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# POWER BOUNDED MATRICES OF FOURIER-STIELTJES TRANSFORMS

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## 1. Introduction.

The purpose of this note is to give a characterization of power bounded matrices, the elements of which are Fourier–Stieltjes transforms. The norm we use will be equivalent to the sum of the total variation norms of the measures that correspond to the elements of the matrix. Such matrices occur in the study of well posed initial value problems for systems of partial differential equations in $L_1$ and $L_\infty$ (for more details and applications of this see [2]).

In the scalar case, necessary and sufficient conditions were obtained by Beurling and Helson [1]. They proved that $f$ is a Fourier–Stieltjes transform with bound powers if and only if

$$f(y) = c\exp(i\langle x,y\rangle),$$

where $|c|=1$ and $x \in R^n$.

In Theorem 3 below we give a corresponding necessary condition for $N\times N$-matrices $\varphi$, the elements of which are Fourier–Stieltjes transforms, that $\|\varphi^m\|$ be bounded. It will be shown that in this case the eigenvalues of $\varphi$ have the same form as $f$ above.

In contrast to the scalar case, however, there seem to be no obvious necessary and sufficient conditions. This is indicated in an example to be given in connection with Theorem 3.

In particular we will construct a $2\times 2$ matrix function $\varphi$, which has eigenvalues 1 and $\exp(2\pi iy)$ and the elements of which are Fourier–Stieltjes transforms. Further $|\varphi^m(y)|$ is uniformly bounded in $y \in R$ and $m = \pm 1, \pm 2,\ldots$, but yet $\|\varphi^m\|$ is not bounded.

In the proof of Theorem 3, we use a local version of the theorem of Beurling and Helson. Although such a version seems to be known (oral communication by Y. Katznelson), no proof of it has been published. So for the convenience of the reader we give the main steps in the proof

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of this theorem (Theorem 1 in Section 2). A corresponding theorem for multipliers on $FL_p$, $p \neq 2$, was proved in Lemma 5 in [2].

Using standard techniques introduced by Cohen and by Beurling and Helson, we then use Theorem 1 to obtain, in Theorem 2, a characterization of the homomorphisms of a class of ideals in $L_1(G)$ into $M(G')$, where $G$ and $G'$ are LCA groups.

Having the applications of Theorem 3 in mind, we have tried to give an essentially self-contained proof of this theorem and of Theorem 1 for $R^n$, without use of the structure theory of locally compact abelian (LCA) groups. This is the main reason for formulating, in Section 2, the lemmas and propositions for $R^n$.

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## 2. On a theorem of Beurling and Helson.

First some notations. $G$ will be a locally compact Abelian (LCA) group and $\Gamma$ its dual group. The characters will be written $(x,y)$, where $x \in G$ and $y \in \Gamma$. In $R^n$, the usual scalar product will be denoted $\langle x,y \rangle$, and on $R^n$ we let the characters be $(x,y) = \exp(2\pi i \langle x,y \rangle)$. The open ball in $R^n$ with center $y_0$ and radius $r$ will be denoted $S(y_0,r)$.

The set of bounded complex measures on $G$ will be denoted $M(G)$. Further, the total variation norm of $\mu$ can be written

$$\|\mu\| = \sup\{\|\mu * f\|_1 \,;\, f \in L_1,\, \|f\|_1 \leq 1\} . \tag{1}$$

This follows immediately if we let $f$ approximate the measure with mass 1 at $x=0$.

For $\mu \in M(G)$ we define the Fourier–Stieltjes transform $\hat{\mu}$ by

$$\hat{\mu}(y) = \int_G (x,y)\, d\mu(x)$$

which is a bounded, uniformly continuous function on $\Gamma$. If $f \in L_1$, its Fourier transform $\hat{f}$ is, in the same way

$$\hat{f}(y) = \int (x,y)\, f(x)\, dx$$

and the set of Fourier–Stieltjes transforms is denoted by $B(\Gamma)$, and for $\mu,\nu \in M(G)$ we have

$$(\mu * \nu)\hat{}\,(y) = \hat{\mu}(y)\, \hat{\nu}(y) ,$$

and so $B(\Gamma)$ is a commutative algebra of functions (containing 1). Normed with $\|\hat{\mu}\| = \|\mu\|$ it is also a Banach algebra.

We are going to study functions which on an open set $\Omega \subset \Gamma$ coincide with a Fourier–Stieltjes transform. We say that $\varphi \in B(\Omega)$ if there is a $\mu \in M(\Gamma)$ so that $\varphi = \hat{\mu}$ on $\Omega$. Inspired by (1), and by some later applications, we introduce a seminorm $\|\varphi\|_\Omega$ on $B(\Omega)$ by

$$\|\varphi\|_\Omega = \sup\{\|\mu * f\|_1 \,;\, f \in L_1, \hat{f} = 0 \text{ outside } \Omega, \|f\|_1 \leqq 1\}$$

which is well defined, since $\varphi\hat{f} = \mu\hat{f}$ does not depend on the behavior of $\hat{\mu}$ outside $\Omega$. If $\Omega = \Gamma$, then $B(\Omega) = B(\Gamma)$ and $\|\varphi\|_\Omega = \|\varphi\|$ by (1). Obviously $\|\cdot\|_\Omega$ is dominated by the ordinary quotient norm in $B(\Omega)$.

The following two well known lemmas will be useful in the following.

