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ON A THEOREM OF ERDÖS AND TURAN

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1.

Let $f(n)$ be an arbitrary positive decreasing function defined for every natural number n with values < 1. Let $x_1, x_2, \ldots, x_P$ be any real numbers and let $N(P)$ denote the number of integers n such that

$$0 < x_n \leqq f(n) \pmod 1, \qquad 1 \leqq n \leqq P .$$

If the numbers $x_1, x_2, \ldots, x_P$ are sufficiently well distributed (mod 1), it is natural to expect $N(P)$ to be asymptotically equal to $\sum_{n=1}^{P} f(n)$. If $f(n) = f$ is constant, there is a well known theorem of Erdös and Turan [1, Theorem III] which gives an estimate for the error term. This theorem is the strongest known quantitative form of the famous Weyl's criteria in the theory of uniform distribution modulo one. In this paper our intention is to generalize the theorem of Erdös and Turan to the above situation. We shall prove the following

THEOREM. *Let*

$$\varphi_q(P) = \max \left\{ \left| \sum_{n=1}^{k} e^{2\pi i q (x_n - f(n))} \right| \;\middle|\; k = 1, 2, \ldots, P \right\},$$

$$\psi_q(P) = \max \left\{ \left| \sum_{n=1}^{k} e^{2\pi i q x_n} \right| \;\middle|\; k = 1, 2, \ldots, P \right\}.$$

Then, for any natural number m, we have

$$\left| N(P) - \sum_{n=1}^{P} f(n) \right| < 32 \left(\frac{P}{m} + \sum_{q=1}^{2m-2} (\varphi_q(P) + \psi_q(P)) q^{-1} \right).$$

2.

As in the paper of Erdös and Turan we use the so-called Dunham Jackson means of the characteristic functions of the intervals in question. Let $J_m(t)$ denote the Dunham Jackson kernel

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$$J_m(t) = \left(\frac{\sin \pi m t}{\sin \pi t}\right)^4$$

$$= m^2 + 4m \sum_{k=1}^{m-1} (m-k) \cos 2\pi k t + 4 \sum_{k=1}^{m-1} \sum_{l=1}^{m-1} (m-k)(m-l) \cos 2\pi k t \cos 2\pi l t .$$

Then we have

$$R_m = \int_0^1 J_m(t)\, dt = \tfrac{1}{3} m(2m^2+1) . \tag{1}$$

For any real numbers α, β such that $\alpha < \beta \leqq \alpha + 1$, we denote

$$\chi_m(x; \alpha, \beta) = \frac{1}{R_m} \int_\alpha^\beta \left(\frac{\sin \pi m(t-x)}{\sin \pi(t-x)}\right)^4 dt = \frac{1}{R_m} \int_{\alpha-x}^{\beta-x} J_m(t)\, dt .$$

Then

$$0 < \chi_m(x; \alpha, \beta) \leqq 1 \tag{2}$$

for every value of x. An elementary computation gives

$$\chi_m(x; \alpha, \beta) = \beta - \alpha + \sum_{q=1}^{2m-2} \frac{d_q}{\pi q} \left(\sin 2\pi q(\beta - x) - \sin 2\pi q(\alpha - x)\right) , \tag{3}$$

where

$$d_q = \begin{cases} 1 - \dfrac{3q}{2m} \dfrac{(2m-q)q+1}{2m^2+1} & \text{for} \quad 1 \leqq q \leqq m-1 , \\[2ex] \dfrac{2m-q}{2m} \dfrac{(2m-q)^2-1}{2m^2+1} & \text{for} \quad m \leqq q \leqq 2m-2 . \end{cases} \tag{4}$$

By (4), it is easy to see that

$$0 < d_q < 1, \qquad 1 \leqq q \leqq 2m-2 . \tag{5}$$

Clearly

$$J_m(t) = J_m(1-t) . \tag{6}$$

Put

$$\delta(s) = \int_{s/m}^{(s+1)/m} J_m(t)\, dt, \qquad s = 0, 1, 2, \ldots, m-1 .$$

For $1 \leqq s \leqq \frac{1}{2}m - 1$, we have

$$\delta(s) < \sin^{-4}\pi\frac{s}{m}\int_{s/m}^{(s+1)/m}\sin^4\pi mt\,dt \tag{7}$$

$$< \frac{1}{16}\left(\frac{m}{s}\right)^4\frac{1}{\pi m}\int_{s\pi}^{(s+1)\pi}\sin^4u\,du = \frac{3}{128}m^3s^{-4}.$$

If m is odd and $\geqq 3$, we have, similarly,

$$\delta\left(\frac{m-1}{2}\right) < \frac{3}{128}\frac{1}{m}\left(\frac{2m}{m-1}\right)^4. \tag{8}$$

