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ON THE MACKEY-TOPOLOGY FOR A VON NEUMANN ALGEBRA

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1. Introduction.

We consider a von Neumann algebra A as a C^*-algebra which is also a dual space as a Banach space. Our interest is aimed at the duality (A,A_*), where A_* is the pre-dual of A. The study of the Mackey-topology $\tau(A,A_*)$ for a von Neumann algebra A was initiated by S. Sakai in [12], [15]. He pointed out that the extremal property of this topology must be a useful tool in the theory of von Neumann algebras. We hope the present paper will support this view.

Basic for the understanding of the properties of the Mackey-topology is our knowledge about the weakly (that is $\sigma(A_*,A)$-)compact subsets of A_*. For the commutative case, characterizations of these sets have been known for some time [3], [4], [8]. That similar characterizations were available in the general, non-commutative case seemed probable, and results in this direction were obtained in [1], [15], [16], [17], and finally by C. Akemann in [2]. For the sake of completeness, we start with presenting these characterizations of $\sigma(A_*,A)$-compact subsets of A_*. The proofs are in several places different from Akemann's. We then go on to study some of the properties of the Mackey-topology. With this topology, A is seen to be complete, and even fully complete (B-complete) in the sense of Ptak [11]. In section 4 we give some properties of the Mackey-topology related to the order-structure of A. In the next section we consider the following problem: If B is a sub-von Neumann algebra of A, when does there exist a linear ultra-weakly continuous projection of A onto B? We obtain a partial answer to this, and apply it to the question whether the Mackey-topologies $\tau(B,B_*)$ and $\tau(A,A_*)$ coincide when the latter is relativized to B.

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2. Notation and basic concepts.

Throughout the paper, A, B will denote von Neumann algebras. By A_*, B_* we denote their pre-duals, and by A^*, B^* their norm-duals, respectively. A^+, A^h, A_1 denote the cone of positive elements, the hermitian elements and the unit sphere in A, respectively. $A_*{}^+$ and $A_*{}^h$ are the cone of positive elements and the hermitian elements in A_*, respectively. We call the elements of A_* *normal* linear functionals. A linear functional f on A is *completely additive* if for any family $\{e_\gamma\}_{\gamma\in\Gamma}$ of mutually orthogonal projections in A we have

$$f(\textstyle\sum_{\gamma\in\Gamma} e_\gamma) = \sum_{\gamma\in\Gamma} f(e_\gamma) .$$

An element $f \in A^*$ is normal if and only if it is completely additive [13] [16].

We consider several locally convex topologies for A, all of which are *admissible* in the sense of [10] relative to the duality (A,A_*):

(1) the norm topology (n) on A as the norm-dual of A_*;

(2) the Mackey-topology (τ) (that is, $\tau(A,A_*)$ is the topology of uniform convergence on relatively $\sigma(A_*,A)$-compact sets);

(3) the $*$-ultra-strong topology (s^*) given by the family of semi-norms $\{\alpha_p,\alpha_p{}^* :\ p \in A_*{}^+\}$ where

$$\alpha_p(x) = p(x^*x)^{\frac{1}{2}}, \quad \alpha_p{}^*(x) = p(xx^*)^{\frac{1}{2}}, \qquad x \in A ;$$

(4) the ultra-strong topology (s) given by the family of semi-norms $\{\alpha_p :\ p \in A_*{}^+\}$, and finally

(5) the ultra-weak topology (σ) (that is, $\sigma(A,A_*)$).

This sequence of topologies is monotone descending with respect to their strength, and the four last are compatible with the duality (A,A_*). For the elementary properties of these topologies (except (2)) we refer to [5]. We note some properties of (τ). Multiplication is separately continuous, indeed the maps $x \to x^*$, ax, xa and a^*xa, $a \in A$, are all continuous [12]. A co-base for this topology is given by the relatively $\sigma(A_*,A)$-compact subsets of $A_*{}^h$. This follows immediately from the fact that the map $f \to f^*$ is $\sigma(A_*,A)$-continuous for $f \in A_*$, and that each such f can be written $f = f_1 + if_2$, where $f_1 = \frac{1}{2}(f+f^*) \in A_*{}^h$ and $f_2 = -\frac{1}{2}i(f-f^*) \in A_*{}^h$.

If e is a projection in A, the von Neumann algebra $\{eae :\ a \in A\}$ will be denoted A_e. The linear map $a \to eae$ is (σ)-continuous. For $f \in A_*$, $a \in A$, we denote by af and f_a the linear functionals $x \to f(ax)$ and $x \to f(a^*xa)$ respectively. af and f_a are normal, and the linear maps $f \to af$ and $f \to f_a$ are $\sigma(A_*,A)$-continuous [12].

If $f \in A_*^+$, the *support* of f is the least projection e in A such that $f_e = f$. Then $f(e) = f(1) = \|f\|$, and $f(1-e) = 0$. Here 1 denotes the identity element of A. If $f \in A_*^h$, it may be uniquely written $f = f^+ - f^-$, where $f^+, f^- \in A_*^+$ and have mutually orthogonal supports e_1, e_2 respectively. We write $|f| = f^+ + f^-$. Then $|f|$ has support $e_1 + e_2$, and $f = (e_1 - e_2)|f|$. Note that $\|f\| = \|f^+\| + \|f^-\| = \||f|\|$. If e is a projection satisfying $e \geqq e_1 + e_2$, we have $f = f_e$. More generally, if $f \in A_*$ there is an element $p \in A_*^+$ and a partial isometry $u \in A$ (that is u^*u and uu^* are projections in A), such that $f = up$ and $p = u^*f$. The support of p is equal to uu^*, and we write $p = |f|$. If f is hermitian, the notation agrees with the one introduced above. We refer to the equation $f = u|f|$ as the polar decomposition of f (cf. [6], [12]).

If $\{a_\gamma\}_{\gamma \in \Gamma}$ is a net in A^+, the notation $a_\gamma \downarrow 0$; $\gamma \in \Gamma$ means that $\{a_\gamma\}$ is monotone decreasing, with $\inf_{\gamma \in \Gamma} a_\gamma = 0$.

A positive, linear functional p on A is *faithful* if $a \in A^+$, $a \neq 0$ implies that $p(a) \neq 0$. Recall that if A is commutative, then it is isometrically isomorphic to a space $L^\infty(S,\mu)$, and its pre-dual to $L^1(S,\mu)$ [19].

3. $\sigma(A_*, A)$-compact subsets of A_*.

Let A be a von Neumann algebra, and B a von Neumann sub-algebra of A. Each normal linear functional on A has a restriction to B, and conversely, each normal linear functional on B has a normal extension to A, by the Hahn–Banach Theorem. Let $r: A_* \to B_*$ denote the restriction map, so $r(f) = f/B$, $f \in A_*$, and r maps A_* onto B_*. It is easily seen that r is continuous with respect to the topologies $\sigma(A_*, A)$ and $\sigma(B_*, B)$ for A_* and B_* respectively.