LEMMA 1. *Suppose that $\Omega$ is a bounded open set in $R^n$ and $\varepsilon$ a positive number. Then there is a function $k \in L_1$, $0 \leqq k \leqq 1$, with $\hat{k} = 1$ on $\Omega$ and with $\hat{k} = 0$ outside some bounded set, and for which $\|k\|_1 \leqq 1 + \varepsilon$.*

For a proof see Section 2.6 in [7].

LEMMA 2. *If $\mu_m \in M(R^n)$, if $\|\mu_m\| \leqq C$, $m = 1, 2, \ldots$, and if $\hat{\mu}_m \to \varphi$ point-wise on $R^n$ as $m \to \infty$, then there exists a $\nu \in M(R^n)$, with $\|\nu\| \leqq C$, such that $\varphi = \hat{\nu}$ a.e. on $R^n$.*

For a proof we refere to 1.9.2 in [7].

Using the above we will now prove the following lemma, in which some facts about $B(\Omega)$ and $\|\cdot\|_\Omega$ are collected.

LEMMA 3. *Suppose $\varphi$ and $\Psi$ belong to $B(\Omega)$, for some open set $\Omega$ in $R^n$.*

(i) *If $\Omega' \subset \Omega$, then $\varphi \in B(\Omega')$ and $\|\varphi\|_{\Omega'} \leqq \|\varphi\|_\Omega$.*

(ii) *If $a \in R - \{0\}$, $y_0 \in R^n$ and if $a^* \varphi(y) = \varphi(ay)$ and $\varphi_{y_0}(y) = \varphi(y + y_0)$, then $\|\varphi\|_\Omega = \|a^* \varphi\|_{a^{-1}\Omega} = \|\varphi_{y_0}\|_{\Omega - y_0}$.*

(iii) *We have $\|\varphi\Psi\|_\Omega \leqq \|\varphi\|_\Omega \|\Psi\|_\Omega$ and if $k \in L_1$, with $\hat{k} = 0$ outside $\Omega$, then $\varphi\hat{k} \in B(R^n)$ and $\|\varphi\hat{k}\| \leqq \|\varphi\|_\Omega \|k\|_1$.*

(iv) *If $\|\varphi\|_S \leqq C$ for each open bounded ball $S$ in $R^n$, then $\varphi \in B(R^n)$ and $\|\varphi\| \leqq C$.*

PROOF. (i), (ii) and the first part of (iii) are obvious consequences of the corresponding facts for elements in $B(R^n)$. The second statement in (iii) follows from (1). We have

$$\begin{aligned}\|\varphi\hat{k}\| &= \sup\{\|\mu * k * f\|_1 \,;\, f \in L_1, \|f\|_1 \leqq 1\} \\ &\leqq \sup\{\|\mu * f\|_1 : f \in L_1, \hat{f} = 0 \text{ outside } \Omega, \|f\|_1 \leqq \|k\|_1\} = \|\varphi\|_\Omega \|k\|_1 .\end{aligned}$$

The proof of (iv) is slightly more complicated. Let $\varepsilon > 0$ be arbitrary. By Lemma 1 we can choose a sequence $\{S_j\}_1^\infty$ of bounded open balls, and an associated sequence of functions $k_j \in L_1$, so that

(a) $\bar{S}_j \subset S_{j+1}$ and $\bigcup_{j=1}^\infty S_j = R^n$,
(b) $\hat{k}_j = 1$ on $S_j$ and $\hat{k}_j$ is 0 outside $S_{j+1}$,
(c) $\|k_j\|_1 \leqq 1 + \varepsilon$.

Let $\varphi = \hat{\mu}_j$ on $S_{j+1}$, where $\mu_j \in M(R^n)$, and let $\hat{\nu}_j = \hat{\mu}_j \hat{k}_j$. Then $\hat{\nu}_j \to \varphi$ on $R^n$ (in fact uniformly on compact sets) and, by (iii).

$$\|\nu_j\| = \|\mu_j * k_j\| = \|\varphi \hat{k}_j\| \leqq \|\varphi\|_{S_{j+1}} \|k_j\|_1 \leqq C(1+\varepsilon) .$$

As $\varphi$ is continuous, by Lemma 2 we get that $\varphi \in B(R^n)$ and that $\|\varphi\| \leqq C(1+\varepsilon)$. Since $\varepsilon > 0$ was arbitrary the lemma is proved.

We will now state the main theorem of this section. It is a generalization of a theorem of Beurling and Helson [1], who proved it for $\Omega = R$. As we already have remarked, the quotient norm in $B(\Omega)$ is larger than $\|\cdot\|_\Omega$, and so the following theorem holds also for the quotient norm.

THEOREM 1. *Let $\Omega$ be an open connected set in $\Gamma$, and assume that $f \in B(\Omega)$, that $|f| = 1$ on $\Omega$, and that there exists a constant $C$ such that*

$$\|f^m\|_\Omega \leqq C, \qquad m = 1, 2, \ldots .$$

*Then there exist a complex number $c$, $|c| = 1$, and an $x_0 \in G$ such that*

$$f(y) = c(x_0, y), \qquad y \in \Omega .$$

The proof of Theorem 1 for $\Gamma = R^n$ will be reduced to the case when $f$ is a sufficiently smooth function (this is the hard part of the proof). Then the following proposition and the structure theoretic Lemma 4 are used to complete the proof of the theorem.

