *order-related C^* subalgebras of A,*
(2) *closed left ideals in A,*

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(3) *weak* closed order ideals in* S,
(4) *elements in* X.

PROOF. (1) $\leftrightarrow$ (2). Let B be an order-related C^* subalgebra of A. Then B^+ is a closed order ideal, and by 1.1 in [4] the set

$$L = \{a \in A \mid a^* a \in B^+\}$$

is a closed left ideal with $B = L^* \cap L$. Conversely, if L is given, define $B = L^* \cap L$. Then B is a C^* algebra with $B^+ = L^+$. To show that L^+ is an order ideal take $a \in A^+$, $b \in L^+$ with $a \leqq b$, and let $\{u_\lambda\}$ be the right approximate identity for L contained in L^+, defined in 1.7.3 of [2]. Then $a^{\frac{1}{2}} u_\lambda \in L$, and since $b^{\frac{1}{2}} \in L$ we have

$$\begin{aligned} \|a^{\frac{1}{2}} u_\lambda - a^{\frac{1}{2}}\|^2 &= \|(1-u_\lambda) a (1-u_\lambda)\| \\ &\leqq \|(1-u_\lambda) b (1-u_\lambda)\| = \|b^{\frac{1}{2}}(1-u_\lambda)\|^2 \to 0 . \end{aligned}$$

It follows that $a^{\frac{1}{2}} \in L$, hence $a \in L^+$.

(1) $\leftrightarrow$ (3). This is 5.1 in [3], just as another proof of (1) $\leftrightarrow$ (2) can be found as theorem 2.4 in the same paper. The correspondence between a closed order ideal B^+ of A^+ and a weak* closed order ideal P of S is given by

$$B^+ = P^\perp, \qquad P = B^{+\perp} .$$

In particular we notice that the smallest closed order ideal containing a subset J of A^+ is $J^{\perp\perp}$.

(1) $\leftrightarrow$ (4). If B is an order-related C^* subalgebra of A, then the above mentioned approximative identity for B converges strongly up to a projection $p \in X$. We have $pB = B$ and p is the smallest projection in A'' with that property. If conversely $p \in X$, define $B = pA''p \cap A$, and we have $pB = B$ and p is upper strong limit of elements from B.

THEOREM 1.2. *A positive functional on an order-related* C^* *subalgebra* B *of* A *has exactly one norm-preserving (hence positive) extension to* A.

PROOF. Let $\{u_\lambda\}$ be the approximative unit in B^+ with u_λ converging strongly to $p \in X$. For any $x \in A''$ we can find a net $\{a_i\} \subset A$ converging strongly to x, and if $x \in pA''p$, then since $\{u_\lambda\}$ is a bounded set, the net $\{u_\lambda a_i u_\lambda\} \subset B$ will converge strongly to $pxp = x$. We conclude that $pA''p$ is the weak closure of B.

Now let f be a state of B, and let $\tilde{f}$ be an extension of f to a state of A. Since A is ultra-weakly dense in A'', there is exactly one normal extension of $\tilde{f}$ (again denoted $\tilde{f}$) to A''. Since B is ultra-weakly dense in $pA''p$,

$\tilde{f}$ is uniquely determined on $pA''p$ as the normal extension of f, and $\tilde{f}(p) = \lim \tilde{f}(u_\lambda) = 1$. But $\tilde{f}$ has norm 1 so that $\tilde{f}(1-p) = 0$, and hence for any $a \in A$ we have

$$\tilde{f}(a) = \tilde{f}(pap) = \lim f(u_\lambda a u_\lambda) .$$

THEOREM 1.3. *The restriction to B of an irreducible representation π of A on the Hilbert space H is irreducible on the closed subspace $\pi(B)H$.*