THEOREM 1. *Let K be a subset of A_*. The following conditions are equivalent*:

(i) *K is relatively $\sigma(A_*, A)$-compact.*

(ii) *$r(K) \subseteq B_*$ is relatively $\sigma(B_*, B)$-compact for every (maximal) commutative, von Neumann sub-algebra B of A.*

(iii) *K is bounded, and each sequence of projections $\{e_n\}_{n \in N} \subseteq A$ such that $e_n \downarrow 0$, converges uniformly on K.*

(iv) *There is a positive, normal linear functional p on A with the following property*: *For each $\varepsilon > 0$, there is a $\delta > 0$ such that if*

$$x \in A, \qquad \|x\| \leqq 1, \qquad p(x^*x + xx^*) < \delta\,,$$

then

$$|f(x)| < \varepsilon \quad \textit{for all} \quad f \in K.$$

REMARK. The equivalence of (i) and (ii) seems first to have been noticed by Takesaki [16]. However, the proof of the implication (ii) $\Rightarrow$ (i) which we give here, is different from his. That (i) implies (iii) is essentially contained in [1]. The more embracing fact, that (i) implies (iv) was first proved by Sakai [15] for algebras of finite type, and then by Akemann in the general case [2].

PROOF OF THE THEOREM. (ii) $\Rightarrow$ (i). To show that K is relatively $\sigma(A_*,A)$-compact when (ii) holds, we will employ the Smulian compactness criterion ([10, 16.6 p. 142]). Thus, we have to prove that K is bounded in the $\sigma(A_*,A)$-topology, and that each linear functional f on A which is bounded on K° is normal, i.e. belongs to A_*. If $x \in A$ is self-adjoint, then it is contained in some (maximal) abelian von Neumann sub-algebra B of A, so

$$\sup_{f\in K}|f(x)| = \sup_{f\in K}|r(f)(x)| < \infty$$

since $r(K)$ is $\sigma(B_*,B)$-bounded. Every element in A is the linear combination of self-adjoint elements, so K is clearly $\sigma(A_*,A)$-bounded. Then K is also norm-bounded, so

$$K \subseteq S_m = \{g \in A_* : \|g\| \leqq m\}$$

for some positive integer m. Hence

$$K^\circ \supseteq S_m{}^\circ = \{x \in A : \|x\| \leqq 1/m\}.$$

So if f is bounded on K°, it is also bounded on $S_m{}^\circ$, which means that $f \in A^*$. Therefore, to prove that f is normal, let $\{e_\gamma\}_{\gamma\in\Gamma}$ be an arbitrary net of commuting projections in A such that $e_\gamma \downarrow 0$, $\gamma \in \Gamma$. We must show that $f(e_\gamma) \to 0$, $\gamma \in \Gamma$. Now take a (maximal) commutative von Neumann sub-algebra B of A which contains the family $\{e_\gamma\}_{\gamma\in\Gamma}$. Let

$$M = \sup\{|f(x)| : x \in K^\circ\}.$$

According to the hypothesis, $r(K) \subseteq B_*$ is relatively $\sigma(B_*,B)$-compact. Hence by the commutative result [4], [8], or [7, ch. IV. 8.11, p. 294], there is a $\gamma_0 \in \Gamma$ such that $|\psi(e_\gamma)| < \varepsilon/M$ for all $\psi \in r(K)$ whenever $\gamma \geqq \gamma_0$. That is $e_\gamma \in K_0\varepsilon/M$ for $\gamma \geqq \gamma_0$, so $|f(e_\gamma)| \leqq M\varepsilon/M = \varepsilon$ for $\gamma \geqq \gamma_0$. Hence $f(e_\gamma) \to 0$, and $f \in A_*$.

(iii) $\Rightarrow$ (ii). This is an immediate consequence of the commutative theorem, for example [7, ch. IV. 8.9, p. 292].

(iv) $\Rightarrow$ (iii). It is evident that if $e_n \downarrow 0$ where $\{e_n\}$ is a monotone sequence of projections in A, then (iv) implies that e_n converges uniformly to zero on K. It is also clear that (iv) implies boundedness of K.

(i) $\Rightarrow$ (iv). We establish this implication through three lemmas. The proofs of the first and the third are due to Akemann [2], the second we prove by a modification of an argument of Sakai [15], and is slightly shorter than the proof given in [2].

LEMMA 1. *Suppose* $\{a_n\}_{n\in N}\subseteqq A_1{}^h$, *and* $a_n \to 0$ (s) *as* $n\to\infty$. *Then, for* $\delta>0$ *given, there is a sequence* $\{e_n\}_{n\in N}$ *of projections in* A *such that* $e_n\to 1$ (s), *and* $\|a_n e_n\|\leqq\delta$ *for* $n=1,2,\ldots$.

PROOF. Let X be the characteristic function of the interval $]-\delta,\delta[$. By the functional calculus we may define $e_n = X(a_n)$ for each n. Then e_n is a projection in A, and we have $a_n{}^2\geqq\delta^2(1-e_n)\geqq 0$. Since $a_n\to 0$ (s) it follows that $1-e_n\to 0$ (s), thus $e_n\to 1$ (s). It is also clear that $\|a_n e_n\|\leqq\delta$ for all n.

LEMMA 2. *Let* K *be a relatively* $\sigma(A_*,A)$*-compact subset of* A_*, *and suppose that* $\{a_n\}_{n\in N}\subseteqq A_1$ *and* $a_n\to 0$ (s^*) *as* $n\to\infty$. *Then* $a_n\to 0$ *uniformly on* K *as* $n\to\infty$.

PROOF. Since $a_n\to 0$ (s^*) we know that $\{a_n\}$ and $\{a_n{}^*\}$ both converge (s) to 0. This implies that the self-adjoint and skew-adjoint parts of the sequence $\{a_n\}$ both converge (s) to 0. Hence we may and shall assume that the a_n are all self-adjoint. Suppose that the lemma is false. Then there is an $\varepsilon>0$ and sequences $\{f_i\}\subseteqq K$, $\{x_i\}\subseteqq A_1$, $i\in N$, where $\{x_i\}$ is a sub-sequence of $\{a_n\}$, such that

$$|f_i(x_i)| > \varepsilon, \qquad i=1,2,\ldots . \tag{1}$$

By the Eberlein Theorem [7, V. 6.1, p. 430] K is relatively sequentially compact in the $\sigma(A_*,A)$-topology, so we may assume that $f_i\to f_0\in A_*$ as $i\to\infty$, with respect to this topology. Now, let $\{e_i\}$ be as in lemma 1, that is, $e_i\to 1$ (s) and $\|x_ie_i\|\leqq\delta$, $i=1,2,\ldots$, for some arbitrarily chosen $\delta>0$. Then we have

$$\begin{aligned}|(f_0-f_j)(x_i)| &\leqq |(f_0-f_j)(x_ie_i)| + |(f_0-f_j)(x_i(1-e_i))|\\ &\leqq 2M\delta + |(f_0-f_j)(x_i(1-e_i))| ,\end{aligned}$$

where $M=\sup\{\|f\| : f\in K\}$. The sequence $\{f_j\}$ converges pointwise to f_0 on the unit sphere A_1 of A, which is $\sigma(A,A_*)$-compact. By the Osgood Theorem [10, ch. 3, 9.6, p. 86] there is a point $x_0\in A_1$ such that $\{f_j\}_{j=0,1,\ldots}$ is equi-continuous in x_0 when restricted to A_1. That is, we can find a $\sigma(A,A_*)$-neighborhood U of 0 in A, such that

$$x\in(x_0+U)\cap A_1 = V$$

implies that

$$|f_j(x)-f_j(x_0)| < \delta \quad \text{for} \quad j=0,1,\ldots .$$

Now choose j_0 such that $j \geqq j_0$ implies $|f_j(x_0) - f_0(x_0)| < \delta$. It follows that $|f_j(x) - f_0(x)| < 3\delta$ for $j \geqq j_0$ whenever $x \in V$.