PROPOSITION 1. *Let $S$ be a bounded open ball in $R^n$. Assume that $\exp(i\varphi) \in B(S)$, that $\varphi$ is a real function in $C^2(S)$, and that there exists a constant $C$ such that*

$$\|\exp(im\varphi)\|_S \leqq C, \qquad m = 1, 2, \ldots .$$

*Then $\varphi$ is a linear function on $S$.*

The reduction to the smooth case will be done by proving the following statement.

PROPOSITION 2. *Let $S = S(0, r)$ and suppose that $f \in B(S)$ and $|f| = 1$ on $S$, and that there exists a constant $C$ such that*

$$\|f^m\|_S \leqq C, \qquad m = 1, 2, \ldots .$$

*Then there exists a ball* $S_0 = S(0, r_0)$, $0 < r_0 \leqq r$, *and a real polynomial* $\varphi$ *such that* $f = \exp(i\varphi)$ *on* $S_0$.

Proposition 1 is proved by an idea used by Hörmander to prove it when $S = R^n$ (by Lemma 3 this case is a consequence of Proposition 1). A proof can be found in Lemma 5 in [2].

The proof of Proposition 2 is more complicated. The idea of the proof is essentially that of Beurling and Helson [1], who used it to prove the proposition for $S = R$. The main tool in the proof is the following lemma, which in an essential way depends on the obvious fact that the only characteristic functions in $B(S)$ are those which are either identically 1 or 0 on $S$ (since $S$ is connected).

MAIN LEMMA. *Let* $S = S(0, r)$. *Suppose that* $f$ *and* $g$ *belong to* $B(S)$ *and that* $|g| = 1$ *on* $S$, *that* $f = g$ *on a closed subset of* $S$ *with positive measure, and that there exists a constant* $C$ *such that*

$$\|f^m\|_S \leqq C, \quad \|g^m\|_S \leqq C, \qquad m = 1, 2, \ldots .$$

*Then* $f = g$ *on* $S$.

Both the proof of this lemma and the rest of Proposition 2 are obvious modifications of the corresponding proof in [1]. Notice that we only have to treat the case when $S$ is bounded.

The general case will be proved by the use of the following lemma.

LEMMA 4. *If a locally compact Abelian group* $\Gamma_0$ *is connected, then the set of all one-parameter subgroups of* $\Gamma_0$ *is dense in* $\Gamma_0$.

PROOF. Since any element in the subgroup of $\Gamma_0$ generated by the one-parameter subgroups can be imbedded in a one-parameter subgroup of $\Gamma_0$, the lemma follows from Theorem 25.20 in [6].

PROOF OF THEOREM 1. We first consider the case $\Gamma = R^n$. Since $\Omega \subset R^n$ is open and connected, and so arc-wise connected, any two points in $\Omega$ can be connected by a chain of open balls contained in $\Omega$. It is sufficient to prove that for every point $y_0 \in \Omega$ there exist an open ball $S_0 = S(y_0, r_0)$ contained in $\Omega$ and a real linear function $\varphi$ such that $f = \exp(i\varphi)$ on $S_0$.

By (ii) of Lemma 3 it is no restriction to assume that $y_0 = 0$. Then, by (i) of Lemma 3 the assumptions in Proposition 2 are satisfied, and so the existence of $S_0$ with the wanted properties follows from Proposition 1, thus proving Theorem 1 in this case.

To prove the general case we will apply Lemma 4. Again we can assume that $0 \in \Omega$. Let $\Gamma_0$ be the largest connected closed subgroup of $\Gamma$

that contains 0. Then $\Gamma_0$ also contains $\Omega$. Let $R$ be any one-parameter subgroup of $\Gamma_0$.

By what we have already proved $f$ coincides on $\Omega \cap R$ with an affine mapping of $R$ into the circle group, that is, $f$ coincides on $\Omega \cap R$ with a constant of modulus 1 times a character. Since $f$ is continuous on $\Omega$, and since by Lemma 4 the set of one-parameter subgroups of $\Gamma_0$ is dense in $\Gamma_0$, it follows that $f$ has the form

$$c(x_0, y), \qquad y \in \Omega ,$$

where $|c| = 1$, for some $x_0 \in G$, and Theorem 1 is proved.

## 3. Homomorphisms of ideals of $L_1(G)$ into $M(G')$.

Let $T$ be a homomorphism of a closed ideal $I$ in $L_1(G)$, with values in $M(G')$. Here $G$ and $G'$ are LCA groups with dual groups $\Gamma$ and $\Gamma'$. It is known that some condition has to be imposed on $T$ in order that it should be generated by a piece-wise affine mapping of $\Gamma'$ into $\Gamma$ (cf. Forelli [4]; if $I = L_1(G)$ this is however always the case, as Cohen [3] proved).

That a mapping $\varphi$ is affine means that

$$\varphi(y + y' - y'') = \varphi(y) + \varphi(y') - \varphi(y'')$$

for all $y$, $y'$ and $y''$ in $\Gamma'$. That $\varphi$ is affine on some subset of $\Gamma'$ means that it coincides on this set with an affine mapping.

Using Theorem 1 and the techniques introduced by Cohen and by Beurling and Helson, we shall prove the following theorem.

