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# A NOTE ON EQUIMORPHISMS OF PROXIMITY SPACES

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It is well known [3] that two realcompact spaces $X$ and $Y$ are homeomorphic if and only if $C(X)$ and $C(Y)$ are isomorphic, where $C(X)$ and $C(Y)$ are the rings of continuous real-valued functions on $X$ and $Y$, respectively. The purpose of this note is to prove: that if $X$ and $Y$ are realcompact proximity spaces, then $X$ and $Y$ are equimorphic if and only if there is an isomorphism of $C(X)$ onto $C(Y)$ which maps the class of $\delta$-functions in $C(X)$ onto the class of $\delta$-functions in $C(Y)$.

The class $P(X)$ of $\delta$-functions in $C(X)$ is generally not an ideal of $C(X)$. The rudimentary algebraic structure of $P(X)$ was observed in [2].

Let $\delta_1$ and $\delta_2$ be proximity relations on $X$ and $Y$, respectively. The proximity spaces $X$ and $Y$ are equimorphic [5] if there exists a mapping $t$ of $X$ onto $Y$ such that both $t$ and $t^{-1}$ are $\delta$-functions.

We define $C(X)$ to be $\delta$-isomorphic to $C(Y)$ if there exists an isomorphism $\gamma$ of $C(X)$ onto $C(Y)$ such that $\gamma[P(X)]=P(Y)$.

THEOREM. *If $X$ and $Y$ are realcompact proximity spaces, then $X$ and $Y$ are equimorphic if and only if $C(X)$ and $C(Y)$ are $\delta$-isomorphic.*

PROOF. If $X$ and $Y$ are equimorphic, it is obvious that $C(X)$ and $C(Y)$ are isomorphic. It is easily verified that $C(X)$ and $C(Y)$ are also $\delta$-isomorphic.

Conversely, let $\gamma$ be a $\delta$-isomorphism of $C(X)$ onto $C(Y)$. Then $f$ is a bounded $\delta$-function in $C(X)$ if and only if $\gamma(f)$ is a bounded $\delta$-function in $C(Y)$. (See theorem 1.7 in [3].) Thus the restriction of $\gamma$ to the ring $P^*(X)$ of bounded $\delta$-functions in $C(X)$ is an isomorphism of $P^*(X)$ onto the corresponding ring $P^*(Y)$.

Every function $f$ in $P^*(X)$ has a unique extension $f^*$ in $C(\delta_1 X)$, where $\delta_1 X$ is the Smirnov compactification of $X$. (See [4] or [6].) Conversely, the restriction to $X$ of a function $f^*$ in $C(\delta_1 X)$ is in $P^*(X)$. Hence $\gamma$ induces a $\delta$-isomorphism $\gamma^*$ of $C(\delta_1 X)$ onto $C(\delta_2 Y)$. Since $\delta_1 X$ and $\delta_2 Y$ are compact, a homeomorphism $t^*$ of $\delta_1 X$ onto $\delta_2 Y$ is induced by $\gamma^*$. Let $t$ be the restriction of $t^*$ to $X$. Then $t$ is an equimorphism of $X$ into

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$\delta_2 Y$, since $X$ is a $\delta$-subspace of $\delta_1 X$. We next show that $t$ maps $X$ onto $Y$.

Let $a \in X$ and let $\bar{M}_a$ be the ideal of functions $f^*$ in $C(\delta_1 X)$ that vanish at $a$. Then $t(a) = b$ if and only if $\gamma^*[\bar{M}_a] = \bar{M}_b$, where $\bar{M}_b$ is the ideal of functions $\gamma^*(f^*)$ that vanish at $b$. But $f^* \in \bar{M}_a$ is equivalent to $f \in M_a \cap P^*(X)$, where $M_a$ is the maximal ideal in $C(X)$ fixed at $a$. Now $M_a$ is real, so that $\gamma[M_a] = M_c$ is a real maximal ideal in $C(Y)$ that is fixed at a point $c \in Y$, since $Y$ is realcompact. (See [3, theorem 8.3].) Thus $\gamma^*(f^*) \in \bar{M}_b$ if and only if $\gamma(f) \in M_c \cap P^*(Y)$. Since $\gamma^*(f^*) = \gamma(f)^*$, it follows that $b$ and $c$ are zeros of every function in $\bar{M}_b$, and since the correspondence between the maximal ideals of $C(\delta_2 Y)$ and the points of $\delta_2 Y$ is one-to-one, we have $b = c$. Thus the values of $t$ are in $Y$. That $t$ is onto $Y$ follows from the realcompactness of $X$. This completes the proof.

REMARKS. Proximity spaces with $\delta$-isomorphic rings of continuous real-valued functions may fail to be homeomorphic. An example is provided by the space $W$ of all countable ordinals with the interval topology together with the Stone–Čech compactification $W^*$ of $W$. If $\delta$ is the proximity on $W$ induced by $C(W)$, then $C(W)$ is $\delta$-isomorphic to $C(W^*)$, and $W$ and $W^*$ are not homeomorphic. Thus the condition that $X$ and $Y$ be realcompact cannot be omitted from the theorem.

There also exist homeomorphic realcompact spaces $X$ and $Y$, so that $C(X)$ and $C(Y)$ are isomorphic, but where $X$ and $Y$ are not equimorphic. (See [5].)

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