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# ON THE ZEROS OF AN ENTIRE FUNCTION III

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Let $f(z)$ be an entire function of order $\varrho$ and lower order $\lambda$ such that $0<\lambda \leqq \varrho < \infty$. Let $\varrho_1$ and $\lambda_1$ denote the exponent of convergence and the lower exponent of convergence of the zeros of $f(z)$, respectively.

It is known that corresponding to every entire function of order $\varrho$, $0<\varrho<\infty$, there exists a function $\varrho(r)$ called its proximate order having the following properties:

i) $\varrho(r)$ is real, continuous and piecewise differentiable for $r \geqq r_0$.
ii) $\varrho(r) \to \varrho \quad$ as $r \to \infty$.
iii) $r\,\varrho'(r) \log r \to 0 \quad$ as $r \to \infty$.
iv) $\log M(r) \leqq r^{\varrho(r)} \quad$ for $r \geqq r_0$,
$\log M(r) = r^{\varrho(r)} \quad$ for a sequence of values of $r$ tending to $\infty$.

S. M. Shah [3] has proved the existence of a function $\lambda(r)$, for an entire function of finite lower order $\lambda$, analogous to $\varrho(r)$ having the following properties:

i)′ $\lambda(r)$ is a non-negative, continuous function of $r$ for $r \geqq r_0$, differentiable except at isolated points at which $\lambda'(r-0)$ and $\lambda'(r+0)$ exist.
ii)′ $\lambda(r) \to \lambda \quad$ as $r \to \infty$.
iii)′ $r\,\lambda'(r) \log r \to 0 \quad$ as $r \to \infty$.
iv)′ $\log M(r) \geqq r^{\lambda(r)} \quad$ for $r \geqq r_0$,
$\log M(r) = r^{\lambda(r)} \quad$ for a sequence of values of $r$ tending to $\infty$.

Let $\varrho_1(r)$ and $\lambda_1(r)$ be the proximate order and the lower proximate order with respect to $n(r,1/f)$, the number of zeros of $f(z)$ in the circle $|z| \leqq r$. Then $\varrho_1(r)$ and $\lambda_1(r)$ have the properties analogous to those of $\varrho(r)$ and $\lambda(r)$, respectively.

We shall prove the following Theorems, where

$$N(r,1/f) = \int_{r_0}^{r} x^{-1} n(x,1/f)\, dx \, .$$

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THEOREM 1. *Let $f(z)$ be an entire function of finite order $\varrho$. Let*

$$\limsup_{r\to\infty} \frac{n(r,1/f)}{N(r,1/f)} = S \text{ where } S<\infty ,$$

$$\liminf_{r\to\infty} \frac{n(r,1/f)}{N(r,1/f)} = T \text{ where } 0<T .$$

*Then*

$$\frac{T^2}{S} \leqq \lambda_1 \leqq \varrho_1 \leqq \frac{S^2}{T}.$$

THEOREM 2.

$$\liminf_{r\to\infty} \frac{n(r,1/f)}{r^{\lambda_1(r)}} \geqq \frac{T^{p+2}\log k_1 \log k_2 \ldots \log k_p}{\lambda_1(k_1k_2\ldots k_p)^{\lambda_1}},$$

*where $p$ is a positive integer, $k_1,k_2,\ldots,k_p$ are constants $>1$, and $\lambda_1(r)$ is the lower proximate order with respect to $N(r,1/f)$.*

COROLLARY.

$$S \geqq \frac{T^{p+2}\log k_1 \log k_2 \ldots \log k_p}{\lambda_1(k_1k_2\ldots k_p)^{\lambda_1}}.$$

$p=0$ gives the first part of Theorem 1.

THEOREM 3.

$$\limsup_{r\to\infty} \frac{n(r,1/f)}{r^{\varrho_1(r)}(\log r)^p} \leqq \frac{S^{p+2}}{\varrho_1},$$

*where $\varrho_1(r)$ is the proximate order with respect to $N(r,1/f)$.*

COROLLARY.

$$\liminf_{r\to\infty} \frac{n(r,1/f)}{N(r,1/f)(\log r)^p} \leqq \frac{S^{p+2}}{\varrho_1}.$$

THEOREM 4. *Let $a$ and $b$, $a\neq b$, be finite numbers and*

$$\limsup_{r\to\infty} \frac{N(r,1/(f-a))+N(r,1/(f-b))}{\log M(r,f)} = \alpha ,$$

$$\liminf_{r\to\infty} \frac{N(r,1/(f-a))+N(r,1/(f-b))}{\log M(r,f)} = \beta .$$

*Let further $\varrho(r)$ be the proximate order and $\lambda(r)$ the lower proximate order with respect to $\log M(r,f)$. Then the following statements hold:*