PROOF. We set $K = \overline{\pi(B)H}$, and consider the restriction of π to B on K. For any pair of vectors $\xi, \eta \in K$ with $\xi \neq 0$ there exists $a \in A$ such that $\pi(a)\xi = \eta$. Since π extends to a normal homomorphism of A'', we have $\pi(u_\lambda)$ converging strongly up to $\pi(p)$ and $K = \pi(p)H$. Finally we have $u_\lambda a u_\lambda \in B$ and

$$\begin{aligned} \|\pi(u_\lambda a u_\lambda)\xi - \eta\| &\leqq \|\pi(u_\lambda a u_\lambda - u_\lambda a p)\xi\| + \|\pi(u_\lambda)\eta - \eta\| \\ &\leqq \|u_\lambda a\| \, \|\pi(u_\lambda)\xi - \xi\| + \|\pi(u_\lambda)\eta - \eta\| \to 0 . \end{aligned}$$

It follows that $\pi(B)$ acts topologically, hence algebraically irreducibly on K, and hence $K = \pi(B)H$.

COROLLARY 1.4. *The restriction to B of a pure state of A is a multiple of a pure state (possibly 0).*

PROOF. If f is pure on A, then there exists an irreducible representation π of A on H, and a vector $\xi \in H$ such that $f(a) = (\pi(a)\xi, \xi)$ for $a \in A$. If η is the projection of ξ on the subspace $\pi(B)H$, then $f(b) = (\pi(b)\eta, \eta)$ for $b \in B$. Since therefore the restriction of f is associated with an irreducible representation of B, it must be pure.

COROLLARY 1.5. *The sum of a maximal, closed left ideal L and a closed right ideal R is closed in A.*

PROOF. The set $B = R^* \cap R$ is an order-related C^* subalgebra, and by 2 in [5], the difference space $A - L$ is a Hilbert space in the quotient norm. Now the left regular representation of A on $A - L$ is irreducible, hence B acts irreducibly on the closed subspace $B(A-L)$. However, the counter image of $B(A-L)$ in A is $BA + L$, and is therefore closed. Now BA is dense in R and we have the inclusions

$$BA + L \subset R + L \subset \overline{BA + L} = BA + L .$$

Hence $R + L$ is closed in A. (Notice that we do not assert that $BA = R$.)

Theorem 1.6. *The map* $\varrho\colon(\pi,H)\to(\pi|B,\pi(B)H)$ *induces a homeomorphism between* $\hat{A}\setminus\operatorname{hull}B$ *and* $\hat{B}$.

Proof. By theorem 1.3, ϱ is a mapping from $\operatorname{Irr}A\setminus\operatorname{hull}B$ into $\operatorname{Irr}B$, and since any irreducible representation of a C^* subalgebra is the restriction of an irreducible representation of A to a subspace (2.10.2 in [2]), ϱ is onto. If (π,H) and (π',H') from $\operatorname{Irr}A\setminus\operatorname{hull}B$ have equivalent restrictions, then by changing, if necessary, (π,H) into an equivalent representation, we may assume $\varrho(\pi,H)=\varrho(\pi',H')$.

Any vector state g on B associated with a unit vector in $\pi(B)H=\pi'(B)H'$ is pure and has via π and π' extensions f and f' to A which are also pure. However, by theorem 1.2 the extension of g is unique, and so $f=f'$. Since therefore π and π' are associated with one and the same pure state, they are equivalent. It follows that ϱ induces a map $\hat{\varrho}$ from $\hat{A}\setminus\operatorname{hull}B$ onto $\hat{B}$ which is one-to-one.

To prove that $\hat{\varrho}$ is a homeomorphism, let F be a set in $\hat{A}\setminus\operatorname{hull}B$ such that $\hat{\varrho}(F)$ is closed. If π belongs to the closure of F, then $\ker F\subset\ker\pi$, and so

$$\ker F\cap B\subset\ker\pi\cap B\,,$$

that is, $\ker\hat{\varrho}(F)\subset\ker\hat{\varrho}(\pi)$. But then $\hat{\varrho}(\pi)\in\hat{\varrho}(F)$, and so $\pi\in F$.