Now put

$$y_i = e_i x_0 e_i + x_i(1 - e_i) .$$

Since by construction e_i and x_i commute, a simple computation shows that $y_i \in A_1$. Moreover, $e_i \to 1$ (s) implies that $y_i \to x_0$ (s), and hence also in the (σ)-topology. Then we can choose $i_0 \in N$ such that $i \geqq i_0$ implies that y_i and $e_i x_0 e_i$ belong to V. It follows that

$$|(f_0 - f_j)(x_i(1 - e_i))| \leqq |(f_0 - f_j) y_i| + |(f_0 - f_j)(e_i x_0 e_i)| < 6\delta$$

for $i \geqq i_0$, $j \geqq j_0$. Consequently

$$|(f_0 - f_j)(x_i)| < (2M + 6)\delta$$

for $i \geqq i_0, j \geqq j_0$. Since δ was arbitrary, and $f_0(x_i) \to 0$, this contradicts (1), and the lemma is proved.

LEMMA 3. *Let K be a relatively $\sigma(A_*, A)$-compact subset of A_*^h. Given any $\varepsilon > 0$ there exists $\delta > 0$ and a finite subset $\{f_1, \ldots, f_m\} \subseteq K$ such that, if $a \in A_1$ and*

$$|f_i| (a^* a + a a^*) < \delta, \quad i = 1, \ldots, n ,$$

then $|f(a)| < \varepsilon$ for all $f \in K$.

PROOF. Suppose the lemma is false for some $\varepsilon > 0$. Then we may construct sequences $\{a_n\} \subseteq A_1$ and $\{f_n\} \subseteq K$, such that if

$$U_n = \{x \in A_1 : |f_i|(x^*x + xx^*) < 2^{-n}, \ i \leqq n\} ,$$

then $a_n \in U_n$, but $|f_{n+1}(a_n)| > \varepsilon$. Put $f = \sum_{i=1}^{\infty} 2^{-i} |f_i|$, and let e be the support projection of f. Now

$$\begin{aligned} f(a_n^* a_n + a_n a_n^*) &= \sum_{i=1}^{\infty} 2^{-i} |f_i|(a_n^* a_n + a_n a_n^*) \\ &\leqq \sum_{i=1}^{n} 2^{-i} |f_i|(a_n^* a_n + a_n a_n^*) + \sum_{i=n+1}^{\infty} 2^{1-i} \|f_i\| \\ &\leqq 2^{-n} + 2^{1-n} M , \end{aligned}$$

where $M = \sup\{\|f\| : f \in K\}$, so

$$f(a_n^* a_n + a_n a_n^*) \to 0 \quad \text{as } n \to \infty .$$

It follows then by Proposition 4, p. 62 in [5], that $\{e a_n e\}$ is (s^*)-convergent to zero. By Lemma 2 this sequence then converges uniformly on K. However

$$|f_{n+1}(ea_n e)| = |f_{n+1}(a_n)| > \varepsilon ,$$

a contradiction, and the proof is complete.

PROOF OF THE IMPLICATION (i) $\Rightarrow$ (iv): It is sufficient to prove it when $K \subseteq A_*{}^h$. Then Lemma 3 applies; let $\varepsilon_n = n^{-1}$, and choose a matching $\delta > 0$ and $\{f_1{}^n, \ldots, f_{m_n}^{\ n}\} \subseteq K$ according to Lemma 3. Put

$$p = \sum_{n=1}^{\infty} 2^{-n} \left(\sum_{i=1}^{m_n} 2^{-i} |f_i{}^n| \right).$$

Then p has the required property with respect to K. The proof is complete.

COROLLARY 1. *The Mackey-topology $\tau(A, A_*)$ coincides with the (s^*)-topology on bounded sets in A.*

PROOF. As noted in Section 2, the Mackey-topology is stronger than the (s^*)-topology. That the converse holds on bounded sets, is now an immediate consequence of the implication (i) $\Rightarrow$ (iv) of the preceding theorem.

COROLLARY 2. *Multiplication is simultaneous (τ)-continuous on A_1.*

PROOF. The mapping $(x,y) \to xy$ of $A_1 \times A_1$ into A_1 is continuous when A_1 is given the (s^*)-topology. The result then follows from Corollary 1.

REMARK. Several other characterizations of relative $\sigma(A_*, A)$-compactness are now easily available. For instance, K is relatively $\sigma(A_*, A)$-compact if and only if each monotone, descending net of self-adjoint operators $x_\gamma \downarrow 0$, also converges uniformly on K. That this is so follows from the fact that this condition is implied by (iv) and implies (iii) in the theorem.

4. Order-properties of the Mackey-topology $\tau(A, A_*)$.

In this section we state some properties of the (τ)-topology related to the order-structure of the von Neuman algebra A.

We say that a net $\{x_\gamma\}_{\gamma \in \Gamma} \subseteq A^h$ is order-convergent (O-convergent) to an element $x \in A^h$, and write $x_\gamma \to x$ (O) if there is a cofinal subset Γ' of the directed set Γ, and two nets $\{y_{\gamma'}\}_{\gamma' \in \Gamma'}$, $\{z_{\gamma'}\}_{\gamma' \in \Gamma'}$ in A^h satisfying

(i) $y_{\gamma'} \downarrow x$; $z_{\gamma'} \uparrow x$; $\gamma' \in \Gamma'$,
(ii) $z_{\gamma'} \leqq x_\gamma \leqq y_{\gamma'}$ if $\gamma \geqq \gamma'$.

A net $\{x_\gamma\}_{\gamma\in\Gamma}\subseteqq A$ is O-convergent to an element $x\in A$, and we write $x_\gamma\to x$ (O) if $\frac{1}{2}(x_\gamma+x_\gamma^*)\to\frac{1}{2}(x+x^*)$ (O), and

$$-\tfrac{1}{2}i(x_\gamma-x_\gamma^*)\to-\tfrac{1}{2}i(x-x^*)\ (O)\,.$$

It is clear that $x_\gamma\to x$ (O) implies $x_\gamma^*\to x^*$ (O), and that $x_\gamma\to x$ (O) if and only if $x_\gamma-x\to 0$ (O). Note also that $x_\gamma\to x$ (O) implies that the net $\{x_\gamma\}$ is eventually bounded in A, because of (ii) above. We observe that if $\{x_\gamma\}$ is a net in A converging to an element $x\in A$ in the norm-topology of A, then $x_\gamma\to x$ (O). Finally, we say that a vector-topology T for A is *order-continuous* (O-continuous) if for a net $\{x_\gamma\}\subseteqq A$ we have $x_\gamma\to x$ (T) whenever $x_\gamma\to x$ (O).

PROPOSITION 1. *The topology* (τ) *is the strongest order-continuous, locally convex topology on* A.