(1) $$\lim_{r\to\infty} \frac{n(r,1/(f-a))}{r^{\varrho(r)}} = 0 \Rightarrow \limsup_{r\to\infty} \frac{n(r,1/(f-b))}{r^{\varrho(r)}} \geqq \beta\varrho .$$

(2) $$\lim_{r\to\infty} \frac{N(r,1/(f-a))}{r^{\lambda(r)}} = 0 \Rightarrow \limsup_{r\to\infty} \frac{n(r,1/(f-b))}{r^{\lambda(r)}} \geqq \alpha\lambda .$$

(3) $$\liminf_{r\to\infty} \frac{N(r,1/(f-a))}{r^{\lambda(r)}} = 0 \Rightarrow \liminf_{r\to\infty} \frac{n(r,1/(f-b))}{r^{\lambda(r)}} \leqq \alpha\lambda .$$

(4) $$\liminf_{r\to\infty} \frac{N(r,1/(f-a))}{r^{\varrho(r)}} = 0 \Rightarrow \liminf_{r\to\infty} \frac{n(r,1/(f-b))}{r^{\varrho(r)}} \leqq \beta\varrho .$$

REMARK. Statement (2) is an improvement of a result of V. Srinivasulu [4], namely of (2) with $\beta\lambda$ instead of $\alpha\lambda$ on the right-hand side.

COROLLARY. *Let*

$$\limsup_{r\to\infty} \frac{n(r,1/(f-b))\log r}{N(r,1/(f-b))\log N(r,1/(f-b))} = J$$

$$\liminf_{r\to\infty} \frac{n(r,1/(f-b))\log r}{N(r,1/(f-b))\log N(r,1/(f-b))} = K .$$

*Then*

(1)′ $$\lim_{r\to\infty} \frac{N(r,1/(f-a))}{r^{\lambda(r)}} = 0 \Rightarrow \limsup_{r\to\infty} \frac{n(r,1/(f-b))}{r^{\lambda(r)}} \geqq K\alpha\lambda_1(b) ,$$

(2)′ $$\lim_{r\to\infty} \frac{N(r,1/(f-a))}{r^{\varrho(r)}} = 0 \Rightarrow \limsup_{r\to\infty} \frac{n(r,1/(f-b))}{r^{\varrho(r)}} \geqq K\beta\lambda_1(b) ,$$

(3)′ $$\liminf_{r\to\infty} \frac{N(r,1/(f-a))}{r^{\varrho(r)}} = 0 \Rightarrow \liminf_{r\to\infty} \frac{n(r,1/(f-b))}{r^{\varrho(r)}} \leqq J\beta\varrho_1(b) ,$$

(4)′ $$\liminf_{r\to\infty} \frac{N(r,1/(f-a))}{r^{\lambda(r)}} = 0 \Rightarrow \liminf_{r\to\infty} \frac{n(r,1/(f-b))}{r^{\lambda(r)}} \leqq J\alpha\varrho_1(b) .$$

PROOF OF THEOREM 1. We have [4]

$$n(r,1/f)r^{-\lambda_1(r)} > o(1) + \lambda_1^{-1}(T-\varepsilon)^2 ,$$

where $\lambda_1(r)$ is the lower proximate order with respect to $N(r,1/f)$. Thus,

$$n(r,1/f)/N(r,1/f) > \lambda_1^{-1}(T-\varepsilon)^2$$

for a sequence of values of $r$ tending to $\infty$. Hence

$$\limsup_{r\to\infty} n(r,1/f)/N(r,1/f) \geqq \lambda_1^{-1}T^2 ,$$

that is, $T^2/S \leqq \lambda_1$. Similarly we get $\varrho_1 \leqq S^2/T$.

PROOF OF THEOREM 2. To $\varepsilon > 0$ there is an $r_0 \geqq 0$ such that

$$n(r,1/f) > (T-\varepsilon)N(r,1/f) \qquad \text{for } r \geqq r_0 .$$

Now, writing $n(x)$ for $n(x,1/f)$, we have, with $k_1 > 1$,

$$N(r,1/f) = \int_{r_0}^{r} \frac{n(x)}{x}\,dx > \int_{r/k_1}^{r} \frac{n(x)}{x}\,dx$$
$$> n(r/k_1)\log k_1,$$

and thus, with $k_2 > 1,\ldots,k_p > 1$,

$$\begin{aligned} n(r) &> (T-\varepsilon)N(r) > (T-\varepsilon)\,n(r/k_1)\log k_1 \\ &> (T-\varepsilon)^2 n(r/(k_1k_2))\log k_1 \log k_2 \\ &\ \ \vdots \\ &> (T-\varepsilon)^p n(r/(k_1k_2\ldots k_p))\log k_1 \log k_2 \ldots \log k_p \\ &> (T-\varepsilon)^{p+1} N(r/(k_1k_2\ldots k_p))\log k_1 \log k_2 \ldots \log k_p \\ &> (T-\varepsilon)^{p+1} \int_{r_0}^{r/(k_1k_2\ldots k_p)} n(x)x^{-1}dx \log k_1 \log k_2 \ldots \log k_p, \\ &> (T-\varepsilon)^{p+2} \int_{r_0}^{r/(k_1k_2\ldots k_p)} N(x)x^{-1}dx \log k_1 \log k_2 \ldots \log k_p . \end{aligned}$$

