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AN EQUATION OF FINITE DIFFERENCES, WHICH HAS SOME CONNECTION WITH THE JACOBIAN THETAFUNCTIONS

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In the following we consider the equation of finite differences

$$u(x)+a_1u(x+1)+a_2u(x+2)(1-\lambda q^{x+1}) = 0 ,$$

where a_1 and $a_2 \neq 0$ are constants, λ a parameter and $|q|<1$. To obtain a solution of this equation we introduce a power series in the parameter λ writing

$$u(x) = \sum c_n \lambda^n {z_n}^x .$$

Putting the coefficient of λ^n equal to zero, we get

$$c_n({z_n}^x+a_1{z_n}^{x+1}+a_2{z_n}^{x+2}) = a_2q^{x+1}z_{n-1}^{x+2}c_{n-1} ,$$

which may be written

$$c_n(1+a_1z_n+a_2{z_n}^2) = a_2qz_{n-1}^2c_{n-1}(qz_{n-1}/z_n)^x .$$

To obtain coefficients which are independent of x, we take

$$z_n = qz_{n-1} = zq^n ,$$

and get

$$(*) \qquad c_n(1+a_1zq^n+a_2z^2q^{2n}) = a_2q^{2n-1}z^2c_{n-1} .$$

Putting $n=0$, we get

$$c_0(1+a_1z+a_2z^2) = 0 .$$

We assume that the equation

$$1+a_1z+a_2z^2 = 0$$

has two different roots r_1 and r_2 and moreover that they do not satisfy an equation

$$r_1 = r_2q^m,$$

where m is an integer. Taking $z=r_1$, we get from (*)

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$$c_n(1-q^n)\left(1-\frac{r_1}{r_2}q^n\right) = \frac{r_1}{r_2}q^{2n-1}c_{n-1}\,,$$

or

$$c_n = c_0 \frac{q^{n^2}\left(\dfrac{r_1}{r_2}\right)^n}{(1-q)(1-q^2)\dots(1-q^n)\left(1-\dfrac{r_1}{r_2}q\right)\dots\left(1-\dfrac{r_1}{r_2}q^n\right)}.$$

In order to abbreviate we introduce the following notations:

$$\Omega_n(q) = \Omega_n = (1-q)(1-q^2)\dots(1-q^n), \quad \Omega_0 = 1\,,$$

and

$$\varphi_n\left(\frac{r_1}{r_2}\right) = \left(1-\frac{r_1}{r_2}q\right)\left(1-\frac{r_1}{r_2}q^2\right)\dots\left(1-\frac{r_1}{r_2}q^n\right), \quad \varphi_0 = 1\,.$$

Then

$$u_1(x) = c_0 r_1{}^x \sum_{n=0}^{\infty} \frac{q^{n^2}(\lambda q^x)^n\left(\dfrac{r_1}{r_2}\right)^n}{\Omega_n\,\varphi_n\left(\dfrac{r_1}{r_2}\right)}.$$

Similarly we get

$$u_2(x) = c_0 r_2{}^x \sum_{n=0}^{\infty} \frac{q^{n^2}(\lambda q^x)^n\left(\dfrac{r_2}{r_1}\right)^n}{\Omega_n\,\varphi_n\left(\dfrac{r_2}{r_1}\right)}.$$

However, we may find another form of the solution which is better adapted for our purpose. Let

$$\psi_n(\lambda) = (1-\lambda)(1-\lambda q)\dots(1-\lambda q^{n-1}), \quad \psi_0 = 1\,.$$

If we expand this after powers of λ, we find

$$\psi_n(\lambda) = \sum_{p=0}^{n}(-1)^p \frac{\Omega_n}{\Omega_p\,\Omega_{n-p}}\, q^{\frac{1}{2}(p^2-p)}\lambda^p\,,$$

where $\Omega_n/(\Omega_p\Omega_{n-p})$ is a generalized binomial coefficient.

Since $|q|<1$, we may also let n tend to infinity, and then we have the formula

$$\prod_{\nu=0}^{\infty}(1-\lambda q^\nu) = \sum_{p=0}^{\infty}(-1)^p \frac{q^{\frac{1}{2}(p^2-p)}}{\Omega_p}\lambda^p\,,$$

an expansion which we shall use in the following.