Hence, for any natural number r, $1 \leqq r \leqq \frac{1}{2}m$, we obtain, by (6), (7), (8),

$$\int_{r/m}^{\frac{1}{2}} J_m(t)\,dt < \frac{3}{128}m^3\sum_{s=r}^{\infty}s^{-4} < \frac{1}{32}\left(\frac{m}{r}\right)^3 \tag{9}$$

and

$$\int_0^{r/m} J_m(t)\,dt > \tfrac{1}{2}R_m - \frac{1}{32}\left(\frac{m}{r}\right)^3. \tag{10}$$

Suppose now that $2r/m \leqq \beta-\alpha \leqq 1$, $r \geqq 1$, and $\alpha + r/m \leqq x \leqq \beta - r/m$. Then, by (1) and (6),

$$\chi_m(x;\alpha,\beta) = \frac{1}{R_m}\int_{\alpha-x}^{\beta-x} J_m(t)\,dt \geqq \frac{2}{R_m}\int_0^{r/m} J_m(t)\,dt$$

$$> 1 - \frac{1}{16}\frac{1}{R_m}\left(\frac{m}{r}\right)^3 > 1 - \frac{3}{32}r^{-3}.$$

Similarly, if $2r/m \leqq 1+\alpha-\beta < 1$, $r \geqq 1$, and $\beta + r/m \leqq x \leqq 1+\alpha-r/m$, then

$$\chi_m(x;\alpha,\beta) = \frac{1}{R_m}\int_{\alpha-x}^{\beta-x} J_m(t)\,dt \leqq \frac{2}{R_m}\int_{r/m}^{\frac{1}{2}} J_m(t)\,dt < \frac{3}{32}r^{-3}.$$

So we obtain the estimates

$$\chi_m(x;\alpha,\beta)\begin{cases} > 1-\frac{3}{32}r^{-3} & \text{for} \quad \alpha+r/m \leqq x \leqq \beta - r/m, \\ < \frac{3}{32}r^{-3} & \text{for} \quad \beta + r/m \leqq x \leqq 1+\alpha - r/m, \end{cases} \tag{11}$$

valid for every natural number r such that $1 \leqq r \leqq m/2$. Because of the periodicity, the x-intervals may, of course, be taken modulo 1.

3.

For any real numbers α, β such that $\alpha < \beta \leqq \alpha + 1$, we write

$$\chi(x;\alpha,\beta) = \begin{cases} 1 & \text{if } \alpha < x \leqq \beta \pmod 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let k and l be any natural numbers such that $1 \leqq k \leqq l \leqq P$ and let γ be any real number. For $m \geqq 3$, we then have, using (3), (5), (11),

$$(12) \qquad \sum_{n=k}^{l} \chi(x_n;\gamma+f(n),\gamma+f(n)+1/m)$$

$$\leqq \tfrac{32}{29} \sum_{n=k}^{l} \chi_m(x_n,\gamma+f(n)-1/m,\gamma+f(n)+2/m)$$

$$\leqq \tfrac{32}{29}\left(\frac{3P}{m} + \sum_{q=1}^{2m-2} \frac{1}{\pi q}\left(\left|\sum_{n=k}^{l} \sin 2\pi q(\gamma+f(n)+2/m-x_n)\right| + \right.\right.$$

$$\left.\left. + \left|\sum_{n=k}^{l} \sin 2\pi q(\gamma+f(n)-1/m-x_n)\right|\right)\right)$$

$$< \frac{4P}{m} + \sum_{q=1}^{2m-2} \frac{1}{q}\left(\left|\sum_{n=k}^{l} \sin 2\pi q(x_n-f(n))\right| + \left|\sum_{n=k}^{l} \cos 2\pi q(x_n-f(n))\right|\right)$$

$$< \frac{4P}{m} + 4\sum_{q=1}^{2m-2} \varphi_q(P)q^{-1}.$$

Similarly, we have

$$(13) \qquad \sum_{n=k}^{l} \chi(x_n;\gamma,\gamma+1/m) < 4P/m + 4\sum_{q=1}^{2m-2} \psi_q(P)q^{-1}.$$

4.

We shall now derive a lower bound for $N(P)$. We obtain

$$(14) \qquad N(P) = \sum_{n=1}^{P} \chi(x_n;0,f(n))$$

$$\geqq \sum_{n=1}^{P} \chi_m(x_n;0,f(n)) - \sum_{n=1}^{P} \chi_m(x_n;0,f(n))\chi(x_n;f(n),1)$$

$$\geqq \sum_{n=1}^{P} f(n) - 1/\pi \sum_{q=1}^{2m-2} (\varphi_q(P)+\psi_q(P))q^{-1} -$$

$$- \sum_{n=1}^{P} \chi_m(x_n;0,f(n))\chi\big(x_n;f(n),\tfrac{1}{2}(1+f(n))\big) -$$