PROOF. We first prove that (τ) actually is O-continuous. Let $\{x_\gamma\}_{\gamma\in\Gamma}$ be a net in A, and let $x_\gamma\to x$ (O). By the preceding remarks, it is not restrictive to assume that $\{x_\gamma\}\subseteqq A_1^h$, and that $x_\gamma\to 0$ (O). To show that $x_\gamma\to 0$ in the topology (τ) it is then by Theorem 1 sufficient to prove that $x_\gamma\to 0$ (s). Let $\Gamma'\subseteqq\Gamma$ and $y_{\gamma'}\downarrow 0$ such that $-y_{\gamma'}\leqq x_\gamma\leqq y_{\gamma'}$ if $\gamma\geqq\gamma'$. Suppose for some $\gamma_0'\in\Gamma'$ we have $\|y_{\gamma'_0}\|=M$. Let the (s)-seminorm $\|\cdot\|_p$ on A be given by

$$\|x\|_p = p(x^*x)^{\frac{1}{2}},\quad \text{where } p\geqq 0,\ p\in A_*\,.$$

We may assume $\|p\|=1$. Now, choose $\gamma'\geqq\gamma_0'$, $\gamma'\in\Gamma'$ such that $\|y_{\gamma'}\|_p<\frac{1}{4}\varepsilon/M$, for a given $\varepsilon>0$. For $\gamma\geqq\gamma'$, $\gamma\in\Gamma$, we have

$$0\leqq y_{\gamma'}-x_\gamma\leqq 2y_{\gamma'}$$

so that

$$0\leqq p(y_{\gamma'}-x_\gamma)\leqq 2p(y_{\gamma'})\leqq 2\|y_{\gamma'}\|_p\,.$$

Now recall that if $x\in A$, $x\geqq 0$ we have $p(x^2)\leqq p(x)\|x\|$. Hence

$$p((y_{\gamma'}-x_\gamma)^2)\leqq p(y_{\gamma'}-x_\gamma)\,\|y_{\gamma'}-x_\gamma\|\leqq 2\|y_{\gamma'}\|_p 2M<\varepsilon$$

so

$$\|x_\gamma\|_p\leqq\|y_{\gamma'}\|_p+\|y_{\gamma'}-x_\gamma\|_p<\varepsilon+\varepsilon^{\frac{1}{2}}\,.$$

Since $\varepsilon>0$ was arbitrary, this proves that $x_\gamma\to 0$ (s), and consequently that the topology (τ) is O-continuous. Now let T be any O-continuous, locally convex topology on A. Since convergence in the norm on A implies O-convergence, T must be weaker than the norm-topology. It follows that a T-continuous linear functional f is bounded on A and thus belongs to A^*. Since T is O-continuous f is completely additive, and consequently normal, i.e. an element of A_*. By the Mackey–Arens

Theorem [10, 18.8, p. 172] it then follows that T is weaker than (τ). The proof is complete.

REMARK. Another proof of the fact that if $\{x_\gamma\}_{\gamma \in \Gamma}$ is a net of bounded self-adjoint operators on a Hilbert space $\mathscr{H}$, O-convergent to a bounded self-adjoint operator x on $\mathscr{H}$, then it is convergent in the strong operator topology, was given by Fell and Kelley [9]. The proposition above generalizes the corresponding well-known statement for the commutative case, which follows from measure theory [3], [7], [8].

REMARK. McShane [18] has proved that if $A = \mathscr{L}(\mathscr{H})$, $\mathscr{H}$ some Hilbert space, and $\{x_\gamma\}$ is a bounded net of self-adjoint-operators such that $x_\gamma \to x$ (s), then $x_\gamma \to x$ (O). It follows from this result and proposition 1, that if $\{x_\gamma\}$ is a bounded net in $A = \mathscr{L}(\mathscr{H})$, then $x_\gamma \to x$ in the topology (τ) if and only if $x_\gamma \to x$ (O).

COROLLARY. *Let U be a neighborhood of 0 in A in the (τ)-topology. If $\{y_\gamma\}_{\gamma \in \Gamma}$ is a net in A such that $y_\gamma \downarrow 0$, $\gamma \in \Gamma$, then there is an index $\gamma_0 \in \Gamma$ such that when $-y_{\gamma_0} \leqq x \leqq y_{\gamma_0}$, $x \in A$, we have $x \in U$.*

PROOF. Let $y_\gamma \downarrow 0$, and suppose the statement above to be false. Then, for each $\gamma \in \Gamma$ there is an element $x_\gamma \in A^h$ satisfying $-Y_\gamma \leqq x_\gamma \leqq y_\gamma$ and $x_\gamma \notin U$. But then $x_\gamma \to 0$ (O), and consequently $x_\gamma \to 0$ in the topology (τ) by Proposition 1. This is a contradiction, and the corollary is proved.

Under additional assumptions, somewhat more can be said in the direction of the last corollary. We first prove

PROPOSITION 2. *If A is a von Neumann abgebra of finite type, and $K \subseteqq A_*$ is relatively $\sigma(A_*, A)$-compact, then $K' = \{|\varphi| : \varphi \in K\}$ is relatively $\sigma(A_*, A)$-compact.*

PROOF. Let $\{x_n\}$ be any sequence in $A_1{}^h$ such that $x_n \to 0$ (s). By Theorem 1, K' is relatively $\sigma(A_*, A)$-compact if we can prove that x_n converges uniformly to 0 on K'. Suppose that it does not. Then there is an $\varepsilon > 0$, a subsequence $\{y_i\}$ of $\{x_n\}$, and a sequence $\{f_i\} \subseteqq K$ such that

$$||f_i|(y_i)| \geqq \varepsilon, \quad i = 1, 2, \ldots .$$

Let $f_i = u_i |f_i|$ be the polar decomposition of f_i, $u_i \in A$. Since $\|u_i\| \leqq 1$ for all i, it follows directly from the Schwarz inequality that $u_i{}^* y_i \to 0$ (s). The $*$-operation is s-continuous on the unit sphere of A [13], so $u_i{}^* y_i \to 0$ (s). It follows that the sequence $\{u_i{}^* y_i\}$ converges uniformly to 0 on K, and hence

$$||f_i|(y_i)| = |f_i(u_i{}^* y_i)| < \varepsilon$$

for sufficiently large i. Since this is a contradiction, the proof is complete.

ADDED IN PROOF. K. Saito [24] has recently shown that Proposition 2 is true for finite algebras only.

One may go ahead at this point and obtain as an easy corollary that the *normal hull*

$$\{f \in A_* :\ 0 \leqq |f| \leqq |g|,\, g \in K\}$$

of a $\sigma(A_*,A)$-compact subset K of the pre-dual of a finite von Neumann algebra is also $\sigma(A_*,A)$-compact. However, the concept of normality seems not to be so useful generally as in the commutative case. The reason for this is that we do not always have

$$|g|(|x|) = \sup\{f(x) :\ 0 \leqq |f| \leqq |g|\}\,,$$

not even for hermitian g and x.