Letting $\lambda_1(r)$ be the lower proximate order with respect to $N(r,1/f)$ we have

$$n(r) > (T-\varepsilon)^{p+2}\log k_1 \log k_2 \ldots \log k_p \int_{r_0}^{r/(k_1k_2\ldots k_p)} x^{\lambda_1(x)-1}\,dx .$$

Hence [2, p. 58]

$$\begin{aligned} n(r) &> \lambda_1^{-1}\,(T-\varepsilon)^{p+2}\log k_1 \log k_2 \ldots \log k_p\,(r/(k_1k_2\ldots k_p))^{\lambda_1(r/(k_1k_2\ldots k_p))} \\ &> \lambda_1^{-1}\,(T-\varepsilon)^{p+2}\log k_1 \log k_2 \ldots \log k_p\; r^{\lambda_1(r)}(k_1k_2\ldots k_p)^{-\lambda_1}, \end{aligned}$$

and thus

$$\liminf_{r\to\infty} n(r,1/f)\lambda_1(r)^{-1} \geqq \lambda_1^{-1}\,T^{p+2}\log k_1 \log k_2 \ldots \log k_p\,(k_1k_2\ldots k_p)^{-\lambda_1} .$$

PROOF OF THEOREM 3. For $r \geqq r_0$ we have

$$\begin{aligned} n(r,1/f) &< (S+\varepsilon)\,N(r,1/f) \\ &< (S+\varepsilon)\,n(r,1/f)\log r \\ &< (S+\varepsilon)^p\,n(r)\,(\log r)^p \\ &< (S+\varepsilon)^{p+1}\,N(r)\,(\log r)^p \\ &< (S+\varepsilon)^{p+1}\,(\log r)^p \int_{r_0}^{r} n(x)x^{-1}\,dx . \end{aligned}$$

Letting $\varrho_1(r)$ be the proximate order with respect to $N(r, 1/f)$ we have

$$n(r, 1/f) < (S+\varepsilon)^{p+2} (\log r)^p \int_{r_0}^{r} N(x)x^{-1}\, dx$$

$$< (S+\varepsilon)^{p+2} (\log r)^p \int_{r_0}^{r} x^{\varrho_1(x)-1}\, dx ,$$

and thus [2, p. 58]

$$n(r, 1/f) < (S+\varepsilon)^{p+2} (\log r)^p\, r^{\varrho_1(r)}/\varrho_1 .$$

Hence

$$\limsup_{r\to\infty} n(r, 1/f)(\log r)^{-p} r^{-\varrho_1(r)} \leqq S^{p+2}/\varrho_1$$

and

$$\liminf_{r\to\infty} n(r, 1/f)(\log r)^{-p}/N(r, 1/f) \leqq S^{p+2}/\varrho_1 .$$

Proof of Theorem 4.

(1) Since

$$\liminf_{r\to\infty} \frac{N(r, 1/(f-a)) + N(r, 1/(f-b))}{\log M(r,f)} = \beta ,$$

to each $\varepsilon > 0$ there is an $r_0$ such that

$$\{N(r, 1/(f-a)) + N(r, 1/(f-b))\}/\log M(r,f) > \beta - \varepsilon \qquad \text{for } r \geqq r_0 .$$

Further, there is a sequence of values of $r$ tending to $\infty$ for which

$$\{N(r, 1/(f-a)) + N(r, 1/(f-b))\} r^{-\varrho(r)} > \beta - \varepsilon ,$$

where $\varrho(r)$ is the proximate order with respect to $\log M(r,f)$. Hence

$$\limsup_{r\to\infty} \{N(r, 1/(f-a)) + N(r, 1/(f-b))\} r^{-\varrho(r)} \geqq \beta .$$