$$- \sum_{n=1}^{P} \chi_m(x_n;0,f(n))\chi\big(x_n;\tfrac{1}{2}(1+f(n)),1\big).$$

We denote the last two sums by Σ_1 and Σ_2, respectively. Put

$$a_n = [\tfrac{1}{2}m(1-f(n))] .$$

Then $\{a_n\}$ is a non-decreasing sequence of non-negative integers. Let $c_0 = -1$, $k_0 = 0$ and determine the integers k_i and c_i, $i = 1, 2, \ldots, s$, so that

$$a_{k_{i-1}+1} = a_{k_{i-1}+2} = \ldots = a_{k_i} = c_i, \quad i = 1, 2, \ldots, s ,$$

where

$$0 \leqq c_1 < c_2 < \ldots < c_s, \qquad 0 < k_1 < k_2 < \ldots < k_s = P .$$

We then obtain

$$\begin{aligned}
\Sigma_1 &= \sum_{n=1}^{P} \sum_{r=1}^{a_n} \chi_m(x_n;\, 0, f(n))\, \chi\left(x_n; f(n) + \frac{r-1}{m}, f(n) + \frac{r}{m}\right) + \\
&\quad + \sum_{n=1}^{P} \chi_m(x_n;\, 0, f(n))\, \chi\left(x_n; f(n) + \frac{a_n}{m}, \tfrac{1}{2}(1 + f(n))\right) \\
&\leqq \sum_{n=1}^{P} \sum_{r=1}^{a_n+1} \chi_m(x_n;\, 0, f(n))\, \chi\left(x_n; f(n) + \frac{r-1}{m}, f(n) + \frac{r}{m}\right) \\
&= \sum_{i=1}^{s} \sum_{r=c_{i-1}+2}^{c_i+1} \sum_{n=k_{i-1}+1}^{P} \chi_m(x_n;\, 0, f(n))\, \chi\left(x_n; f(n) + \frac{r-1}{m}, f(n) + \frac{r}{m}\right) \\
&= \sum_{i=1}^{s} X_i ,
\end{aligned}$$

say. Put

$$\Phi = \frac{4P}{m} + 4 \sum_{q=1}^{2m-2} \varphi_q(p) q^{-1} ,$$

$$\Psi = \frac{4P}{m} + 4 \sum_{q=1}^{2m-2} \psi_q(P) q^{-1} .$$

For $i \geqq 3$, we have $c_{i-1} \geqq 1$, $c_i \geqq 2$, so that (11) and (12) imply

$$\text{(15)} \qquad \begin{aligned} X_i &\leqq \tfrac{3}{32} \Phi \left(\sum_{r=c_{i-1}+2}^{c_i} (r-1)^{-3} + (c_i - 1)^{-3} \right) \\ &\leqq \tfrac{3}{16} \Phi \sum_{r=c_{i-1}+2}^{c_i-1} r^{-3} . \end{aligned}$$

For $i = 1, 2$, we have similarly

$$\text{(16)} \qquad X_1 \leqq \Phi \left(2 + \tfrac{3}{32} \sum_{r=1}^{c_1-1} r^{-3} \right)$$

and

$$X_2 \leqq \Phi\left(1+\tfrac{3}{32}\sum_{r=c_1+1}^{c_2-1} r^{-3}\right). \tag{17}$$

Hence, by (15), (16), (17),

$$\Sigma_1 \leqq \Phi\left(3+\tfrac{3}{16}\sum_{r=1}^{\infty} r^{-3}\right) < 4\Phi . \tag{18}$$

In the same way one obtains the estimate

$$\Sigma_2 < 4\Psi . \tag{19}$$

Hence (14), (18), (19) imply the required result

$$N(P) > \sum_{n=1}^{P} f(n) - 32\left(\frac{P}{m}+\sum_{q=1}^{2m-2}(\varphi_q(P)+\psi_q(P))q^{-1}\right). \tag{20}$$

5.

Starting from the expression

$$\begin{aligned} N(P) &= \sum_{n=1}^{P}\chi(x_n;\, 0,f(n)) \\ &= \sum_{n=1}^{P}\chi_m(x_n;\, 0,f(n)) + \sum_{n=1}^{P}\left(1-\chi_m(x_n;\, 0,f(n))\right)\chi(x_n;\, 0,f(n)) \end{aligned}$$

one can apply a similar procedure and obtain the upper bound

$$N(P) < \sum_{n=1}^{P} f(n) + 32\left(\frac{P}{m}+\sum_{q=1}^{2m-2}(\varphi_q(P)+\psi_q(P))q^{-1}\right). \tag{21}$$

The Theorem follows from (20) and (21).

LITERATURE

1. P. Erdös and P. Turan, *On a problem in the theory of uniform distribution*, Indag. Math. 10 (1948), 370–378, 406–413.

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