Conversely if F is closed in $\hat{A}\setminus\operatorname{hull}B$, and we have $\pi\in\hat{A}\setminus\operatorname{hull}B$, with $\hat{\varrho}(\pi)$ in the closure of $\hat{\varrho}(F)$, then as before $\ker F\cap B\subset\ker\pi\cap B$. By theorem 1.1 there is a left ideal L such that $B=L^*\cap L$, and $a\in L$ iff $a^*a\in B^+$. It follows that

$$\ker F\cap L\subset\ker\pi\cap L\,.$$

Since $\ker\pi$ is a primitive ideal, it is also prime, and so $\ker F\subset\ker\pi$ or $L\subset\ker\pi$. Since by assumption the latter possibility is ruled out, we have $\ker F\subset\ker\pi$, and so $\pi\in F$, that is, $\hat{\varrho}(F)$ is closed.

2. C^* integrals.

If A is a commutative C^* algebra without unit, that is, of the form $C_0(T)$ with T locally compact Hausdorff, then the order-related C^* subalgebras of A are no other than the closed ideals of A, and the elements in X correspond to the open sets in T.

Hence for a non-commutative C^* algebra A without unit the subset

$$Y=\{p\in X\mid \exists a\in A\colon p\leqq a\}$$

becomes of particular interest since its elements are the non-commutative analogues of open sets in the underlying space, with compact closure.

For $p \in Y$ let $B(p)$ be the corresponding order-related C^* subalgebra of operators from A with range projections below p, and let K be the smallest order-related *subalgebra of A containing all $B(p)$. Then [4, Theorem 1.3] K is a dense, order-related, two-sided ideal in A, minimal among all such.

If M is a subset of A^+, we let $\operatorname{Conv} M$ denote the convex hull of M. Since we want not only convex sets, but also sets which have the hereditary property that with an element they contain all elements below, we introduce the set

$$\text{h-Conv}\, M = \{a \in A^+ \mid \exists b \in \operatorname{Conv} M:\ a \leqq b\}.$$

(Notice that since A does not satisfy the Riesz decomposition property, $\operatorname{Conv} M$ may not have the hereditary property even if M has it.)

Furthermore we introduce the set

$$\operatorname{Sym} M = \operatorname{Conv} \bigcup \{\theta M \mid \theta \in \mathbf{C},\ |\theta| = 1\}$$

and for $\varepsilon > 0$

$$M_\varepsilon = \{a \in M \mid \|a\| < \varepsilon\}.$$

In this notation we have

$$K = \operatorname{Sym} \text{h-Conv} \bigcup \{B(p)^+ \mid p \in Y\}.$$

Let U denote the group of unitary operators in the C^* algebra obtained by adjoining an identity to A. For $u \in U$ the map $p \to u^*pu$ is clearly an automorphism of Y, and so U introduces an equivalence relation in Y. We let $\tilde{Y}$ be the set of equivalence classes with elements

$$\tilde{p} = \{u^*pu \mid u \in U\}.$$

The set of maps from $\tilde{Y}$ into the (strictly) positive real numbers is denoted Δ, and for $\delta \in \Delta$ define

$$E_\delta{}^+ = \text{h-Conv} \bigcup \{B(p)^+_{\delta(\tilde{p})} \mid p \in Y\}$$

and

$$E_\delta = \operatorname{Sym} E_\delta{}^+.$$

Clearly the sets $E_\delta{}^+$ and E_δ are convex, absorbing sets in K^+ and K, respectively. Moreover E_δ is symmetric and $(E_\delta)^+ = E_\delta{}^+$.

A vector space topology on K is called locally hereditary-convex if it has a basis of symmetric, convex neighbourhoods around 0 whose positive parts satisfy the hereditary property.

THEOREM 2.1. *The sets E_δ, $\delta \in \Delta$, and their translates form a basis for a locally hereditary-convex topology τ on K. It is the strongest locally heredi-*

tary-convex topology in which multiplication is jointly continuous, uniformly over normbounded sets, and in which all injections from the C^* algebras $B(p)$, $p \in Y$, into K are continuous.

PROOF. Clearly the sets E_δ constitute a basis for a locally hereditary-convex topology on K in which all injections $B(p) \to K$ are continuous. To prove that multiplication is uniformly continuous over norm-bounded sets, it suffices to show that $E_\delta A_1 \subset E_{16\delta}$.