PROPOSITION 3. *If $K' = \{|f| : f \in K\}$ is relatively $\sigma(A_*,A)$-compact whenever K is relatively $\sigma(A_*,A)$-compact, there are local bases $\mathscr{U}$ and $\mathscr{V}$ for the Mackey-topology (τ) such that*

(a) $$-y \leqq x \leqq y,\ 0 \leqq y \in U \ \Rightarrow\ x \in U,\ U \in \mathscr{U}\,,$$

(b) $$x^*x \leqq y^*y,\ y \in V \ \Rightarrow\ x \in V,\ V \in \mathscr{V}\,.$$

In particular, $y \in V \Rightarrow |y| \in V$. Hence, for each neighborhood U of 0, there is a neighborhood V of 0 such that

$$0 \leqq |x| \leqq |y|, \qquad y \in V, \qquad x \in A^h \Rightarrow x \in U\,.$$

PROOF. Let $\mathscr{F}$ be the collection of relatively $\sigma(A_*,A)$-compact sets such that if $f \in K \in \mathscr{F}$, then $f \geqq 0$. Let $\mathscr{U}$ be the collection of sets

$$U = \{x \in A :\ |f(x)| \leqq 1,\ f \in K\}\,,$$

and let $\mathscr{V}$ be the collection of sets

$$V = \{x \in A : f(x^*x)^{\frac{1}{2}} \leqq 1,\ f \in K\}\,,$$

where K runs through $\mathscr{F}$. We claim that $\mathscr{U}$ and $\mathscr{V}$ are local bases around 0 satisfying the requirements of the proposition. It is easily seen that $\mathscr{U}$ and $\mathscr{V}$ are local bases for two locally convex topologies $\mathscr{T}_{\mathscr{U}}$ and $\mathscr{T}_{\mathscr{V}}$ respectively. By the Schwarz inequality, $\mathscr{T}_{\mathscr{V}}$ is stronger than $\mathscr{T}_{\mathscr{U}}$. Now, let K be any $\sigma(A_*,A)$-compact set in $A_*{}^h$, and put

$$K^+ = \{f^+ : f \in K\}, \qquad K^- = \{f^- : f \in K\}\,.$$

Then K^+, K^- belong to $\mathscr{F}$, and hence also $2K^+$, $2K^-$ and $K_1 = 2K^+ \cup 2K^-$. Take

$$U = \{x \in A :\ |g(x)| \leqq 1,\ g \in K_1\}\,.$$

Then, if $x \in U$, we have for $f \in K$

$$|f(x)| = |f^+(x)-f^-(x)| \leqq |f^+(x)| + |f^-(x)| \\ = \tfrac{1}{2}(|2f^+(x)| + |2f^-(x)|) \leqq 1 ,$$

since $2f^+$ and $2f^- \in K_1$. Hence $U \subseteqq K^0$, which shows that $\mathscr{T}_{\mathscr{U}}$ is stronger than the Mackey-topology $\tau(A,A_*)$. Now let f be a $\mathscr{T}_{\mathscr{V}}$-continuous linear functional on A. f is clearly bounded. If $\{e_\gamma\}_{\gamma\in\Gamma}$ is a net of commuting projections in A such that $e_\gamma \downarrow 0$, $\gamma \in \Gamma$, then $e_\gamma \to 0$ $(\mathscr{T}_{\mathscr{V}})$, and it follows that f is completely additive, and hence belongs to A_*. By the Mackey–Arens theorem it then follows that $\mathscr{T}_{\mathscr{V}}$ is weaker than (τ). Hence $\mathscr{U}$ and $\mathscr{V}$ are local bases for the Mackey-topology (τ).

Now, if $-y \leqq x \leqq y$, $0 \leqq y \in U$, for an $U \in \mathscr{U}$, we have for $f \in K \in \mathscr{F}$ and $U=K^0$

$$-f(y) \leqq f(x) \leqq f(y), \quad \text{so} \quad |f(x)| \leqq |f(y)| \leqq 1,$$

that is, $x \in U$, proving (a). If $0 \leqq x^*x \leqq y^*y$, $y \in V$, then

$$f(x^*x)^{\frac{1}{2}} \leqq f(y^*y)^{\frac{1}{2}} \leqq 1 ,$$

so $x \in V$, and in particular $y \in V$ implies that

$$|y| = (y^*y)^{\frac{1}{2}} \in V ,$$

proving (b). Now, let U' be any (τ)-neighborhood of 0 in A. Take U, V in $\mathscr{U}$, $\mathscr{V}$ respectively such that $V \subseteqq U \subseteqq U'$. Then, if $y \in V$, we know that $|y| \in V$, and hence $|y| \in U$. So, if $|x| \leqq |y|$, we have $|x| \in U$ by (a). If $x \in A^h$, we have $|x| = x^+ + x^-$, so

$$|f(x)| = |f(x^+)-f(x^-)| \leqq f(|x|) \leqq 1 \quad \text{if} \quad f \in K ,$$

and $U=K^0$. Hence, $x \in U$ and the proof is complete.

Corollary. *If A is of finite type, the conclusion of Proposition* 3 *holds.*

For reference, we also note the following fact.

Lemma. *If A is of finite type, and K is a $\sigma(A_*,A)$-compact subset of A_*, then there is a positive, normal, linear functional p on A such that $f=f_e$ for all $f \in K$, where $e=$*support *p.*

Proof. We may assume that $K \subseteqq A_*{}^h$. By Proposition 2, $K' = \{|f| : f \in K\}$ is relatively $\sigma(A_*,A)$-compact. So, by Theorem 1, there is a positive, normal, linear functional p on A, which in particular has the property that $p(e')=0$ implies that $|f|(e')=0$, e' any projection in A. The lemma follows.

5. The restriction problem.

We are going to consider the following problem. Let A be a von Neumann algebra, and let B be a von Neumann sub-algebra of A. When is the restriction of the Mackey-topology for A to B equivalent to the Mackey-topology for B? That is, when do we have $\tau(A,A_*)|B = \tau(B,B_*)$? As can easily be verified, this restriction property holds for each of the other topologies usually considered for von Neumann algebras. The ultra-weak topology for B is the topology $\sigma(B,B_*)$, but is also the restriction to B of the topology $\sigma(A,A_*)$, the norm on B is determined either as the dual norm with respect to B_*, or as the restricted norm on A to B, and so forth. However, it is not obvious that the same is true for the Mackey-topologies for A and B relative their pre-duals. Indeed, for locally convex spaces in general, it is false:

A subspace of a Mackey-space need not be a Mackey-space, since each locally convex space is topologically isomorphic to a subspace of a product of semi-normed spaces, which is a Mackey-space.

We single out a class of sub-algebras B of A having this restriction property, so that $\tau(A,A_*)|B = \tau(B,B_*)$, characterizing them by a "minimal distance property", which may be of some interest in itself. It is easily seen that the equivalence of $\tau(A,A_*)|B$ and $\tau(B,B_*)$ depends on the existence of a $\sigma(A_*,A)$-compact set $K_1 \subseteq A_*$ such that $r(K_1) = K$ for each $\sigma(B_*,B)$-compact set $K \subseteq B_*$, where r is the restriction map $f \to f|B$, $f \in A_*$. What we want to do, is therefore to construct a continuous map φ of B_* into A_* such that $r \circ \varphi$ is the identity map on B_*, that is, we are looking for a continuous cross-section for r.

In all what follows, A and B will denote von Neumann algebras, $B \subseteq A$ will mean that B is a von Neumann sub-algebra of A. Let 1_B and 1_A denote the identity elements of B and A respectively, and if e is a projection in A, let (A_*,e) denote the set of positive, normal, linear functionals p on A satisfying $p(1_A - e) = 0$. By a *projection* P of A on B we simply mean a linear map of A onto B satisfying $P \circ P = P$.