THEOREM 2. *Suppose that $T$ is a homomorphism of a closed ideal $I$ of $L_1(G)$ into $M(G')$, and let $\Delta_2 = \{y \,;\, (Tf)\hat{}\,(y) \neq 0,$ some $f \in I\}$. Then there exist a real continuous function $\varphi$ from $\Gamma'$ to $\Gamma$, which is affine in the interior of each component of $\Delta_2$, such that*

$$(Tf)\hat{}\,(y) = 0, \qquad y \notin \Delta_2 ,$$
$$(Tf)\hat{}\,(y) = \hat{f}(\varphi(y)), \qquad y \in \Delta_2 .$$

When $I = L_1(G)$ and $T$ was onto $L_1(G')$ this was proved by Beurling and Helson [1], and as mentioned above a sharper result was proved by Cohen [3] under the assumption $I = L_1(G)$. Assuming that $\|T\| = 1$, Forelli [4] proved that $T$ is generated by an affine mapping (i.e. the same map on all components of $\Delta_2$). The above theorem is neither contained in nor contains the theorem proved by Forelli.

We need some well known results and notations, mainly from the ideal theory of $L_1$. Most of the proofs can be found in [6] and [7].

LEMMA 5. *Let $\Delta_1$ be the set of $y \in \Gamma$ such that $\hat{f}(y) \neq 0$ for some $f \in I$. Then each $y \in \Delta_1$ defines a non-trivial homomorphism into the complex numbers, whose value at $f \in I$ is $\hat{f}(y)$. Conversely, each non-trivial complex homomorphism is obtained in this way.*

A proof for locally compact groups can be found in [6], Theorem 23.4.

Each $y \in \Delta_2$ defines a non-trivial complex homomorphism of $I$, the value of which at $f \in I$ is $(Tf)\hat{}(y)$. From Lemma 6 it follows that there exists a $\varphi(y) \in \Delta_1$, such that $(Tf)\hat{}(y) = \hat{f}(\varphi(y))$. The function $\varphi\colon \Delta_2 \to \Delta_1$ determines $T$:

$$\begin{aligned} (Tf)\;\,(y) &= 0, & y &\notin \Delta_2\,, \\ (Tf)\hat{}(y) &= \hat{f}(\varphi(y)), & y &\in \Delta_2\,. \end{aligned}$$

From the uniqueness theorem and the theorem on the closed graph, it follows that $T$ is continuous, i.e.

$$\|Tf\| \leqq C\|f\|_1, \qquad f \in I\,,$$

for some constant $C$ (and so $\varphi$ is continuous on $\Delta_2$).

We want to prove that $\varphi$ is affine in the interior of each component of $\Delta_2$. To proceed to the next step in the proof of this we need the following lemma.

LEMMA 6. *For each $y_0 \in \Delta_1$, there exist an open set $S_0$ containing $y_0$ and a function $g_0 \in I$, such that $\hat{g}_0 = 1$ on $S_0$ and $\hat{g}_0 = 0$ in a neighborhood of $\Gamma - \Delta_1$.*

For a proof we refere to Section 7.2 in [7], especially Corollary (a) of 7.2.5.

We will next consider the case when $\Gamma' = R$.

To get shorter statements, we introduce some notation. We let $\hat{I} = \{\hat{f}; f \in I\}$ and let $\hat{T}$ be the homomorphism with values in $B(R)$ defined by

$$\hat{T}(\hat{f}) = (Tf)\hat{}, \qquad \hat{f} \in \hat{I}\,.$$

The main step in the proof of Theorem 2 is now to show that $\hat{T}$ can be continuously extended to a mapping from $B(\Gamma)$ to $B(S)$, $S$ some open set in $R$. More precisely we prove the following proposition.

PROPOSITION 3. *Suppose $y_0 \in \Delta_2$. Then there exists an open interval $S_0$ containing $y_0$ such that $T$ can be extended to a continuous mapping from $B(\Gamma)$ to $B(S_0)$.*

PROOF. Let $S_0'$ be a neighborhood of $\varphi(y_0)$, and $g_0$ a function in $I$, as in Lemma 8. Since $\varphi$ is continuous we can take $S_0$ containing $y_0$ so that $\varphi(S_0) \subset S_0'$. Let $\mu \in M(G)$ and define

$$\lambda(y) = \hat{\mu}(\varphi(y)), \qquad y \in \Delta_2 .$$

We shall prove that $\lambda \in B(S_0)$ and that

$$\|\lambda\|_{S_0} \leqq C' \|\mu\| ,$$

where $C'$ is independent of $\mu$. Now $\hat{\mu}\hat{g}_0 \in I$ (since $\hat{g}_0 \in \hat{I}$ and $\hat{\mu}\hat{g}_0 = \hat{k}\hat{\mu}\hat{g}_0$, $k \in L_1$ and $\hat{k} = 1$ on the set where $g_0 \neq 0$), and so, on $\Delta_2$

$$\hat{T}(\hat{g}_0)\lambda = \hat{g}_0(\varphi)\,\hat{\mu}(\varphi) = (\hat{g}_0\hat{\mu})(\varphi) = \hat{T}(\hat{g}_0\hat{\mu}) .$$

Since $\hat{g}_0 = 1$ on $S_0$, we find for any $h \in L_1$, with $\hat{h} = 0$ outside $S_0$, that

$$\lambda\hat{h} = \hat{T}(\hat{g}_0\hat{\mu})\hat{h} .$$

Thus

$$\|\lambda\|_{S_0} \leqq C \|g_0\|_1 \|\mu\| = C' \|\mu\| ,$$

and the proposition is proved.