To this end we pick $a \in E_\delta{}^+$, $b \in A_1$. Then

$$ab = \tfrac{1}{4} \sum_{n=0}^{3} i^n (1 - i^n b)^* a (1 - i^n b) .$$

Each element $1 - i^n b$ has norm less than 2, and thus has a representation as a sum of 4 elements from U. But

$$(u_1 + u_2 + u_3 + u_4)^* \, a \, (u_1 + u_2 + u_3 + u_4) \leqq 4(u_1{}^* a u_1 + u_2{}^* a u_2 + u_3{}^* a u_3 + u_4{}^* a u_4) ,$$

and since by definition $E_\delta{}^+$ is invariant under unitary transformations, we conclude that for each n

$$(1 - i^n b)^* a (1 - i^n b) \in 16 E_\delta{}^+$$

and thus $ab \in E_{16\delta}$. Finally

$$E_\delta A_1 = \operatorname{Sym}(E_\delta{}^+) A_1 \subset \operatorname{Sym}(E_\delta{}^+ A_1) \subset \operatorname{Sym} E_{16\delta} = E_{16\delta} .$$

If E is a neighbourhood around 0 in another topology σ of the above mentioned type, there is, since multiplication is uniformly continuous over norm-bounded sets, another neighbourhood E' in σ such that $u^* E' u \subset E$ for any $u \in U$. Since the injections $B(p) \to K$ are σ-continuous, there exists for each $p \in Y$ an $\varepsilon(p) > 0$ such that $B(p)^+_{\varepsilon(p)} \subset E'$. Hence for any function $\delta \in \Delta$ such that $\delta(\tilde{p}) = \varepsilon(p)$ for some $p \in \tilde{p}$ we have

$$\bigcup \{ u^* B(p)^+_{\delta(\tilde{p})} u \mid u \in U \} \subset E .$$

Since we may suppose E hereditary-convex and symmetric, we conclude that $E_\delta \subset E$, and thus, τ is stronger than σ.

The elements of the dual of (K, τ) are called the C^* integrals of A. If $1 \in A$, then also $1 \in Y$, and so $K = A$, and τ coincides with the norm topology. Hence the C^* integrals of A are just the elements of the dual of A.

If $1 \notin A$ and A is commutative, that is, $A = C_0(T)$, then τ will just be the inductive limit topology on $K(T)$ induced by the mappings $C_0(p) \to$

$K(T)$, where p ranges over the relatively compact open subsets of T. By the theorem of F. Riesz the C^* integrals of A are then the Radon measures on T.

Returning to the non-commutative case, we call a functional f on K unitarily bounded, if for all $a \in K$

$$\sup \{|f(u^*au)| \mid u \in U\} < \infty .$$

THEOREM 2.2. *The positive C^* integrals of A are exactly the unitarily bounded, positive functionals on K.*

PROOF. Since the neighbourhoods E_δ are invariant under unitary transformations, any C^* integral will be unitarily bounded. Conversely, if f is a positive functional on K, then it is bounded on each of the C^* algebras $B(p)$ with a norm $\|f\|_p$. And if f is also unitarily bounded, then

$$\|f\|_{\tilde{p}} = \sup \{\|f\|_p \mid p \in \tilde{p}\} < \infty$$

and so for a $\delta \in \Delta$, with $\delta(\tilde{p})\|f\|_{\tilde{p}} < 1$ for all $\tilde{p} \in \tilde{Y}$, we have

$$|f(E_\delta)| = f(E_\delta{}^+) = \operatorname{Conv}\{f(B(p)^+_{\delta(\tilde{p})}) \mid p \in Y\} < 1 ,$$

and f is τ-continuous.

THEOREM 2.3. *Any C^* integral can be decomposed as a linear combination of at most four positive C^* integrals.*

PROOF. If f is a C^* integral, then the complex conjugate function is also a C^* integral, and we have the usual decomposition of f in real and imaginary parts. So we may as well assume that f is a real valued, continuous functional on (K^R, τ). (For any *algebra B, we write B^R for the self-adjoint elements in B.)