Now,

$$\limsup_{r\to\infty} N(r, 1/(f-a)) r^{-\varrho(r)} + \limsup_{r\to\infty} N(r, 1/(f-b)) r^{-\varrho(r)}$$

$$\geqq \limsup_{r\to\infty} \{N(r, 1/(f-a)) r^{-\varrho(r)} + N(r, 1/(f-b)) r^{-\varrho(r)}\}$$

and as

$$\lim_{r\to\infty} N(r, 1/(f-a)) r^{-\varrho(r)} = 0 ,$$

we obtain

$$\limsup_{r\to\infty} N(r, 1/(f-b)) r^{-\varrho(r)} \geqq \beta \Rightarrow \limsup_{r\to\infty} n(r, 1/(f-b)) r^{-\varrho(r)} \geqq \beta\varrho .$$

(2) Since

$$\limsup_{r\to\infty} \frac{N(r,1/(f-a))+N(r,1/(f-b))}{\log M(r)} = \alpha ,$$

there is a sequence of values of $r$ tending to $\infty$ for which

$$\{N(r,1/(f-a))+N(r,1/(f-b))\}/\log M(r) > \alpha-\varepsilon .$$

Now, let $\lambda(r)$ be the lower proximate order with respect to $\log M(r)$. Then we have

$$\{N(r,1/(f-a))+N(r,1/(f-b))\}r^{-\lambda(r)} > \alpha-\varepsilon ,$$

and as

$$\lim_{r\to\infty} N(r,1/(f-a))r^{-\lambda(r)} = 0 ,$$

we get

$$\limsup_{r\to\infty} N(r,1/(f-b))r^{-\lambda(r)} \geqq \alpha \Rightarrow \limsup_{r\to\infty} n(r,1/(f-b))r^{-\lambda(r)} \geqq \alpha\lambda .$$

(3) To each $\varepsilon>0$ there is an $r_0$ such that

$$\{N(r,1/(f-a))+N(r,1/(f-b))\}/\log M(r) < \alpha+\varepsilon \quad \text{for } r\geqq r_0 ,$$

and there is a sequence of values of $r$ tending to $\infty$ for which

$$\{N(r,1/(f-a))+N(r,1/(f-b))\}r^{-\lambda(r)} < \alpha+\varepsilon ,$$

where $\lambda(r)$ is the lower proximate order with respect to $\log M(r,f)$. Thus,

$$\liminf_{r\to\infty} \{N(r,1/(f-a))+N(r,1/(f-b))\}r^{-\lambda(r)} \leqq \alpha .$$

Now,

$$\liminf_{r\to\infty} N(r,1/(f-a))r^{-\lambda(r)} + \liminf_{r\to\infty} N(r,1/(f-b))r^{-(\lambda)r}$$
$$\leqq \liminf_{r\to\infty} \{N(r,1/(f-a))+N(r,1/(f-b))\}r^{-\lambda(r)} ,$$

and as

$$\liminf_{r\to\infty} N(r,1/(f-a))r^{-\lambda(r)} = 0 ,$$

we get

$$\liminf_{r\to\infty} N(r,1/(f-b))r^{-\lambda(r)} \leqq \alpha \Rightarrow \liminf_{r\to\infty} n(r,1/(f-b))r^{-\lambda(r)} \leqq \alpha\lambda .$$

The proof of (4) is similar.

PROOF OF COROLLARY. (1)′ Since

$$\limsup_{r\to\infty}\frac{n(r,1/(f-b))}{r^{\lambda(r)}}\limsup_{r\to\infty}\frac{\log r}{\log N(r,1/(f-b))}\cdot$$

$$\cdot\limsup_{r\to\infty}\frac{N(r,1/(f-b))\log N(r,1/(f-b))}{n(r,1/(f-b))\log r}\geqq\limsup_{r\to\infty}\frac{N(r,1/(f-b))}{r^{\lambda(r)}},$$

we have

$$\limsup_{r\to\infty} n(r,1/(f-b))r^{-\lambda(r)}\geqq\limsup_{r\to\infty} N(r,1/(f-b))r^{-\lambda(r)}K\lambda_1(b),$$

and the statement is a consequence of (2) in Theorem 4.

Similarly (2)′ follows.

(3)′ Since

$$\liminf_{r\to\infty}\frac{n(r,1/(f-b))}{r^{\varrho(r)}}\liminf_{r\to\infty}\frac{\log r}{\log N(r,1/(f-b))}\cdot$$

$$\cdot\liminf_{r\to\infty}\frac{N(r,1/(f-b))\log N(r,1/(f-b))}{n(r,1/(f-b))\log r}\leqq\liminf_{r\to\infty}\frac{N(r,1/(f-b))}{r^{\varrho(r)}},$$

we have

$$\liminf_{r\to\infty} n(r,1/(f-b))r^{-\varrho(r)}\leqq\liminf_{r\to\infty} N(r,1/(f-b))r^{-\varrho(r)}J\varrho_1(b).$$

and the statement is a consequence of (4) in Theorem 4.

Similarly (4)′ follows.

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