PROOF OF THEOREM 2. We first prove the theorem when $\Gamma' = R$. Let $\Delta_2'$ be an open connected set in $\Delta_2$. Let $\hat{\mu}_{x,m}(y) = (x, my)$, which belongs to $B(\Gamma)$ and has norm 1 for $m = 1, 2, \ldots$ and $x \in G$. Let $y_0$ be an abitrary point in $\Delta_2'$, and let $S_0$ be the corresponding interval, as in Proposition 3. Then we have

$$\lambda^m(y) = \hat{\mu}_{x,m}(\varphi(y)) = (x, m\varphi(y)) \in B(S_0)$$

and

$$\|\lambda^m\|_{S_0} \leqq C', \qquad m = 1, 2, \ldots .$$

By Theorem 1 this means that $(x, \varphi(y))$ is affine on $S_0$. Since $y_0$ in $\Delta_2'$ was arbitrary as was $x \in G$, it follows that $\varphi$ is affine on $\Delta_2'$, and Theorem 2 is proved for $\Gamma' = R$.

The proof for general LCA groups $\Gamma'$ now follows from Lemma 4, as in the proof of Theorem 1.

## 4. Fourier transforms of matrices.

In this section we shall give the counterpart of Theorem 1 for matrices, the elements of which are bounded measures. This we do in Theorem 3, which describes the eigenvalues of power bounded matrices of Fourier–Stieltjes transforms. We only treat the case corresponding to $\Omega = R^n$ in Theorem 1, although it is not hard to change the proof, using the results from section 2, to cover the general situation, i.e. when $\Omega$ is any open connected set.

We first discuss some properties of the algebra of matrices, the elements of which are bounded measures on $R^n$.

If $v=(v_1,\ldots,v_N)$ is a complex $N$-vector, $|v|$ will denote its Euclidean norm,

$$|v| = (\Sigma_{j=1}^N |v_j|^2)^{\frac{1}{2}} ,$$

and if $A$ is an $N\times N$-matrix $|A|$ will denote the norm of $A$,

$$|A| = \sup\{|Av|;\ v=(v_1,\ldots,v_N),\ |v|\leqq 1\} .$$

By $\mathscr{L}_1(R^n)$ (or simply $\mathscr{L}_1$) we mean the set of all complex $N$-vector functions $v=(v_1,\ldots,v_N)$ on $R^n$ such that $v_j\in L_1$, $j=1,\ldots,N$. In $\mathscr{L}_1(R^n)$ we use the norm

$$\|v\|_1 = \int_{R^n} |v(x)|\,dx .$$

Similiarly $\mathscr{M}(R^n)$ denotes the set of $N\times N$-matrices with elements in $M(R^n)$. If $\mu\in\mathscr{M}(R^n)$ its norm is

$$\|\mu\| = \sup\{\|\mu * v\|_1;\ v\in\mathscr{L}_1,\ \|v\|\leqq 1\} ,$$

where $\mu * v$ is defined as the vector with components $(\mu=(\mu_{ij}))$

$$(\mu * v)_k = \sum_{j=1}^N \mu_{kj} * v_j, \qquad k=1,\ldots,N .$$

Analogously $\mu * \nu\in\mathscr{M}(R^n)$ is defined for any $\mu,\nu\in\mathscr{M}(R^n)$ via the usual matrix multiplication, and we get

$$\|\mu * \nu\| \leqq \|\mu\|\,\|\nu\| .$$

We also note that

$$\|\mu_{ij}\| \leqq \|\mu\| \leqq \sum_{j=1}^N \sum_{k=1}^N \|\mu_{kj}\| .$$

It follows that $\mathscr{M}(R^n)$ is a Banach algebra with unit (non-commutative for $N>1$).

For matrices and vectors with elements belonging to $\mathscr{M}(R^n)$ (or $\mathscr{L}_1$) we define the Fourier–Stieltjes transform (or Fourier transform) by taking the transform element by element. If $\mu\in\mathscr{M}(R^n)$, its Fourier–Stieltjes transform is denoted $\hat{\mu}$. The matrix function $\hat{\mu}$ is uniformly continuous and bounded on $R^n$, since its elements are. The set of Fourier–Stieltjes transforms of elements in $\mathscr{M}(R^n)$ is denoted by $\mathscr{B}(R^n)$. If we introduce the norm $\|\hat{\mu}\|=\|\mu\|$, $\mu\in\mathscr{M}(R^n)$, it follows from

$$(\mu * \nu)^\wedge = \hat{\mu}\hat{\nu}$$

that $\mathscr{B}(R^n)$ is a Banach algebra with unit $E$ under pointwise (matrix-) multiplication and addition. It is non-commutative for $N > 1$.

We notice that for each $y_0 \in R^n$ we have

$$|\hat{\mu}(y_0)| \leqq \|\hat{\mu}\| = \|\mu\| . \tag{7}$$

To prove this first choose $v_0$ so that $|\hat{\mu}(y_0)v_0| = |\hat{\mu}(y_0)|$, and then, as in the proof of (1), approximate $v_0$ by the Fourier transform of a vector in $\mathscr{L}_1$. As a consequence of (7) we note that, if $\hat{\mu}^m$ is uniformly bounded in $\mathscr{B}$ for $m=1,2,\ldots$, then $|\hat{\mu}^m(y_0)|$ is uniformly bounded in $m$ and $y_0$. Hence the eigenvalues of $\hat{\mu}(y)$ have modulus at most 1 ($y \in R^n$). When the eigenvalues have modulus 1, we give a complete description of them. This is a consequence of the following theorem, which is the main result of this paper.