Now let S_1 denote the set of positive linear functionals on A different from 0, and with norm less than or equal 1. Then S_1 is a locally compact Hausdorff space in the weak* topology, and we have an isometric injection of A^R into $C_0{}^R(S_1)$. We identify A^R with its image in $C_0{}^R(S_1)$, and define for $a \in K^+$, $\delta \in \Delta$:

$$F(a) = \{x \in C_0{}^R(S_1) \mid |x| \leqq a\} ,$$
$$F = \bigcup \{F(a) \mid a \in K^+\} ,$$
$$F_\delta = \bigcup \{F(a) \mid a \in E_\delta{}^+\} .$$

Then F is a real vector space, and the sets F_δ and their translates form a basis for a locally convex topology on F. For each $\delta \in \Delta$ we have

$$E_\delta{}^R \subset F_\delta \cap K^R \subset E_{2\delta}^R$$

so that the restriction of the topology in F to the subspace K^R gives the topology τ on K^R.

Let F_δ be a neighbourhood of 0 in F such that $|f(F_\delta \cap K^R)| \leqq 1$. The Minkowski functional Φ defined by

$$\Phi(x) = \inf\{\alpha > 0 \mid \alpha^{-1}x \in F_\delta\}$$

is a norm on F, and $|f(x)| \leqq \Phi(x)$ for $x \in K^R$. Let $\tilde{f}$ be a Hahn-Banach extension of f from K^R to F, with respect to the norm Φ. If we can prove that $\tilde{f}$ is relatively bounded on F, then since F is a vector lattice, we know [1, Chap. II, § 2, Théorème 1] that $\tilde{f}$ splits into the difference of positive parts.

Since for any chosen $x \in F^+$ there is a constant α such that $\alpha x \in F_\delta{}^+$, we may as well assume $x \in F_\delta{}^+$. But then any $y \in F$ with $|y| \leqq x$ will also belong to F_δ, and it follows that

$$h(x) = \sup\{|f(y)| \mid |y| \leqq x\} \leqq 1 .$$

By the above mentioned theorem there exist two positive functionals f_1 and f_2 with $\tilde{f} = f_1 - f_2$ and $h = f_1 + f_2$. Since $a \in E_\delta{}^+$ implies $u^*au \in E_\delta{}^+$ for all $u \in U$, we conclude that

$$\sup\{h(u^*au) \mid u \in U\} \leqq 1$$

so that the restrictions of f_1 and f_2 to K^R are unitarily bounded positive functionals, and hence, by theorem 2.2, C^* integrals.

If A is the algebra $B_0(H)$ of compact operators on the Hilbert space H, then K consists of the operators of finite rank, X contains all projections on H, and Y is the set of finite dimensional projections. The set of equivalence classes $\tilde{Y}$ is therefore isomorphic to N, and neighbourhoods around 0 in τ are given by sets of the form

$$\text{Sym h-Conv} \bigcup \{a \in K^+ \mid \dim a \leqq n,\ \|a\| < \delta_n\}$$

for various sequences $\{\delta_n\}$. By elementary calculations this system is proved to be equivalent to the well-known system of neighbourhoods

$$\{a \in K \mid \operatorname{tr}(a^*a)^{\frac{1}{2}} < \varepsilon\}$$

for various ε.

The positive C^* integrals were determined in [4, Theorem 3.8] by an isomorphism with $B^+(H)$, and by theorem 2.3 we now have $(K,\tau)^*$ isomorphic to $B(H)$, where the integral f and the operator $b \in B(H)$ are linked by the formula $f(a) = \operatorname{tr}(ba)$ for all $a \in K$.

3. An example.

Let A be the C^* algebra generated by two projections p and q on a Hilbert space H. We put $a=pqp$ and have $0 \leqq a \leqq 1$. We are going to show that, apart from minor modifications, A is completely determined up to *isomorphisms by $\mathrm{Sp}(a)$. The situation should be compared with the well-known result in the converse direction: To any operator a on a Hilbert space H with $0 \leqq a \leqq 1$ there exist projections p and q on a larger space H' such that $pqp=a$ on $H=pH'$.