DEFINITION. We say that $B \subseteq A$ has the *minimal distance property* (m.d.p.) with respect to A, if for each $q \in {B_*}^+$ there is $p \in (A_*, 1_B)$ which extends q and satisfies the following condition: For every $a \in A$ there is $a_1 \in a + B$ such that

$$p({a_1}^* a_1) \leqq p(x^* x) \quad \text{for all} \quad x \in a + B .$$

The following result shows that the m.d.p. comes close to being necessary for ultra-weakly continuous projections to exist.

PROPOSITION 4. *Let P be an ultra-weakly continuous projection of A on B with $\|P\|=1$. Then B has the* m.d.p.

PROOF. Let $P_2=P|A_e$, where $e=1_B$. By a result of Tomiyama [21], P_2 is positive, and satisfies

$$P_2(xy) = xP_2(y) \quad \text{for all} \quad x \in B,\ y \in A_e .$$

Let P_1 be the linear map $a \to eae$ of A onto A_e, and take $P'=P_2 \circ P_1$ so P' is a positive projection of A onto B satisfying

$$P'(1_A) = e \quad \text{and} \quad P'(xa) = xP'(a),\ x \in B,\ a \in A .$$

Let q be any element of $B_*{}^+$, and put $p=q \circ P'$. Then $p \in (A_*,e)$. Now take any $a \in A$, and put $a_1=a-P'a$. Let x be any element of $a+B$, it can then be written $x=a_1+b$, $b \in B$. The form

$$(y,z)_p = p(z^*y), \quad y,z \in A ,$$

is positive and conjugate bilinear on $A \times A$, and introduces a semi-norm

$$\|y\|_p = (y,y)_p{}^{\frac{1}{2}}, \quad y \in A ,$$

on A. We now obtain

$$(a_1,b) = p(b^*a_1) = p \circ P'(b^*a_1) = p(b^*P'a_1) = 0 .$$

Hence, by the Pythagorean equality it follows that

$$\|x\|_p{}^2 = \|a_1\|_p{}^2+\|b\|_p{}^2, \quad \text{so} \quad p(x^*x) \geqq p(a_1{}^*a_1),$$

and the proof is finished.

NOTE. In the first draft of this paper the above result was stated in a less general form. We are indebted to W. Arveson, who made us acquainted with Tomiyamas result and his own work [20]. The last part of the proof above is modeled after one of his arguments.

REMARK. An immediate consequence of this proposition is that if e is a projection in A, then A_e has the m.d.p. with respect to A. In particular, this is true for any (σ)-closed two-sided ideal in A. Kadison and Singer [22] showed that a positive projection of norm 1 always exists if B is abelian. It is also known [20], [23] that if A is of finite type and countably decomposable, then a normal projection of this kind exists on any $B \subseteqq A$.

We are now going to study the problem to which extent the condition m.d.p. on B is sufficient for ultra-weakly continuous projections to exist.

LEMMA 1. *Let $0 \leqq p \in A_*$ and suppose that p is faithful when restricted* *to* $B \subseteqq A$. *Then there is at most one element* $a_1 \in a+B$, $a \in A$, *satisfying* $p(a_1{}^*a_1) \leqq p(x^*x)$, $x \in a+B$.

Proof. Let $a \in A$ be given, and suppose that

$$p(a_1{}^*a_1) = p(a_2{}^*a_2) = d^2 \leqq p(x^*x) ,$$

for all $x \in a+B$, and $a_1, a_2 \in a+B$. Then $\frac{1}{2}(a_1+a_2) \in a+B$, so

$$d \leqq \|\tfrac{1}{2}(a_1+a_2)\|_p \leqq \tfrac{1}{2}\|a_1\|_p + \tfrac{1}{2}\|a_2\|_p = d ,$$

so $\|a_1+a_2\|_p = 2d$. For the semi-norm $\|\cdot\|_p$ the parallellogram equality holds. Hence

$$\|a_1-a_2\|_p{}^2 = -\|a_1+a_2\|_p{}^2 + 2(\|a_1\|_p{}^2 + \|a_2\|_p{}^2) = 0 .$$

Now $a_1-a_2=B$, and $\|\cdot\|_p$ restricted to B is a norm since $p\,|\,B$ is faithful, so this implies that $a_1=a_2$, and the proof is complete.

Lemma 2. *If* $A \supseteqq B$, *and* B *has the* m.d.p. *with respect to* A *and* e *is a projection in* B, *then* B_e *has the* m.d.p. *with respect to* A_e.

Proof. Let $0 \leqq q_1 \in (B_e)_*$, and define p_1 on B by $p_1(x) = q_1(exe)$; $x \in B$. Then $p_1 \in B_*{}^+$, and therefore by assumption has a suitable extension $q \in (A_*, 1_B)$. Now take $p = q\,|\,A_e$. For $x \in B_e$, we get

$$p(x) = q(x) = p_1(x) = q_1(exe) = q_1(x) ,$$

so p is an extension of q_1 to A_e. Note that

$$q(1_B - e) = p_1(1_B - e) = p\big(e(1_B-e)e\big) = 0 ,$$

so $q \in (A_*, e)$.

Now, let $a \in A_e$ be arbitrarily chosen. Since B has the m.d.p. in A, there is an $x_1 \in a+B$ such that

$$q(x_1{}^*x_1) \leqq q(x^*x), \qquad x \in a+B,\ x \in A .$$

Hence, also

$$q(x_1{}^*x_1) \leqq p(x^*x), \qquad x \in a+B_e,\ x \in A_e, \tag{2}$$

since $a+B_e \subseteqq a+B$. Now $x_1 = a+b$, $b \in B$, so

$$ex_1e = eae+ebe = a+ebe ,$$

$$a_1 = ex_1e \in a+B_e, \quad \text{and} \quad a_1 \in A_e .$$

Further,

$$\begin{aligned} p(a_1{}^*a_1) = q(a_1{}^*a_1) &= q\big((ex_1e)^*(ex_1e)\big) \\ &= q(ex_1{}^*ex_1e) = q(x_1{}^*ex_1), \end{aligned}$$

since the support of q is contained in e. Now

$$q(x_1{}^*ex_1) \leqq q(x_1{}^*x_1) \leqq p(x_1{}^*x_1),$$

by (2) which proves that B_e has the m.d.p. in A_e.

LEMMA 3. *If $A \supseteqq B$, and B has the* m.d.p. *in A, and there is an element $q \in B_*{}^+$ which is faithful on B, then there is a linear projection P of A onto B and an element $p \in (A_*, 1_B)$ extending q such that P is continuous with respect to the semi-norm $\|\cdot\|_p$.*

PROOF. Let $p \in (A_*, 1_B)$ be an extension of q, chosen so that by assumption and Lemma 1, for each $a \in A$, there is a unique $a_1 \in a+B$ satisfying

$$p(a_1{}^*a_1) \leqq p(x^*x), \qquad x \in a+B.$$

By a standard Hilbert-space argument, a_1 is orthogonal to B with respect to the conjugate bilinear form $(.,.)_p$. Define $Pa=a-a_1$. Then $Pa \in B$, so

$$\|a\|_p{}^2 = \|a_1\|_p{}^2+\|Pa\|_p{}^2 \geqq \|Pa\|_p{}^2.$$

We observe that if $a \in B$, then $Pa=a$ since $a_1=0$ in this case. It is easily verified that P is linear so it satisfies the conditions of the lemma.