THEOREM 3. *If $\varphi \in \mathscr{B}(R^n)$ and if for some constant $C$ we have*

$$\|\varphi^m\| \leqq C, \quad m = \pm 1, \pm 2, \ldots , \tag{8}$$

*then there exist functions $\lambda_1, \ldots, \lambda_N$ of the form*

$$\lambda_j(y) = c_j \exp(i\langle x_j, y\rangle), \quad y \in R^n, \ 1 \leqq j \leqq N , \tag{9}$$

*where $x_j \in R^n$ and $|c_j| = 1$, and such that $\lambda_1(y), \ldots, \lambda_N(y)$ are the eigenvalues of $\varphi(y)$, $y \in R^n$, counted with proper multiplicities.*

As a corollary of this theorem, we have the following.

COROLLARY 1. *If $\varphi \in \mathscr{B}(R^n)$ and if the eigenvalues of $\varphi(y)$ have absolute value* 1 *for all $y \in R^n$, and if*

$$\|\varphi^m\| \leqq C, \quad m = 1, 2, \ldots ,$$

*then there exist functions $\lambda_1, \ldots, \lambda_N$ of the form*

$$\lambda_j(y) = c_j \exp(i\langle x_j, y\rangle), \quad y \in R^n, \ 1 \leqq j \leqq N ,$$

*where $x_j \in R^n$ and $|c_j| = 1$, and such that $\lambda_1(y), \ldots, \lambda_N(y)$ are the eigenvalues of $\varphi(y)$, ($y \in R^n$), counted with proper multiplicities.*

PROOF OF COROLLARY 1. By Theorem 3 it is sufficient to prove that all powers of $\varphi$ are uniformly bounded in $\mathscr{B}(R^n)$. For $m \geqq 1$ we have

$$\varphi^{-m} = \tilde{\varphi}^{(m)}(\det \varphi^m)^{-1} = \tilde{\varphi}^{(m)}\, \overline{\det \varphi^m}$$

since all the eigenvalues of $\varphi$ have modulus 1. The elements of $\tilde{\varphi}^{(m)}$ are sums of products of the elements of $\varphi^m$, and so

$$\|\tilde{\varphi}^m\| \leqq \textstyle\sum_k \sum_l \|\tilde{\varphi}_{kl}^{(m)}\| \leqq \sum_k \sum_l (N-1)!\, \sup_{r,s} \|\varphi_{rs}^{(m)}\|^{N-1} \leqq N N!\, C^{N-1} = C' ,$$

and also

$$\|\overline{\det\varphi^m}\| = \|\det\varphi^m\| \leqq N!\,\sup_{k,l}\|\varphi_{k,l}^{(m)}\|^N \leqq N!\,C^N = C''\,.$$

Hence the powers of $\varphi$ are uniformly bounded in $\mathscr{B}(R^n)$, and the corollary is proved.

Before we give the proof of Theorem 3, we make some remarks.

By the remarks preceding Theorem 3, if $\varphi \in \mathscr{B}(R^n)$ satisfies (8), then the eigenvalues of $\varphi(y)$ have modulus 1, for all $y \in R^n$. Hence Theorem 3 and Corollary 1 are equivalent.

It will be clear from proof of Theorem 3, that if $\varphi$ has bounded positive powers in $\mathscr{B}(R^n)$, and if for all $y$ in some open connected set $\Omega$ $\varphi(y)$ has an eigenvalue of modulus 1, then $\varphi(y)$ has an eigenvalue of the form (9) on the whole of $R^n$.

If $\varphi$ satisfies (8), and if $p$ and $p^{-1}$ belong to $\mathscr{B}(R^n)$, then also $\psi = p\varphi p^{-1}$ satisfies (8).

The next example shows that there is no obvious converse of Theorem 3. We shall construct a matrix function $\varphi$ in $\mathscr{B}(R)$ which has eigenvalues 1 and $\exp(2\pi i y)$, i.e. eigenvalues of the form (9). Further $|\varphi^m(y)|$ is uniformly bounded for $y \in R$ and $m = \pm 1, \pm 2, \ldots$, but $\|\varphi^m\| \to \infty$ as $m \to \infty$.

Let $\chi_1(y) = 0$ for $y \leqq 0$, $\chi_1(y) = \exp(2\pi i/y)$ for $0 < y < 1$, and $\chi_1(y) = 1$ for $y \geqq 1$, and let $\lambda(y) = \exp(2\pi i y)$. Multiply $\chi_1$ with a $C^\infty$-function with compact support which is 1 on $[-1, 2]$, and call this new function $\chi$. It is easy to verify that $\chi(1-\lambda)$ is Lipschitz continuous of order 1 for $0 \leqq y \leqq 1$. Hence we see that $\chi(1-\lambda) \in B(R)$. Let

$$\varphi = \begin{pmatrix} 1 & \chi(1-\lambda) \\ 0 & \lambda \end{pmatrix} \in \mathscr{B}(R)\,.$$

We have

$$\varphi^m = \begin{pmatrix} 1 & \chi(1-\lambda^m) \\ 0 & \lambda^m \end{pmatrix}.$$

Since

$$|\chi(y)\big(1-\lambda^m(y)\big)| \leqq 2$$

we see that

$$|\varphi^m(y)| \leqq 4\,,$$

and so $|\varphi^m(y)|$ is uniformly bounded.