Our first step is to find the structure of the (not necessarily proper) closed two-sided ideal A_0, the closure of the set of all polynomials in p and q with no first degree terms. For this purpose we introduce the notations

$$\mathrm{Sp}'(a) = \mathrm{Sp}(a)\setminus\{0\} \quad \text{and} \quad \mathrm{Sp}''(a) = \mathrm{Sp}'(a)\setminus\{1\}.$$

THEOREM 3.1. *$\hat{A}_0$ is homeomorphic to $\mathrm{Sp}'(a)$. The representation corresponding to* 1 *(if $1 \in \mathrm{Sp}'(a)$) is one-dimensional, while the remaining elements of $\hat{A}_0$ are two-dimensional.*

PROOF. The order-related C^* subalgebra pA_0p is the closure of the set of polynomials in a and hence $pA_0p=C_0(\mathrm{Sp}'(a))$. By theorem 1.6 we have $\mathrm{Sp}'(a)$ homeomorphic to $\hat{A}_0\setminus\mathrm{hull}(pA_0p)$, but since $\pi \in \mathrm{hull}(pA_0p)$ implies $\pi(pq)=0$ hence $\pi(A_0)=0$, we conclude that $\mathrm{hull}(pA_0p)=\emptyset$.

The order-related C^* subalgebra $(1-p)A_0(1-p)$ is the closure of the set of polynomials in $(1-p)q(1-p)$, so that $(1-p)A_0(1-p)$ is also commutative. Since for any $b \in A_0{}^+$ we have

$$b \leqq 2(pbp+(1-p)b(1-p)),$$

and since the restriction to pA_0p or $(1-p)A_0(1-p)$ of an irreducible representation of A_0 is one-dimensional, we conclude that $\hat{A}_0$ consists of at most two-dimensional representations.

Thus if π_α is the element in $\hat{A}_0$ corresponding to $\alpha \in \mathrm{Sp}'(a)$, we have

$$\pi_\alpha(a) = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}.$$

Since all π_α extend canonically to representations of A on the same space, we conclude that apart from unitary equivalence the only possible images for $\pi_\alpha(p)$ and $\pi_\alpha(q)$ are

$$\pi_\alpha(p) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \pi_\alpha(q) = \begin{pmatrix} \alpha & (\alpha-\alpha^2)^{\frac{1}{2}} \\ (\alpha-\alpha^2)^{\frac{1}{2}} & 1-\alpha \end{pmatrix}.$$

From this the theorem follows.

For any $b \in A_0$ we define complex functions b_{ij}, $i,j=1,2$, on $\mathrm{Sp}(a)$ by the definitions

$$\begin{aligned} b_{ij}(0) &= 0\,, \\ b_{ij}(\alpha) &= \pi_\alpha(b)_{ij} \quad \text{for } \alpha \in \mathrm{Sp}''(a)\,, \\ b_{11}(1) &= \pi_1(b) \quad \text{if } 1 \in \mathrm{Sp}(a)\,, \\ b_{ij}(1) &= 0 \quad \text{for } i,j \neq 1,1\,. \end{aligned}$$

The following equations are immediate:

$$a^n{}_{ij}(\alpha) = \alpha^n p_{ij}(\alpha), \qquad (pq)^n{}_{ij}(\alpha) = \alpha^{n-1}(pq)_{ij}(\alpha)$$

It follows that when b runs through all polynomials in p and q, then b_{11} in turn gives all polynomials in α, b_{12} and b_{21} give all polynomials in α including those with constant terms, but all multiplied by a factor $(\alpha-\alpha^2)^{\frac{1}{2}}$, and b_{22} gives all polynomials in α multiplied by $1-\alpha$.