LEMMA 4. *Let E be a Frechet space, and let $T: E \to B$ be a linear map of E into the von Neumann algebra B. Suppose that $p \in B_*{}^+$ is faithful on B, and that T is continuous with respect to the norm $\|\cdot\|_p$ on B. Then T is continuous with respect to the C^*-norm on B.*

PROOF. We prove that T is closed, that is, if $x_n \to x$, in E and $Tx_n \to y(n)$ in B, then $Tx=y$. We first observe that each $f \in B_*$ is continuous with respect to $\|.\|_p$ when restricted to B_1. It is sufficient to prove this for $f \in B_*{}^+$, and it is also sufficient to prove that $f|B_1$ is $\|.\|_p$-continuous at 0. (Cf. [10, 13.5, p. 113].) So let $f \in B_*{}^+$ be given, and suppose

$$\|z_n\|_p \to 0 \quad \text{as } n \to \infty, \quad z_n \in B_1.$$

Since p is faithful, f is absolutely continuous with respect to p, and it follows from [5, Prop. 5, p. 62] that $f(z_n{}^*z_n) \to 0$. Now

$$|f(z_n)|^2 \leqq f(1)f(z_n{}^*z_n),$$

so $f(z_n) \to 0$ as $n \to \infty$, proving that $f|B_1$ is continuous with respect to $\|.\|_p$ for all $f \in B_*$.

Since $Tx_n \to y(n)$, the set $\{Tx, Tx_n\}_{n\in N}$ is bounded in norm, we may assume by 1, that is, $\{Tx, Tx_n\}_{n\in N} \subseteqq B_1$. Now $x_n \to x$, so by assumption $\|Tx_n - Tx\|_p \to 0$. It follows from the observation above, that

$$f(Tx_n) \to f(Tx) \quad \text{for each} \quad f \in B_* .$$

On the other hand; $Tx_n \to y(n)$, so in particular $f(Tx_n) \to f(y)$ for $f \in B_*$. Hence, $f(y)=f(Tx)$ for all $f \in B_*$, proving that $y=Tx$. By the closed graph theorem it follows that T is continuous with respect to the C^*-norm on B.

LEMMA 5. *Let $T : A \to B$ be a linear map of A into B. Let p_1, p_2 be positive, normal linear functionals on A, B respectively, and suppose that p_2 is faithful on B. If T is continuous with respect to the semi-norm $\|.\|_{p_1}$ on A and the norm $\|.\|_{p_2}$ on B, then T is ultra-weakly continuous.*

PROOF. By [5, Theorem 1, p. 40], it is sufficient to prove that $f \circ T$ is ultra-strongly continuous on A_1 for each $f \in B_*$. Clearly T fulfills the hypothesis of Lemma 4, so there is a constant $\lambda > 0$ such that $T(A_1) \subseteq \lambda B_1$. We may without loss of generality suppose that $\lambda = 1$, so $T(A_1) \subseteq B_1$. Now, if $f \in B_*$, we observed in the proof of Lemma 4 that $f|B_1$ is $\|.\|_{p_2}$-continuous. It follows that $f \circ T$ is $\|.\|_{p_1}$-continuous on A_1, and so — a fortiori — ultra-strongly continuous on A_1. The proof is complete.

Combining Lemma 3 and Lemma 5 we get the following result:

PROPOSITION 5. *If $A \supseteq B$, and B has the* m.d.p. *with respect to A, and there is an element $p \in B_*^+$ which is faithful on B, then there is a linear projection P of A onto B which is ultra-weakly continuous.*

COROLLARY. *Let $A \supseteq B$, and suppose B to have the* m.d.p. *in A. Let p be a positive, normal linear functional on B, with* support$(p)=e \in B$. *Then there is an ultra-weakly continuous linear projection P of A onto B_e. Moreover, if $P_* : (B_e)_* \to A_*$ is the adjoint of P, then P_* is $\sigma((B_e)_*, B_e)$-$\sigma(A_*, A)$-continuous, $r \circ P_*$ is the identity mapping on $(B_e)_*$, where r is the restriction mapping $f \to f|B_e$, $f \in A_*$.*

PROOF. The map $P_1 : a \to eae$ is an ultra-weakly continuous projection of A onto A_e. Since B has the m.d.p. in A, and $e \in B$, we know by Lemma 2 that B_e has the m.d.p. in A_e. p is faithful on B_e, so by Proposition 5 it follows that there is a projection P_2 of A_e onto B_e which is ultra-weakly continuous. Now take $P = P_2 \circ P_1$ and the first statement of the corollary is proved. The adjoint $P_* : (B_e)_* \to A_*$ is then automatically $\sigma((B_e)_*, B_e)$-$\sigma(A_*, A)$-continuous. If $i : B \to A$ is the injection map, then $P \circ i$ is the identity map on B_e. It follows that

$$(P \circ i)_* = i_* \circ P_* = r \circ P_*$$

is the identity map on $(B_e)_*$.

NOTE. If $g \in (B_e)_*$, then $(P_* g)_e = P_* g$. Indeed, if $x \in A$, we have for instance

$$
\begin{aligned}
(P_* g)(ex(1-e)) &= g\big(P(ex(1-e))\big) \\
&= g\big(P_2 \circ P_1(ex(1-e))\big) \\
&= g\Big(P_2\big(e(ex(1-e))e\big)\Big) = g(P_2(0)) = 0 ,
\end{aligned}
$$

and likewise

$$(P_* g)\big((1-e)xe\big) = 0 ,$$

so

$$(P_* g)(x) = (P_* g)(exe), \qquad x \in A .$$

THEOREM 2. *Let $A \supseteqq B$. The Mackey-topologies $\tau(A, A_*)$ and $\tau(B, B_*)$ coincide on B in each of the following cases:*

(a) *B has the* m.d.p. *and each $\sigma(B_*, B)$-compact set is metrizable;*
(b) *B has the* m.d.p. *and is of finite type;*
(c) *B has the* m.d.p. *and is countably decomposable;*
(d) *A is of finite type and countably decomposable;*
(e) *1_B is finite and countably decomposable in A.*