We observe that

$$\psi_n(\lambda) = (1-q^{-n})\psi_{n-1}(\lambda q) + q^{-n}\psi_n(\lambda q) .$$

We now assume

$$u_1(x) = r_1^x \sum c_n \psi_n(\lambda q^x) .$$

Introducing this series, we get

$$\sum \left(c_n \psi_n(\lambda q^x) + a_1 r_1 c_n \psi_n(\lambda q^{x+1}) + a_2 r_1^2 c_n \psi_{n+1}(\lambda q^{x+1})\right) = 0 .$$

Substituting

$$\psi_n(\lambda q^x) = (1-q^{-n})\psi_{n-1}(\lambda q^{x+1}) + q^{-n}\psi_n(\lambda q^{x+1})$$

we get

$$(1-q^{-(n+1)})c_{n+1} + c_n q^{-n} + a_1 r_1 c_n + a_2 r_1^2 c_{n-1} = 0 .$$

Since

$$a_1 = -\frac{1}{r_1} - \frac{1}{r_2}, \qquad a_2 = \frac{1}{r_1 r_2},$$

the equation may be written

$$(1-q^{-(n+1)})c_{n+1} - \frac{r_1}{r_2}c_n - (1-q^{-n})c_n + \frac{r_1}{r_2}c_{n-1} = 0 .$$

This equation is evidently satisfied if

$$(1-q^{-n})c_n - \frac{r_1}{r_2}c_{n-1} = 0$$

or

$$c_n = -\frac{\frac{r_1}{r_2}q^n}{1-q^n}c_{n-1}$$

and hence if

$$c_n = (-1)^n \frac{q^{\frac{1}{2}(n^2+n)}}{\Omega_n}\left(\frac{r_1}{r_2}\right)^n .$$

We thus get a solution in the form

$$u_1(x) = r_1^x \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}(n^2+n)}}{\Omega_n}\left(\frac{r_1}{r_2}\right)^n \psi_n(\lambda q^x) ,$$

and similarly

$$u_2(x) = r_2^x \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}(n^2+n)}}{\Omega_n}\left(\frac{r_2}{r_1}\right)^n \psi_n(\lambda q^x) .$$

To compare these solutions with the solution given above, we put $\lambda=0$. Then $\psi_n=1$, and

$$u_1(x) = r_1^x \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}(n^2+n)}}{\Omega_n} \left(\frac{r_1}{r_2}\right)^n = r_1^x \prod_{\nu=1}^{\infty} \left(1-\frac{r_1}{r_2}q^\nu\right).$$

If we put $\lambda=0$ in the first form of the solution, we get

$$u_1(x) = c_0 r_1^x .$$

By comparing the two values we find

$$c_0 = \prod_{\nu=1}^{\infty} \left(1-\frac{r_1}{r_2}q^\nu\right)$$

and

$$r_1^x \sum (-1)^n \frac{q^{\frac{1}{2}(n^2+n)}}{\Omega_n} \left(\frac{r_1}{r_2}\right)^n \psi_n(\lambda q^x) = r_1^x \prod_{\nu=1}^{\infty} \left(1-\frac{r_1}{r_2}q^\nu\right) \sum \frac{q^{n^2}\left(\frac{r_1}{r_2}\right)^n (\lambda q^x)^n}{\Omega_n \varphi_n\left(\frac{r_1}{r_2}\right)}.$$

Consider now the determinant

$$D(x) = \begin{vmatrix} u_1(x) & u_1(x+1) \\ u_2(x) & u_2(x+1) \end{vmatrix}.$$

From equation (1) we have

$$\begin{aligned} u_1(x)+a_1u_1(x+1) &= -a_2(1-\lambda q^{x+1})u_2(x+2) \\ u_2(x)+a_1u_2(x+1) &= -a_2(1-\lambda q^{x+1})u_2(x+2) \end{aligned}$$

which give

$$\begin{aligned} D(x) &= -a_2(1-\lambda q^{x+1}) \begin{vmatrix} u_1(x+2) & u_1(x+1) \\ u_2(x+2) & u_2(x+1) \end{vmatrix} \\ &= a_2(1-\lambda q^{x+1})\, D(x+1) . \end{aligned}$$

This is an equation of finite differences of the first order, and if we put

$$D(x) = \sum c_n \lambda^n (zq^n)^x$$

we easily get

$$z = r_1 r_2$$

and

$$c_n = c_0(-1)^n q^{\frac{1}{2}(n^2+n)}/\Omega_n$$

which gives

$$D(x) = (r_1r_2)^x c_0 \sum (-1)^n q^{\frac{1}{2}(n^2+n)} (\lambda q^x)^n = c_0 (r_1r_2)^x \prod_{\nu=1}^{\infty} (1-\lambda q^{x+\nu}) .$$

It remains to find the value of c_0. If we put $\lambda=0$, we get

$$u_1(x,0) = r_1{}^x \prod_{\nu=1}^{\infty}\left(1-\frac{r_1}{r_2}q^\nu\right)$$

and

$$u_1(x+1,0) = r_1{}^{x+1} \prod_{\nu=1}^{\infty}\left(1-\frac{r_1}{r_2}q^\nu\right).$$