For all $b \in A_0$ we have by 3.3.6 in [2]

$$\|b_{12}\| \leqq \|b\| \quad \text{but} \quad \|b_{12}\| = \|pb(1-p)\|$$

and similar expressions for other choices of i,j, so that a net of operators converges iff the corresponding four nets of functions converge. An application of the Stone-Weierstrass theorem now yields the following

THEOREM 3.2.

$$A_0 = \begin{pmatrix} C_0(\mathrm{Sp}'(a)) & C_0(\mathrm{Sp}''(a)) \\ C_0(\mathrm{Sp}''(a)) & C_0(\mathrm{Sp}''(a)) \end{pmatrix}.$$

For any operator b let $[b]$ denote the range projection of b. We can then state the following

LEMMA 3.3.

$$p-[pqp] \perp q \quad \textit{and} \quad q-[qpq] \perp p\,.$$

PROOF. $p-[pqp]$ is the lower strong limit of polynomials $(p-pqp)^n$ and hence

$$\begin{aligned} q(p-[pqp])q &\leqq q(p-pqp)^n q \\ &= qpq(q-qpq)^n \to qpq(q-[qpq]) = 0\,. \end{aligned}$$

We are now able to give the precise description of A:

THEOREM 3.4. *For* $0 \notin \overline{\mathrm{Sp}'(a)}$ *we have*

(1) $A = A_0$ *for* $[pqp] = p$ *and* $[qpq] = q$,
(2) $A = A_0 \oplus \mathsf{C}$ *if either* $[pqp] \neq p$ *or* $[qpq] \neq q$,
(3) $A = A_0 \oplus \mathsf{C} \oplus \mathsf{C}$ *if both* $[pqp] \neq p$ *and* $[qpq] \neq q$.

For $0 \in \overline{\mathrm{Sp}'(a)}$ *we have (regardless of* $[pqp]$ *and* $[qpq]$)

$$A = \begin{pmatrix} C(\mathrm{Sp}(a)) & C_0(\mathrm{Sp}''(a)) \\ C_0(\mathrm{Sp}''(a)) & C_0(\mathrm{Sp}(a) \setminus \{1\}) \end{pmatrix}. \tag{4}$$

PROOF. If 0 is isolated in $\mathrm{Sp}(a)$, then since $\mathrm{Sp}'(pqp) = \mathrm{Sp}'(qpq)$, 0 is isolated in $\mathrm{Sp}(qpq)$ and we have $[pqp] \in A_0$ and $[qpq] \in A_0$. An application of the lemma now proves the first three cases.

If 0 is a limit point in $\mathrm{Sp}(a)$, then neither $[pqp]$ nor $[qpq]$ belongs to A_0. It follows that the C^* algebra generated by these two projections is *isomorphic to A, and thus we may as well assume $p = [pqp]$ and $q = [qpq]$.

If we think of all π_α, $\alpha \in \mathrm{Sp}''(a)$, as representations on one and the same two-dimensional Hilbert space, then their weak limit as $\alpha \to 0$ is also a representation π_0 for which

$$\pi_0(A_0) = 0, \quad \pi_0(p) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi_0(q) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows that we can define functions p_{ij} and q_{ij} on $\mathrm{Sp}(a)$ such that p_{11} and q_{22} are non-vanishing for $\alpha = 0$, which proves the theorem.

Clearly π_0 can be decomposed into two complex homomorphisms π_p and π_q, and since $A/A_0 = \mathbf{C} \oplus \mathbf{C}$, these are the only representations in $\mathrm{hull} A_0$. Hence we have the following

COROLLARY 3.5. *In case* (4), $\hat{A}$ *is homeomorphic to* $\mathrm{Sp}'(a) \cup \{\pi_p\} \cup \{\pi_q\}$, *where both* π_p *and* π_q *are limit points when* $\alpha \to 0$.

Since A_0 is a C^* algebra with continuous trace (in fact every element has continuous trace) we infer from [4, Theorem 1.5] that $K(A_0)$ consists of those $b \in A_0$ for which b_{ij} vanishes in a neighbourhood of 0 for all i,j.

If we turn to A and consider only the interesting case (4), then A is a CCR algebra with compact, but non-Hausdorff structure space, and we have no general theorems about $K(A)$. However by definition $p,q \in K(A)$, and since $K(A)$ is an ideal, we also have $bpc \in K(A)$ for all $b,c \in A$. But

$$(bpc)_{ij} = \begin{pmatrix} b_{11}c_{11} & b_{11}c_{12} \\ b_{21}c_{11} & b_{21}c_{12} \end{pmatrix},$$

hence $ApA = A_0 + p\mathbf{C}$ and $K(A) = A$.

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