We suppose that

$$\|\varphi^m\| \leqq C, \qquad m = 1, 2, \ldots\,.$$

From this we shall derive a contradiction. From the assumption above we conclude that

$$\|\chi(1-\lambda^m)\| \leqq C, \qquad m=1,2,\ldots .$$

Now $\lambda^m(y)=\lambda(my)$, and so we get, with $\chi(y/m)=\chi_m(y)$,

$$\|\chi_m(1-\lambda)\| = \|\chi(1-\lambda^m)\| \leqq C, \qquad m=1,2,\ldots .$$

Let $\Omega=(\frac{1}{3},\frac{2}{3})$. Since $(1-\lambda)^{-1}$ is analytic on $\Omega$, it belongs to $B(\Omega)$. Lemma 3(i) then gives

$$\|\chi_m(1-\lambda)\|_\Omega \leqq \|\chi_m(1-\lambda)\| \leqq C, \qquad m=1,2,\ldots ,$$

and so

$$\|\chi_m\|_\Omega \leqq \|(1-\lambda)^{-1}\|_\Omega C = C', \qquad m=1,2,\ldots .$$

But in $\Omega$ we have $\chi_m(y) = \exp(2\pi i m/y) = \chi^m(y)$. Hence

$$\|\chi^m\|_\Omega \leqq C, \qquad m=1,2,\ldots .$$

This contradicts Theorem 1, and so $\|\varphi^m\|$ can not be bounded as $m \to \infty$.

The rest of this section is devoted to the proof of Theorem 3. We first prove the counterpart of the Main Lemma from Section 2 in the matrix case. The Main Lemma and Theorem 1 are the essential tools in the proof of Theorem 3.

MAIN LEMMA. *Let $\varphi \in \mathscr{B}(R^n)$ and suppose that $g \in B(R^n)$ and that $g(y)$ is an eigenvalue of $\varphi(y)$ for all $y$ in a set of positive measure. Assume further that there exists a constant $C$ such that*

$$\|\varphi^m\| \leqq C, \qquad m=1,2,3,\ldots ,$$
$$\|g^m\| \leqq C, \qquad m=\pm 1, \pm 2,\ldots .$$

*Then $g(y)$ is an eigenvalue of $\varphi(y)$, for all $y \in R^n$, with constant multiplicity a.e. on $R^n$.*

PROOF. Let $\Phi=g^{-1}\varphi$. Then $\Phi \in \mathscr{B}(R^n)$ and

$$\|\Phi^m\| \leqq \|g^{-m}\|\,\|\varphi^m\| \leqq C^2, \qquad m=1,2,\ldots .$$

Further let

$$\psi = \tfrac{1}{2}(E+\Phi)\,.$$

Then $\psi \in \mathscr{B}(R^n)$ and

$$\|\psi^m\| \leqq 2^{-m}\sum_{j=0}^{m}\binom{m}{j}\|\Phi^j\| \leqq C^2, \qquad m=1,2,\ldots .$$

Further the only eigenvalues of $\psi$ of modulus 1 are the 1's (the other eigenvalues having modulus less then 1). Using (7) we see that $\psi$ has bounded powers, and so

$$\tilde{\psi}(y) = \lim_{m\to\infty}\psi^m(y)$$

exists for all $y \in R^n$, and the matrix $\tilde{\psi}$ is of the form $PDP^{-1}$, where $D$ is a diagonal matrix with 1's and 0's in the main diagonal.

Let $\Lambda$ be the set of all $y \in R^n$ such that $g(y)$ is an eigenvalue of $\varphi(y)$. Then $\Lambda$ is a closed set of positive measure, by the assumption. If $y \in \Lambda$ then $\Phi(y)$ has 1 as an eigenvalue, and so also $\Psi(y)$ has an eigenvalue 1. It follows that for $y \in \Lambda$ we have an eigenvalue 1 of $\tilde{\Psi}(y)$. The multiplicity of $g(y)$ as an eigenvalue of $\varphi(y)$ is the same as that of 1 of $\tilde{\Psi}(y)$. If $y \notin \Lambda$ then $g(y)$ is not an eigenvalue of $\varphi(y)$, hence 1 is not an eigenvalue of $\Phi(y)$. It follows that there are no eigenvalues of modulus 1 of $\Psi(y)$ when $y \notin \Lambda$, and so $\tilde{\Psi}(y) = 0$.

An application of Lemma 2 to the elements of $\Psi^m$ shows that there is a $\nu \in \mathscr{M}(R^n)$ such that

$$\tilde{\Psi} = \hat{\nu}, \qquad \text{a.e. on } R^n .$$

The eigenvalues of $\hat{\nu}(y)$ can be chosen as continuous functions of $y \in R^n$. Since the eigenvalues of $\tilde{\Psi}(y)$ only take the values 0 and 1, it follows that these functions must be constants, either 1 or 0, on the whole of $R^n$. In particular the multiplicity of 1 as an eigenvalue of $\hat{\nu}(y)$ is constant.