PROOF. Let $i : B \to A$ be the injection of B into A, and $s = i_* : A_* \to B_*$ the restriction map $f \to f|B, f \in A_*$. We must show that i is a homeomorphism of B into A. Let U be $\tau(A, A_*)$-neighborhood of 0 in A. Then $K = U^0 \subseteqq A_*$ is $\sigma(A_*, A)$-compact, so $K_1 = s(K)$ is $\sigma(B_*, B)$-compact. Clearly $i(K_1{}^0) \subseteqq U^{00}$, proving that i is continuous. To prove that i is relatively open, let U be a $\tau(B_*, B)$-neighborhood of 0 in B. We may assume that $U = K^0$, where K is a $\sigma(B_*, B)$-compact subset of B_*. The theorem will be proved if we can find a $\sigma(A_*, A)$-compact subset K_1 of A_* such that $s(K_1) = K$. For if this is the case, we have $K_1{}^0 \cap i(B) = U$, which shows that i is relatively open. So let the $\sigma(B_*, B)$-compact subset K of B_* be given, we may without loss of generality assume that $K \subseteqq B_*{}^h$. If B is of finite type, we know by Lemma, Section 4, that there is a positive, normal linear functional p on B, such that $f = f_e$ for all $f \in K$, where $e = \operatorname{support} p$. If K is metrizable, the same is true. Indeed, since K is compact it contains a countable, $\sigma(B_*, B)$-dense subset $\{f_n\}_{n \in N}$. Put $p = \sum_{n=1}^{\infty} 2^{-n}|f_n|$, and take $e = \operatorname{support} p$. Now, if $f \in K$, there is a subsequence $\{f_{n_i}\}_{i \in N}$ converging in the $\sigma(B_*, B)$-topology to f. Hence, if $x \in B$, we have

$$f(exe) = \lim_i f_{n_i}(exe) = \lim_i f_{n_i}(x) = f(x) ,$$

so

$$f = f_e \quad \text{for} \quad f \in K .$$

Let p have the property described above, it is then faithful on B_e, so by the preceding corollary there is a $\sigma((B_e)_*, B_e)$-$\sigma(A_*, A)$-continuous linear map of $(B_e)_*$ into A_*. Let $t: B_* \to (B_e)_*$ and $r: A_* \to (B_e)_*$ be

$$\begin{tikzcd} & A_* \arrow[d, "s"] \arrow[dl, shift right, "r"'] \\ (B_e)_* \arrow[ur, dashed, shift right, "P_*"'] & B_* \arrow[l, "t"] \end{tikzcd}$$

the restriction maps of B_* and A_* onto $(B_e)_*$, respectively. Then K_1, defined by

$$K_1 = (P_* \circ t)(K) \subseteqq A_* ,$$

is $\sigma(A_*, A)$-compact, and we want to prove that s maps K_1 onto K. So take an element $f \in K$, put

$$h = (P_* \circ t)(f) \in K_1 \quad \text{and} \quad f' = s(h) .$$

Then

$$\begin{aligned} t(f') = (t \circ s)(h) = r(h) &= r((P_* \circ t)(f)) \\ &= (r \circ P_*)(t(f)) = t(f) , \end{aligned}$$

so f and f' have the same restriction to B_e. Hence, for $x \in B$ we have

$$f(x) = f_e(x) = f(exe) = f'(exe) = f'(x) ,$$

where the last equality follows from the note preceding the theorem. Hence $f = f'$ which proves that $s(K_1) = K$. This proves the theorem under the conditions (a) and (b).

Remark. The condition in (a) that $\sigma(B_*, B)$-compact sets shall be metrizable, holds if B has a countable, total subset L (that is, $f(x) = 0$ for all $x \in L$ implies $f = 0$, where $f \in B_*$). This is true if B acts on a separable Hilbert space, since in this case B_1 has a countable dense subset in the $\sigma(B, B_*)$-topology (cf. [5, p. 34]), and such a set is certainly total.

More generally, if B is countably decomposable, there is a positive, normal linear functional on B with support equal to 1_B. This follows easily from Prop. 6, p. 6, in [5]. By examining the proof above it becomes clear that the condition that B is finite or $\sigma(B_*, B)$-compact sets are metrizable, may be replaced by the condition that B is countably decomposable. So the theorem holds under (c). That it holds under (d) follows once we have shown that it holds under (e). But if (e) is satisfied, the algebra A_{1_B} is of finite type and countably decomposable, so we know by the remark following Proposition 4 that B has the m.d.p. Since it is countably decomposable, the result follows from (c). The proof is complete.

6. Completeness.

The following proposition and its corollaries are simple consequences of the general theory of locally convex spaces. However, the truth of these facts seems not to have been noticed in connection with von Neumann algebras.

Ptak [11], has generalized the open-mapping theorem in terms of full completeness. A locally convex Hausdorff space E is *fully complete* (B-complete) if it satisfies the following condition: If M is a linear subspace of the dual E^* of E such that for each circled, convex $\sigma(E^*,E)$-closed equi-continuous subset K of E^* the set $K \cap M$ is $\sigma(E^*,E)$-closed, then M itself is $\sigma(E^*,E)$-closed. Ptak proved that this property is equivalent to the validity of the open-mapping theorem, in the following sense. A linear map Φ of a linear topological space E into another F, is *almost open* if $\overline{\Phi(U)}$ is a neighborhood of 0 in F whenever U is a neighborhood of 0 in E. The theorem states that if E and F are locally convex spaces, and $\Phi : E \to F$ a linear, continuous, almost open and surjective map, then Φ is open if and only if E is fully complete.

As can be seen from the Grothendieck completeness theorem [10, 16.9, p. 145], if E is fully complete then it is complete. The converse is generally false, although it holds for metrizable, locally convex spaces.

Let A be a von Neumann algebra, and A_* its pre-dual.

PROPOSITION 6. *If C is a convex subset of A_*, then C is $\sigma(A_*,A)$-closed if and only if its intersection with each circled, convex, $\sigma(A_*,A)$-compact set in A_* is $\sigma(A_*,A)$-closed.*

PROOF. The condition is clearly necessary. Suppose now that it is satisfied, and let f be a point in the $\sigma(A_*,A)$-closure of C. Since C is convex, we may find a sequence $\{f_n\} \subseteq C$ such that

$$\|f_n - f\| \to 0 \quad \text{as } n \to \infty .$$

It follows in particular that $\{f_n\}$ is relatively $\sigma(A_*,A)$-compact. A_* is complete, so by the Krein theorem [7, V 6.3, p. 434], the bi-polar $K = \{f_n\}^{00}$ is $\sigma(A_*,A)$-compact, circled and convex. Now $f \in K$, $\{f_n\} \subseteq K \cap C$, so by assumption $f \in C$, and the proof is complete.

COROLLARY 1. *A is complete and fully complete in the Mackey-topology $\tau(A,A_*)$.*

A locally convex space E is said to be *barrelled* if each absorbing, circled, convex and closed subset of E is a neighborhood of 0. A Banach space is barrelled. A linear, surjective map of a locally convex space to a barrelled space is almost open [10]. Hence we have:

COROLLARY 2. *If A is a von Neumann algebra equipped with the topology $\tau(A,A_*)$, and Φ is a continuous, linear map of A onto a barrelled space E, then Φ is open.*

COROLLARY 3. *Let A be a *-algebra of operators on a Hilbert space $\mathscr{H}$, and suppose that the identity operator is in A. Then the following are equivalent:*

(i) $A=A''$.
(ii) *A is complete in the Mackey-topology of A'' relative its pre-dual.*

PROOF. If $A=A''$, A is complete by Corollary 1. On the other hand, if A is complete in the Mackey-topology of A'', it is closed in A'', and hence also ultra-weakly closed in A''. Since A is ultra-weakly dense in A'', it follows that $A=A''$.

REMARK. If A is infinite-dimensional, the Mackey-topology $\tau(A,A_*)$ can never be metrizable. Indeed, if it were, then each bounded linear functional on A would be continuous, [10, 19.4 p. 183 and 22.3 p. 210]. A set in A is bounded in the (τ) topology if and only if it is norm-bounded, so this would imply that $A^*=A_*$, so that A became reflexive. But a reflexive C^*-algebra is finite dimensional [14, Prop. 2, p. 661].

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