For u_2 we have to exchange r_1 and r_2. For $\lambda=0$ we then have

$$D(x)_{\lambda=0} = (r_1r_2)^x(r_2-r_1)\prod_{\nu=1}^{\infty}\left(1-\frac{r_1}{r_2}q^\nu\right)\left(1-\frac{r_2}{r_1}q^\nu\right).$$

The result is that

$$D(x) = (r_1r_2)^x(r_2-r_1)\prod_{\nu=1}^{\infty}(1-\lambda q^{x+\nu})\left(1-\frac{r_1}{r_2}q^\nu\right)\left(1-\frac{r_2}{r_1}q^\nu\right).$$

From this formula we conclude that the two solutions are independent if

$$r_1 \neq r_2 q^m,$$

where m is a positive integer or zero.

We shall now consider the special case $x=0$ and $\lambda=1$. We then have

$$D(0,1) = (r_2-r_1)\prod_{\nu=1}^{\infty}(1-q^\nu)\left(1-\frac{r_1}{r_2}q^\nu\right)\left(1-\frac{r_2}{r_1}q^\nu\right).$$

On the other hand we get

$$u_1(0,1) = 1, \qquad u_1(1,1) = r_1\sum(-1)^n q^{\frac{1}{2}(n^2+n)}\left(\frac{r_1}{r_2}\right)^n$$

and similarly

$$u_2(0,1) = 1, \qquad u_2(1,1) = r_2\sum(-1)^n q^{\frac{1}{2}(n^2+n)}\left(\frac{r_2}{r_1}\right)^n,$$

and consequently

$$D(0,1) = \sum(-1)^n q^{\frac{1}{2}(n^2+n)}\left(\frac{r_2{}^{n+1}}{r_1{}^n}-\frac{r_1{}^{n+1}}{r_2{}^n}\right).$$

Replacing q by q^2, we get the fundamental formula

$$\sum_{n=0}^{\infty}(-1)^n q^{n^2+n}\left(\frac{r_2{}^{n+1}}{r_1{}^n}-\frac{r_1{}^{n+1}}{r_2{}^n}\right) = (r_2-r_1)\prod_{\nu=1}^{\infty}(1-q^{2\nu})\left(1-\frac{r_1}{r_2}q^{2\nu}\right)\left(1-\frac{r_2}{r_1}q^{2\nu}\right)$$

from which we may deduce the fundamental formulae for the Jacobian thetafunctions.

We now choose

$$r_2 = e^{i\pi v}, \qquad r_1 = e^{-i\pi v}.$$

Then

$$\sum_{n=0}^{\infty} (-1)^n q^{n^2+n} 2i \sin(2n+1)\pi v = 2i \sin\pi v \prod_{\nu=1}^{\infty} (1-q^{2\nu}) \prod_{\nu=1}^{\infty} (1-2q^{2\nu} \cos 2\pi v + q^{4\nu}) .$$

Multiplying by the factor $q^{\frac{1}{4}}$ and cancelling the factor i, we get

$$2\sum_{n=0}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} \sin(2n+1)\pi v = 2q^{\frac{1}{4}} \sin\pi v \prod_{\nu=1}^{\infty} (1-q^{2\nu}) \prod_{\nu=1}^{\infty} (1-2q^{2\nu} \cos 2\pi v + q^{4\nu}) ,$$

which is the fundamental formula for the Jacobian function $\vartheta_1(v,q)$. Then we take $r_2 = e^{i\pi v}$ and $r_1 = qe^{-i\pi v}$ and get

$$\sum_{n=0}^{\infty} (-1)^n q^{n^2} e^{(2n+1)i\pi v} - \sum_{n=0}^{\infty} (-1)^n q^{(n+1)^2} e^{-(2n+1)i\pi v} = e^{i\pi v}\Big(1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2n\pi v\Big) .$$

On the right hand side we get

$$e^{i\pi v} \prod_{\nu=1}^{\infty} (1-q^{2\nu})(1-qe^{-2i\pi v}) \prod_{\nu=1}^{\infty} (1-q^{2\nu+1}e^{-2i\pi v})(1-q^{2\nu-1}e^{2i\pi v}) .$$

This gives the formula

$$1+2\sum_{n=1}^{\infty} (-1)^2 q^{n^2} \cos 2n\pi v = \prod_{\nu=1}^{\infty} (1-q^{2\nu}) \prod_{\nu=1}^{\infty} (1-2q^{2\nu-1} \cos 2\pi v + q^{4\nu-2}) ,$$

which is the fundamental formula for the function $\vartheta_0(v,q)$.

The two remaining thetafunctions are obtained by replacing v by $v+\frac{1}{2}$.

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