Now $\tilde{\Psi} \neq 0$ on $\Lambda$. As $\Lambda$ has positive measure, the above implies that $\hat{\nu} \neq 0$ on $R^n$. Since $R^n - \Lambda$ is open and $\hat{\nu} = 0$ a.e. on this set, $\Lambda = R^n$. Furthermore we see that $g(y)$ has the same multiplicity a.e. on $R^n$ as an eigenvalue of $(y)$, as 1 has as an eigenvalue of $\hat{\nu}(y)$. This completes the proof of the Main lemma.

We will need a version of the Wiener–Levy theorem.

LEMMA 7. (*Wiener–Levy*) *Let $S$ be an open set in $R^n$, let $\varphi_1, \ldots, \varphi_N \in B(S)$ and let $y_0 \in S$. If $F$ is a holomorphic function of $N$ complex variables in a neighbourhood of $(\varphi_1(y_0), \ldots, \varphi_N(y_0))$ then there exists, for any neighbourhood $S_1 \subset S$ of $y_0$, an open ball $S_0 = S(y_0, r_0) \subset S_1$ such that $F(\varphi_1, \ldots, \varphi_N)$ coincides on $S_0$ with the Fourier transform of an $L_1$-function $f$, with $\hat{f} = 0$ outside $S_1$.*

We will also need the following variant of the Wiener theorem.

LEMMA 10. *If $v \in \mathscr{L}_1(R^n)$ and if $\hat{v} \neq 0$ on an open ball $S = S(y_0, r)$, then there exist an open ball $S_0 = S(y_0, r_0)$, $0 < r_0 \leqq r$, and a constant $C$ such that for every $f \in L_1$ with $\hat{f} = 0$ outside $S_0$ we have the inequality*

$$\|f\|_1 \leqq C \|f \star v\|_1 . \tag{10}$$

We can now give a proof of Theorem 3.

PROOF OF THEOREM 3. Let $\varphi(y) = \hat{\mu}(y)$, $\mu \in \mathscr{B}(R^n)$ and let $\lambda(y)$ be an eigenvalue of $\varphi(y)$, for all $y \in R^n$. Let $\Omega$ be any non-void open connected set in $R^n$. We shall first make use of the fact that the elements of $\varphi$ belong to $B(R^n)$. So we choose a ball $S_1 = S(y_0, r_1)$, $r_1 > 0$, in $\Omega$ such that $\lambda$ and a corresponding eigenvector $u$ are analytic functions of the elements of $\varphi$ when restricted to $S_1$. We can assume that $u \neq 0$ on $S_1$. By Lemma 7 there exists a function $g \in L_1$ and a vector $v \in \mathscr{L}_1$ and a ball $S_2 = S(y_0, r_2)$, $r_1 > r_2 > 0$, such that $\hat{g} = \lambda$ and $\hat{v} = u$ on $S_2$. If we now apply Lemma 8 to $v$ and $S_2$ we finally find a ball $S = S(y_0, r)$, $r > 0$, such that (10) holds for all $f \in L_1$ with $\hat{f} = 0$ outside $S$. For such a function $f$ we have

$$\begin{aligned} \varphi^m(y)\,\hat{v}(y)\,\hat{f}(y) \;=\; \varphi^m(y)\,u(y)\,\hat{f}(y) \;&=\; \lambda^m(y)\,u(y)\,\hat{f}(y) \\ &=\; \hat{g}^m(y)\,\hat{v}(y)\,\hat{f}(y) \end{aligned}$$

for $m = 1, 2, \ldots$. We can estimate the norm of the left hand side by

$$\|\varphi^m \hat{v}\hat{f}\| \;=\; \|\mu^{*m} * v * f\|_1 \;\leqq\; \|\mu^{*m}\|\,\|f * v\|_1 \;\leqq\; C\,\|v\|_1\,\|f\|_1\,; \tag{11}$$

where $\mu^{*m}$ is the convolution of $\mu$ with itself $m$ times. Using (10) we get an estimate to below for the norm of $\|\varphi^m \hat{v}\hat{f}\|$,

$$\|g^{*m} * f\|_1 \;\leqq\; C\,\|g^{*m} * v * f\|_1 \;=\; C\,\|\varphi^m \hat{v}\hat{f}\|\,. \tag{12}$$

From (11) and (12) we get, with a new constant $C$,

$$\|g^{*m} * f\|_1 \;\leqq\; C\,\|f\|_1\,. \tag{13}$$

By the definition of $\|\cdot\|_S$, however, (13) means that

$$\|\lambda^m\|_S \;=\; \|\hat{g}^m\|_S \;\leqq\; C, \qquad m = 1, 2, \ldots\,. \tag{14}$$

By the remarks preceding the theorem, we have $|\lambda| = 1$ and therefore we can apply Theorem 1 to $\lambda$. It follows that $\lambda$ is the restriction to $S$ of a function of the form

$$h(y) \;=\; c\,\exp(i\langle x, y\rangle), \qquad |c| = 1 \text{ and } x \in R^n 1\,,$$

that is, a function that has bounded powers in $B(R^n)$. The Main Lemma then shows that $h$ is an eigenvalue of $\varphi$ on the whole of $R^n$. Since the non-void open set $\Omega$ was arbitrary, as was the eigenvalue $\lambda$, the fact that $h$ has constant multiplicity a.e. on $R^n$ as an eigenvalue of $\varphi$ concludes our proof.

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